The Elements of Infinitesimal Calculus

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Preface

These notes started the fall of 2004, when I taught Maths 165, Differential Calculus, at Community College of Philadelphia.

The students at that course were Andrea BATEMAN, Kelly BLOCKER, Alexandra LOUIS, Cindy LY, Thoraya SABER, Stephanilee MAHONEY, Brian McCLINTON, Jessica MENDEZ, Labaron PALMER, Leonela TROKA, and Samneak SAK. I would like to thank them for making me a better teacher with their continuous input and questions.

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1 Preliminaries

This chapter introduces essential notation and terminology that will be used throughout these notes. We will often use the symbol $\iff$ for "if and only if", and the symbol $\implies$, "implies." The symbol $\approx$ means *approximately*.

1.1 Real Numbers

In these notes we consider the following sets of numbers, assigning to them special notation.

1 Definition The set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

is the set of *natural numbers*.

Natural numbers allow us to count objects. The sum and product of two natural numbers is also a natural number, and so we say that natural numbers are *closed* under addition and subtraction. So, for example, $1 + 1$ is a natural number, which we write as $1 + 1 \in \mathbb{N}$, read "one times one belongs to the natural numbers." Similarly, $1 \cdot 1 \in \mathbb{N}$. The natural numbers are not closed under subtraction or division, since, for example, $1 - 2 \notin \mathbb{N}$, which we read "one minus two is not in the natural numbers." In order to have a set closed under subtraction, we must adjoin the opposite of the natural numbers, creating thus the following set.

2 Definition The set

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

is the set of *integers*.

The integers are not closed under division, since for example, $1 \div 2 \notin \mathbb{Z}$. Starting from the integers, in order to have a set closed under division, we must adjoin all the quotients of integers, creating thus the following set.

3 Definition The set

$$\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\right\}$$

is the set of *rational numbers*. This is read as "$\mathbb{Q}$ is the set of all fractions a over b such that a is an integer, b is an integer, and b is different from 0."

In other words, fractions, that is, rational numbers, are divisions that we are too lazy to perform.

Notice that we do not allow division by 0. What would happen if we were not that lazy, and actually performed the implicit divisions in the rational numbers? We would get objects like

$$\frac{2}{1} = 2.0, \qquad \frac{1}{2} = 0.5, \qquad \frac{1}{3} = 0.33333\ldots = 0.\overline{3},$$

where in this last division, the division goes on forever, but we see that we repeatedly obtain 3 in the quotient, that is, we get a *periodic decimal*. It can be proved—but we will not do so in these notes—that the set of rational numbers $\mathbb{Q}$ is precisely the set of numbers whose decimal expansion is either finite, or a periodic decimal.

In $\mathbb{Q}$ we have a very elegant system of numbers that is closed under addition, subtraction, multiplication, and division (except division by 0). Do we need more numbers? What happens with numbers like the *Champernowne-Mahler* constant

$$0.12345678910111213141516171819202 1\ldots,$$

which is the decimal number obtained by consecutively writing all the natural number? This number is clearly not a periodic decimal, and hence it is not rational.

To accommodate infinite non-periodic decimals, we must create the following set.

4 Definition The set $\mathbb{R}$ is the set of real numbers, that is, the set of all numbers with either

1. a finite decimal expansion, or
2. an infinite periodic decimal expansion, or
3. an infinite non-periodic decimal.

A real number which is not rational is called *irrational.*

We must remark that "looking into the decimal expansion" of a number is not enough to *prove* that a number is irrational. For example, it was known since the times of Pythagoras that the number $\sqrt{2}$ is irrational. This guarantees that its decimal expansion

$$\sqrt{2} = 1.41421356\ldots$$

does not repeat. If we started, however, with the number $1.41421356\ldots$ we would not know whether it is rational or irrational, for, it may have a very long decimal period, so long that our calculators and computers could not store it. Again, although Archimedes suspected that

$$\pi = 3.14159265\ldots$$

was irrational, a proof of this was not obtained until the eighteenth century by Lambert.

Homework

1.1.1 Problem Give an example of a rational number between $\dfrac{1}{10} = 0.1$ and $\dfrac{1}{9} = 0.\overline{1}$. Give an example of an irrational number between $\dfrac{1}{10} = 0.1$ and $\dfrac{1}{9} = 0.\overline{1}$.

