LAURENT POLYNOMIAL APPROXIMATIONS?

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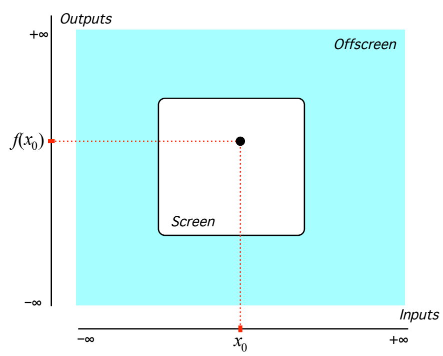
Laurent polynomials (`https://en.wikipedia.org/wiki/Laurent_polynomial`) are nice because, like decimal numbers in arithmetic, they allow us to *approximate* and therefore to get what we want at the least possible cost[1]. But how does one sell such an idea in a culture which still prefers miles, feet and inches to kilometers, meters and centimeters, and even more to the point, fractions to decimals? Being an obdurate curmudgeon, though, I will try to make a case for "approximate algebra" in a series of pieces.

I will begin by showing here how Laurent polynomial approximations provide a nice alternative to the highly questionable but unfortunately usual recipe for graphing a function: "*Pick a few inputs, compute the outputs, and join the plot points smoothly*". (If nothing else, how about the fact that there could be any number of *poles* inbetween the plot points not to mention *oscillations*? Moreover, how can a recipe foster any *understanding of the function's behavior?*)

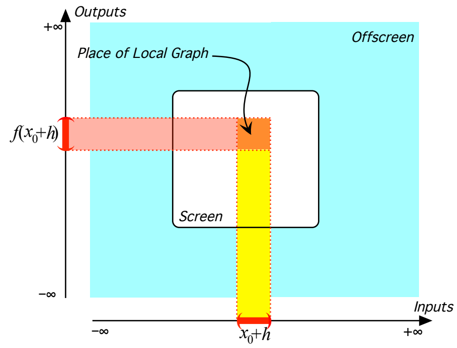
The general idea will be to "thicken the plot", that is, given an input, instead of going for the *plot point at* the given input



which registers only the output *at* the given input, we will want the *local graph near* the given input

[1]Why *Laurent* polynomials? If only because

$$8\,765.432 = 8x^{+3} + 7x^{+2} + 6x^{+1} + 5x^{0} + 4x^{-1} + 3x^{-2} + 2x^{-3}\big|_{x\leftarrow 10}$$



because, in addition to registering with its *height* the outputs for inputs *near* the given input, a local graph also registers with its *slope* and its *concavity* how the outputs *change near* the given input.

So, instead of just computing *outputs at* given inputs, we will compute *approximate outputs near* given inputs. But then the immediate question is: near which given inputs? The answer, though, is rather reasonable: look at what's *large*. More precisely, *large* inputs but also *bounded* inputs whose nearby inputs have *large* outputs. In other words, we will want the local graphs near ∞ and near *poles* if any.

For instance, in the case of the function $x \xrightarrow{\ f\ } f(x) = \frac{x^2-4}{x^2+x-6}$:

1. We declare that x is near ∞:

$$x \text{ near } \infty \xrightarrow{\ f\ } f(x) = \frac{x^2-4}{x^2+x-6}$$

We approximate separately Numerator $f(x)$ and Denominator $f(x)$ keeping in mind that x is *large*:

$$= \frac{x^2 + [...]}{x^2 + [...]}$$

We divide (*short* division):

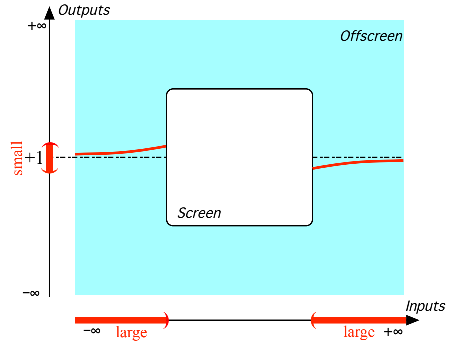
$$= +1 + [...]$$

where [...], read "something too small to matter here", is a proto Bachmann-Landau little o.

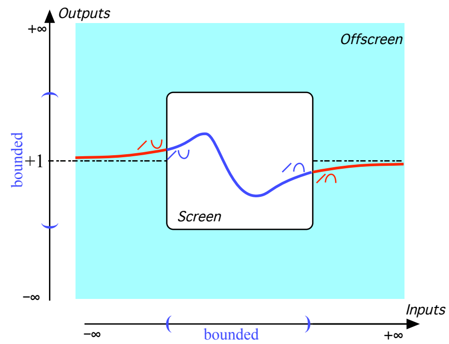
However, since here we don't want just the *height* (sometimes, though, that's all we need as when we just want the *sign* of the output near ∞) but also the *slope* and the *concavity*, what we ignored above was not "too small to matter" and we need the *long* division which we stop as soon as we get to a term with *slope* and *concavity*:

$$x \text{ near } \infty \xrightarrow{\ f\ } = \frac{x^2-4}{x^2+x-6}$$
$$= +1 - x^{-1} + [...]$$

which gives us the local graph of f near ∞



2. The question now is whether we may just join the local graph near ∞ smoothly across the screen as in



or if there might not be bounded inputs near which f returns *large* outputs (AKA poles). But for $f(x_0 + h)$ to be *large*, either