1.2 Intervals

5 Definition An *interval* I is a subset of the real numbers with the following property: if $s \in I$ and $t \in I$, and if $s < x < t$, then $x \in I$. In other words, intervals are those subsets of real numbers with the property that every number between two elements is also contained in the set. Since there are infinitely many decimals between two different real numbers, intervals with distinct endpoints contain infinitely many members. Table 1.1 shews the various types of intervals.

Observe that we indicate that the endpoints are included by means of shading the dots at the endpoints and that the endpoints are excluded by not shading the dots at the endpoints. [1]

[1] It may seem like a silly analogy, but think that in $[a;b]$ the brackets are "arms" "hugging" a and b, but in $]a;b[$ the "arms" are repulsed. "Hugging" is thus equivalent to including the endpoint, and "repulsing" is equivalent to excluding the endpoint.

Interval Notation	Set Notation	Graphical Representation
$[a\,;b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$[2]	
$]a\,;b[$	$\{x \in \mathbb{R} : a < x < b\}$	
$[a\,;b[$	$\{x \in \mathbb{R} : a \leq x < b\}$	
$]a\,;b]$	$\{x \in \mathbb{R} : a < x \leq b\}$	
$]a\,;+\infty[$	$\{x \in \mathbb{R} : x > a\}$	
$[a\,;+\infty[$	$\{x \in \mathbb{R} : x \geq a\}$	
$]-\infty\,;b[$	$\{x \in \mathbb{R} : x < b\}$	
$]-\infty\,;b]$	$\{x \in \mathbb{R} : x \leq b\}$	
$]-\infty\,;+\infty[$	$\mathbb{R}$	

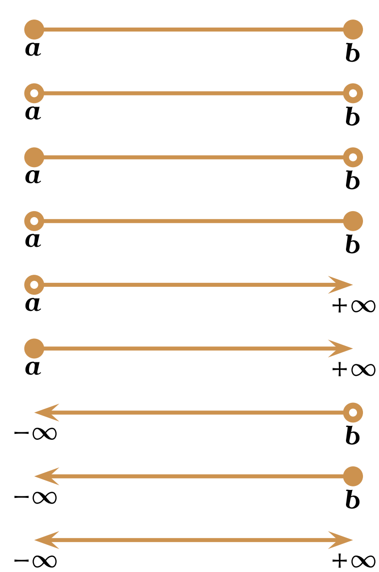


Table 1.1: Intervals.

1.3 Inequalities

Vocabulary Alert! *We will call a number x* positive *if $x \geq 0$ and* strictly positive *if $x > 0$. Similarly, we will call a number y* negative *if $y \leq 0$ and* strictly negative *if $y < 0$. This usage differs from most Anglo-American books, who prefer such newspeak terms as* non-negative *and* non-positive.

The set of real numbers $\mathbb{R}$ is endowed with a relation $>$ which satisfies the following axioms.

6 Axiom (Trichotomy Law) For all real numbers x, y exactly one of the following holds:

$$x > y, \quad x = y, \quad \text{or} \quad y > x.$$

7 Axiom (Transitivity of Order) For all real numbers x, y, z,

$$\text{if} \quad x > y \quad \text{and} \quad y > z \quad \text{then} \quad x > z.$$

8 Axiom (Preservation of Inequalities by Addition) For all real numbers x, y, z,

$$\text{if} \quad x > y \quad \text{then} \quad x + z > y + z.$$

9 Axiom (Preservation of Inequalities by Positive Factors) For all real numbers x, y, z,

$$\text{if} \quad x > y \quad \text{and} \quad z > 0 \quad \text{then} \quad xz > yz.$$

10 Axiom (Inversion of Inequalities by Negative Factors) For all real numbers x, y, z,

$$\text{if} \quad x > y \quad \text{and} \quad z < 0 \quad \text{then} \quad xz < yz.$$

$x < y$ means that $y > x$. $x \leq y$ means that either $y > x$ or $y = x$, etc.

The above axioms allow us to solve several inequality problems.

11 Example Solve the inequality

$$2x - 3 < -13.$$

Solution: ► *We have*

$$2x - 3 < -13 \implies 2x < -13 + 3 \implies 2x < -10.$$

The next step would be to divide both sides by 2. *Since* $2 > 0$, *the sense of the inequality is preserved, whence*

$$2x < -10 \implies x < \frac{-10}{2} \implies x < -5.$$

The solution set is thus the interval $]-\infty\,;-5[$. ◄

12 Example Solve the inequality

$$-2x - 3 \leq -13.$$

Solution: ► *We have*

$$-2x - 3 \leq -13 \implies -2x \leq -13 + 3 \implies -2x \leq -10.$$

The next step would be to divide both sides by -2. *Since* $-2 < 0$, *the sense of the inequality is inverted, and so*

$$-2x \leq -10 \implies x \geq \frac{-10}{-2} \implies x \geq -5.$$

The solution set is therefore $[-5\,;+\infty[$. ◄

The method above can be generalised for the case of a product of linear factors. To investigate the set on the line where the inequality

$$(a_1x + b_1)\cdots(a_nx + b_n) > 0, \tag{1.1}$$

holds, we examine each individual factor. By trichotomy, for every k, the real line will be split into the three distinct zones

$$\{x \in \mathbb{R} : a_kx + b_k > 0\} \cup \{x \in \mathbb{R} : a_kx + b_k = 0\} \cup \{x \in \mathbb{R} : a_kx + b_k < 0\}.$$

Here the sign $\cup$, read "union" means that elements of all the sets involved are considered. We will call the real line with punctures at $x = -\dfrac{a_k}{b_k}$ and indicating where each factor changes sign the *sign diagram* corresponding to the inequality (1.1).

13 Example Consider the inequality

$$x^2 + 2x - 35 < 0.$$

1. Form a sign diagram for this inequality.
2. Write the set $\{x \in \mathbb{R} : x^2 + 2x - 35 < 0\}$ as an interval or as a union of intervals.
3. Write the set $\left\{x \in \mathbb{R} : x^2 + 2x - 35 \geq 0\right\}$ as an interval or as a union of intervals.
4. Write the set $\left\{x \in \mathbb{R} : \dfrac{x+7}{x-5} \geq 0\right\}$ as an interval or as a union of intervals.

5. Write the set $\left\{x \in \mathbb{R} : \dfrac{x+7}{x-5} \leq -2\right\}$ as an interval or as a union of intervals.

Solution: ►

1. *Observe that* $x^2 + 2x - 35 = (x-5)(x+7)$, *which vanishes when* $x = -7$ *or when* $x = 5$. *In neighbourhoods of* $x = -7$ *and of* $x = 5$, *we find:*

$x \in$	$]-\infty\,; -7[$	$]-7\,; 5[$	$]5\,; +\infty[$
$x+7$	$-$	$+$	$+$
$x-5$	$-$	$-$	$+$
$(x+7)(x-5)$	$+$	$-$	$+$

On the last row, the sign of the product $(x+7)(x-5)$ *is determined by the sign of each of the factors* $x+7$ *and* $x-5$.

2. *From the sign diagram above we see that*
$$\{x \in \mathbb{R} : x^2 + 2x - 35 < 0\} =]-7\,; 5[\,.$$
3. *From the sign diagram above we see that*
$$\left\{x \in \mathbb{R} : x^2 + 2x - 35 \geq 0\right\} =]-\infty\,; -7] \cup [5\,; +\infty[\,.$$
Notice that we include both $x = -7$ *and* $x = 5$ *in the set, as* $(x+7)(x-5)$ *vanishes there.*
4. *From the sign diagram above we see that*
$$\left\{x \in \mathbb{R} : \frac{x+7}{x-5} \geq 0\right\} =]-\infty\,; -7] \cup]5\,; +\infty[\,.$$
Notice that we include $x = -7$ *since* $\dfrac{x+7}{x-5}$ *vanishes there, but we do not include* $x = 5$ *since there the fraction* $\dfrac{x+7}{x-5}$ *would be undefined.*
5. *We must add fractions:*
$$\frac{x+7}{x-5} \leq -2 \iff \frac{x+7}{x-5} + 2 \leq 0 \iff \frac{x+7}{x-5} + \frac{2x-10}{x-5} \leq 0 \iff \frac{3x-3}{x-5} \leq 0.$$
We must now construct a sign diagram puncturing the line at $x = 1$ *and* $x = 5$:

$x \in$	$]-\infty\,; 1[$	$]1\,; 5[$	$]5\,; +\infty[$
$3x-3$	$-$	$+$	$+$
$x-5$	$-$	$-$	$+$
$\dfrac{3x-3}{x-5}$	$+$	$-$	$+$