- Numerator $f(x_0 + h)$ would have to be *large*. But since Numerator f is a polynomial function Numerator $f(x)$ can be *large* only near ∞,

or

- Denominator $f(x_0 + h)$ would have to be *small*. But since

$$x \leftarrow x_0 + h \xrightarrow{\text{Denominator } f} \text{Denominator } f(x_0 + h) = (x_0 + h)^2 + (x_0 + h) - 6$$
$$= [x_0^2 + x_0 - 6] + [2x_0 + 1]h + [1]h^2$$

in order for Denominator $f(x_0 + h)$ to be *small*, the constant term $[x_0^2 + x_0 - 6]$, the only term that is not *small*, must be 0.
We compute Discriminant $[x^2 + x - 6] = +25$ which gives us that $[x_0^2 + x_0 - 6]$ will be 0 when x_0 is $+2$ or -3. So, Denominator $f(x_0 + h)$ will be *small* near $+2$ and -3 from which we get that $f(x_0 + h)$ *may* be *large* near $+2$ and/or -3.

To decide, we look at the approximate outputs near $+2$ and near -3:

- We declare that x is near $+2$:

$$x \leftarrow +2 + h \xrightarrow{f} f(+2 + h) = \frac{(+2 + h)^2 - 4}{(+2 + h)^2 + (+2 + h) - 6}$$

We approximate separately Numerator $f(+2+h)$ and Denominator $f(+2+h)$ keeping in mind that h is small :

$$= \frac{+4h + [...]}{+5h + [...]}$$

We divide (*short* division):

$$= +\frac{4}{5} + [...]$$

So, $+2$ is not a pole (and, by the way and the same token, nor is it a zero) and we need not bother with the local graph near $+2$.

- We declare that x is near -3:

$$x \leftarrow -3 + h \xrightarrow{\quad f \quad} f(-3+h) = \frac{(-3+h)^2 - 4}{(-3+h)^2 + (-3+h) - 6}$$

We approximate separately Numerator $f(-3+h)$ and Denominator $f(-3+h)$ keeping in mind that h is small :

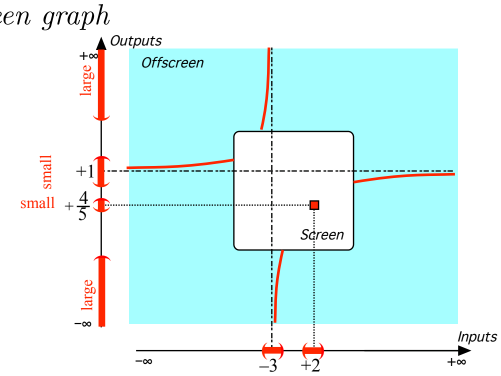
$$= \frac{+5 + [...]}{-5h + [...]}$$

We divide (*short* division):

$$= -h^{-1} + [...]$$

So, -3 is a pole and we get the local graph near -3.

We thus have the *offscreen graph*



and I will leave the essential bounded graph, where by "essential" I mean *forced* by the offscreen graph, to the reader's imagination.

3. The *essential* bounded graph gives us some rather useful information about f namely that f has no *essential* extremum, that f has an *essential* zero between -3 and $+2$, that f is *essentially* piecewise increasing, that there is an *essential* concavity sign-change at -3—which we already knew from just the local graph near ∞ but we now have its *location*: $x_{\text{concavity sign-change}} = -3$.

4. However, this *essential* information is only about what would be visible from far away and the *essential* bounded graph says nothing about the existence or non-existence of *non-essential* features (i.e. deformable to the vanishing point) such as *oscillations* and/or

waverings which to be detected require the Laurent polynomial approximaton of the *outputs* *near* a *generic input*:

$$x \leftarrow x_0 + h \xrightarrow{\ f\ } f(x_0+h) = \frac{(x_0+h)^2 - 4}{(x_0+h)^2 + (x_0+h) - 6}$$
$$= \frac{[x_0^2 - 4] + [2x_0]h + [+1]h^2}{[x_0^2 + x_0 - 6] + [2x_0 + 1]h + [+1]h^2}$$

where the *long* division (in ascending powers of h) is usually a somewhat formidable affair. But, at least, the *constant* term

$$= \frac{[x_0^2 - 4] + [...]}{[x_0^2 + x_0 - 6] + [...]}$$
$$= \frac{x_0^2 - 4}{x_0^2 + x_0 - 6} + [...]$$

shows that f is *continuous* and *differentiability* requires only the *linear* term.

$\therefore$

We all have our tricks. Laurent polynomial approximations, though, are not a trick and there would be no point in your trying anything like the above in your Monday morning Precalculus class because it would have no more of a chance to mean anything to your students in the long run than were you to decide in a Developmental Arithmetic course to devote an hour to decimal approximations. We are talking about a mindset so, in both cases, you have to *reconstruct the whole course content.*

Of course, as physicist David Hestenes of Geometric Algebra fame said at the outset of his 2002 Oersted lecture:

Course content is taken [by many] as given, so the research problem is how to teach it most effectively. This approach [...] has produced valuable insights and useful results.However, it ignores the possibility of improving pedagogy by ***reconstructing course content***. (Emphasis added.)

But, as Kipling would have said, that is another story—which I will try to tell in subsequent pieces.