We deduce that
$$\left\{x \in \mathbb{R} : \frac{x+7}{x-5} \leq -2\right\} = [1\,; 5[\,.$$
Notice that we include $x = 1$ *since* $\dfrac{3x-3}{x-5}$ *vanishes there, but we exclude* $x = 5$ *since there the fraction* $\dfrac{3x-3}{x-5}$ *is undefined.*

◄

Homework

1.3.1 Problem

1. Determine a sign diagram for the set
$$\{x \in \mathbb{R} \quad : \quad x(x+1)(x-1) < 0\}.$$
2. Using the sign diagram obtained, write the set $\{x \in \mathbb{R} : x(x+1)(x-1) < 0\}$ as a union of intervals.
3. Write the set $\left\{x \in \mathbb{R} : \dfrac{x(x-1)}{x+1} \geq 0\right\}$ as a union of intervals.
4. Write the set $\left\{x \in \mathbb{R} : \dfrac{x(x-1)}{x+1} \geq -1\right\}$ as an interval.

1.4 Functions

We will take a very narrow approach here to the definition of a function, one that will be useful to our purposes.

14 Definition A *function* $f : \mathbf{Dom}(f) \to \mathbb{R}$, $x \mapsto f(x)$ from the set $\mathbf{Dom}(f)$ to the real numbers is the collection of the following ingredients:

1. A collection of inputs $\mathbf{Dom}(f)$ called the *domain of definition* of the function, being the largest set of real numbers for which the output of the function is a real number.
2. A name for the function, typically f, g, h, etc.
3. A name for a typical input, normally x.
4. An assignment rule or formula, that associates to every input x a unique output $f(x)$.

We will give here some examples of functions.

15 Example Consider the function

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x - x^2 \end{array}.$$

Observe that the domain of definition is $\mathbb{R}$ since the expression $x - x^2$ is a real number for every $x \in \mathbb{R}$. We have, for instance,

$$f(0) = 0 - 0^2 = 0, \qquad f(2) = 2 - 2^2 = -2, \qquad f(1+\sqrt{2}) = 1+\sqrt{2}-(1+\sqrt{2})^2 = 1+\sqrt{2}-(1+2\sqrt{2}+2) = -2-\sqrt{2}.$$

16 Example Consider the function

$$f : \begin{array}{ccc} \mathbb{R} \setminus \{0,1\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{1}{x - x^2} \end{array}.$$

Since the denominator vanishes when $x - x^2 = 0 \implies x(1-x) = 0 \implies x \in \{0,1\}$, the domain of definition is all real numbers excluding those two numbers, which we write as $\mathbb{R} \setminus \{0,1\}$, pronounced "$\mathbb{R}$ without the set containing 1 and 2." We have, for instance,

$$f\left(\frac{1}{2}\right) = \frac{1}{\frac{1}{2} - \frac{1}{4}} = 4.$$

17 Example Consider the function

$$f : \begin{array}{ccc}]-\infty\,;-7] \cup [5\,;+\infty[& \to & \mathbb{R} \\ x & \mapsto & \sqrt{x^2+2x-35} \end{array}.$$

By example 13, the quantity under the square root is strictly negative for $x \in]-7\,;5[$, and hence, the formula will not give a real number output. We must consider only those numbers for which the quantity under the square root is positive and hence the domain of definition of the function is $]-\infty\,;-7] \cup [5\,;+\infty[$.

18 Definition Let f and g be two functions and let the point x be in both of their domains. Then $f+g$ is their sum, defined at each point x by

$$(f+g)(x) = f(x) + g(x).$$

The difference $f-g$ is defined by

$$(f-g)(x) = f(x) - g(x),$$

and their product fg is defined by

$$(fg)(x) = f(x) \cdot g(x).$$

Furthermore, if $g(x) \neq 0$, then their quotient is defined as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

The composition $f \circ g$ (“f composed with g”) is defined at the point x by

$$(f \circ g)(x) = f(g(x)).$$

19 Example Let

$$f : \begin{array}{ccc} \mathbb{R} \setminus \{0\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{1}{x} \end{array}, \qquad g : \begin{array}{ccc} [-5\,;5] & \to & \mathbb{R} \\ x & \mapsto & \sqrt{25-x^2} \end{array}.$$

Find

1. $(f+g)(2)$
2. $(fg)(2)$
3. $(f \circ g)(2)$
4. $(g \circ f)(2)$

Solution: ▶ *We have*

1. $(f+g)(2) = f(2) + g(2) = \dfrac{1}{2} + \sqrt{21}$
2. $(fg)(2) = \dfrac{1}{2} \cdot \sqrt{21} = \dfrac{\sqrt{21}}{2}$
3. $(f \circ g)(2) = f(g(2)) = f(\sqrt{21}) = \dfrac{1}{\sqrt{21}}$
4. $(g \circ f)(2) = g(f(2)) = g\left(\dfrac{1}{2}\right) = \sqrt{25 - \dfrac{1}{4}} = \dfrac{3\sqrt{11}}{2}$

◀

We conclude this section with some special names that will be used throughout these notes.

20 Definition A function $p : \mathbb{R} \to \mathbb{R}$ whose formula is of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where the $a_k \in \mathbb{R}$ are constants, $a_n \neq 0$, is called a *polynomial function* of degree n. If $n = 0$ then

$$p(x) = a$$

for some real constant numbers a, this is called a *constant function*. If $n = 1$ then

$$p(x) = a + bx$$

for some real numbers a, b, with $b \neq 0$, this is called an *affine function*. If $n = 2$ then

$$p(x) = a + bx + cx^2$$

for some real numbers a, b, c, with $c \neq 0$, this is called a *quadratic function*.

21 Definition Let $p(x)$ and $q(x)$ be two polynomials with real coefficients, and let $Z = \{x \in \mathbb{R} : q(x) = 0\}$. The function $r : \mathbb{R} \setminus Z \to \mathbb{R}$ with formula $r(x) = \dfrac{p(x)}{q(x)}$ is called a *rational function*.

22 Definition The function with assignment rule $x \mapsto f(x)$ is said to be *algebraic* if $y = f(x)$ is a solution of an equation of the form

$$p_n(x)y^n + \cdots + p_1(x)y + p_0(x) = 0,$$

where the $p_0(x), p_1(x), \ldots, p_n(x)$ are polynomials in x. A function that satisfies no such equation is said to be *transcendental*.

23 Example The function

$$p : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 - x + 1 \end{array}$$

is a polynomial function of degree 3.

The function

$$r : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{x}{x^2 + 2x + 2} \end{array}$$

is a rational function.

The function

$$a : \begin{array}{ccc} [0\,;+\infty[& \to & \mathbb{R} \\ x & \mapsto & \sqrt{x + \sqrt{x + \sqrt{x}}} \end{array}$$

is a algebraic function.

Homework

1.4.1 Problem Determine the domain of definition of the assignment rule $x \mapsto \sqrt{x} + \sqrt{-x}$.

1.4.2 Problem Determine the domain of definition of the assignment rule $x \mapsto \dfrac{1}{\sqrt{x} + \sqrt{-x}}$.

1.4.3 Problem Determine the domain of definition of the assignment rule $x \mapsto \dfrac{x+1}{(x-1)\sqrt{x}}$.

1.4.4 Problem Consider the functions

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \sqrt{x^2+1} \end{array}, \qquad g : \begin{array}{ccc}]-\infty\,;-1] \cup [1\,;+\infty[& \to & \mathbb{R} \\ x & \mapsto & \sqrt{x^2-1} \end{array}.$$

1. Determine $(f+g)(2)$.
2. Determine $(fg)(2)$.
3. Determine $(f \circ g)(2)$.
4. Determine $(g \circ f)(2)$.

1.5 Infinitesimals

We now want to informally introduce the concepts of *nearness* and *smallness*.

What does it mean for one point to be "near" another point? We could argue that 1 is near to 0, but, for some purposes, this distance could be too "far." We could certainly say that 0.5 is closer to 0 than 1 is, but then again, for some purposes, even this distance could be "far." Mentioning a specific number "near" 0, like 1 or 0.5 fails in what we desire for "nearness" because mentioning a specific point immediately gives a "static" quality to "nearness": once you mention a specific point, you could mention infinitely many more points which are closer than the point you mentioned. The points in the sequence

$$0.1, \qquad 0.01, \qquad 0.001, \qquad 0.0001, \qquad \ldots$$

get closer and closer to 0 with an arbitrary precision. Notice that this sequence approaches 0 through values > 0. This arbitrary precision is what will be the gist of our concept of "nearness." "Nearness" is dynamic: it involves the ability of getting closer to a point with any desired degree of accuracy. It is not static.

Again, the points in the sequence

$$-\frac{1}{2}, \qquad -\frac{1}{4}, \qquad -\frac{1}{8}, \qquad -\frac{1}{16}, \qquad \ldots$$

are arbitrarily close to 0, but they "approach" 0 from the left. Once again, the sequence

$$+\frac{1}{2}, \qquad -\frac{1}{3}, \qquad +\frac{1}{4}, \qquad -\frac{1}{5}, \qquad \ldots$$

approaches 0 from both above and below. After this long preamble, we may formulate our first definition.

24 Definition The notation $x \to a$, read "x tends to a," means that x is very close, with an arbitrary degree of precision, to a. Here x can approach a through values smaller or larger than a. We write $x \to a+$ (read "x tends to a from the right") to mean that x approaches a through values larger than a and we write $x \to a-$ (read "x tends to a from the left") we mean that x approaches a through values smaller than a.

25 Definition Given a function f and a point $a \in \mathbb{R}$, we write $f(a+)$ for the value that $f(x)$ attains when $x \to a+$. Similarly, we write $f(a-)$ for the value that $f(x)$ attains when $x \to a-$.

26 Example Given $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \sqrt{4-x} & \text{if } x < 4, \\ 1 & \text{if } x = 4, \\ 3 & \text{if } x > 4. \end{cases}$$

Then

$$f(4-) = \sqrt{4-4} = 0, \qquad f(4) = 1, \qquad f(4+) = 3.$$

Observe that also, for example,

$$f(0-) = 2, \qquad f(0) = 2, \qquad f(0+) = 2,$$

and

$$f(5-) = 3, \qquad f(5) = 3, \qquad f(5+) = 3.$$

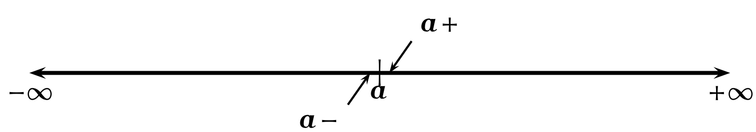


Figure 1.1: A neighbourhood of a.

27 Definition A *neighbourhood of a point* a is an interval containing a.

Notice that the definition of neighbourhood does not rule out the possibility that a may be an endpoint of the the interval. Our interests will be mostly on arbitrarily small neighbourhoods of a point. Schematically we have a diagram like figure 1.1.

28 Definition (Infinitesimal) We say that a given formula $f(x)$ in the variable x is *infinitesimal* as $x \to 0$, written $f(x) = o(1)$ if $f(x) \to 0$ as $x \to 0$. We also say that f *is small oh of* 1.

We make the following observations, which we will take as axioms.

29 Axiom If $x \to 0$ and if $n \geq 1$, $n \in \mathbb{N}$, then $x^n = o(1)$.

30 Axiom (The Sum of Infinitesimals is an Infinitesimal) If $x \to 0$ and if $a(x) = o(1)$ and $b(x) = o(1)$, then $a(x) + b(x) = o(1)$.

31 Axiom (The Product of an Infinitesimal and a Constant is an Infinitesimal) If $x \to 0$ and if $a(x) = o(1)$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$, then $\lambda a(x) = o(1)$.

32 Axiom (The Product of Infinitesimals is an Infinitesimal) If $x \to 0$ and if $a(x) = o(1)$ and $b(x) = o(1)$, then $a(x)b(x) = o(1)$.

33 Example Justify the assertion that $f(x) = x^3 - x = o(1)$ as $x \to 0$.

> **Solution:** ► *We have $x^3 = o(1)$ and $x = o(1)$ as $x \to 0$ by Axiom 29. Also, $-x = o(1)$ by Axiom 31. Finally, $x^3 - x = x^3 + (-x) = o(1)$ by Axiom 30.* ◄

We would like to compare now how "fast" two infinitesimals formulas tend toward zero. In order to motivate the following definitions, let us consider a numerical example. Consider a very small number, say,

$$0.0000000100023 = \frac{1}{10^8} + \frac{2}{10^{12}} + \frac{3}{10^{13}}.$$

In comparison to $\frac{1}{10^8}$, the sum $\frac{2}{10^{12}} + \frac{3}{10^{13}}$ is very small, practically negligible. In other words, the number $\frac{1}{10^8}$ overwhelms or dominates the sum $\frac{2}{10^{12}} + \frac{3}{10^{13}}$.

34 Definition Let m and n be non-zero natural numbers with $m < n$. We say that x^m *dominates* x^n as $x \to 0$. We also say that x^n goes *faster* to 0 than x^m, or that x^m goes *slower* to 0 than x^n. We write this as $x^n = o(x^m)$. In general, if $\frac{f(x)}{g(x)} \to 0$, we write $f(x) = o(g(x))$, and if $a(x) - b(x) = o(c(x))$, we write $a(x) = b(x) + o(c(x))$. The term $o(c(x))$ is called the *error term*.

35 Axiom Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $f(x) = o(1)$ and $g(x) = o(1)$. The $o()$ symbol has the following properties: as $x \to 0$,

$$\begin{aligned}
\lambda o(f(x)) &= o(f(x)) && (1.2)\\
o(\lambda f(x)) &= o(f(x)) && (1.3)\\
o(f(x)) \pm o(f(x)) &= o(f(x)) && (1.4)\\
o(o(f(x))) &= o(f(x)) && (1.5)\\
f(x)g(x) &= o(f(x)) && (1.6)\\
f(x)g(x) &= o(g(x)) && (1.7)\\
o(f(x) + o(f(x))) &= o(f(x)) && (1.8)\\
o(f(x)^n) &= o\left(f(x)^k\right), \quad n \geq 2, n \in \mathbb{N}, k = 1, 2, \ldots, n-1 && (1.9)\\
o(f(x))^n &= o((f(x))^n), \quad n \geq 1, n \in \mathbb{N} && (1.10)\\
(f(x))^n o(f(x)) &= o\left((f(x))^{n+1}\right), \quad n \in \mathbb{N} && (1.11)\\
\frac{o((f(x))^n)}{f(x)} &= o\left((f(x))^{n-1}\right), \quad n \geq 2, n \in \mathbb{N} && (1.12)
\end{aligned}$$

36 Example As $x \to 0$ we have $o\left(x^3\right) + o\left(x^4\right) = o\left(x^3\right)$, since the x^3 term dominates over the x^4 for small x.

37 Definition Let $n_1 < n_2 < \cdots < n_k$ be natural numbers. The *dominant term* as $x \to 0$ of the polynomial

$$p(x) = a_{n_1}x^{n_1} + a_{n_2}x^{n_2} + \cdots + a_{n_k}x^{n_k}$$

is $a_{n_1}x^{n_1}$. We write this as $p(x) \sim a_{n_1}x^{n_1}$, read "p of x is asymptotic to a sub one times x to the power n sub 1."

38 Example Given that $a(x) = 2x^2 + x^3 + o\left(x^3\right)$ and $b(x) = x - 4x^4 + o\left(x^4\right)$, estimate the product $a(x)b(x)$ as $x \to 0$.

Solution: ► *We have*

$$\begin{aligned} a(x)b(x) &= \left(2x^2 + x^3 + \mathrm{o}\left(x^3\right)\right)\left(x - 4x^4 + \mathrm{o}\left(x^4\right)\right) \\ &= 2x^3 - 8x^6 + \mathrm{o}\left(2x^6\right) + x^4 - 4x^7 + \mathrm{o}\left(x^7\right) + \mathrm{o}\left(x^4\right) + \mathrm{o}\left(-4x^7\right) + \mathrm{o}\left(x^7\right) \\ &= 2x^3 + x^4 + \mathrm{o}\left(x^4\right), \end{aligned}$$

since all other error terms are faster than $\mathrm{o}\left(x^4\right)$. ◄

39 **Example** As $x \to 0$, $3x^2 + 2x + x \sim x$.

40 **Axiom (Product of Asymptotic Estimates)** If $p(x)$ and $q(x)$ are polynomials, with $p(x) \sim ax^n$ and $q(x) \sim bx^m$ as $x \to 0$, then $p(x)q(x) \sim abx^{m+n}$ as $x \to 0$.

41 **Example** As $x \to 0$, $x(2x + 3x^2)(3 + 4x)(5 + 6x) \sim x \cdot 2x \cdot 3 \cdot 5 = 20x^2$.

Homework

1.5.1 Problem Given $a(x) = 2 + x^2 + \mathrm{o}\left(x^2\right)$ and $b(x) = x + \mathrm{o}(x)$, as $x \to 0$, estimate $a(x)b(x)$ as $x \to 0$.

1.6 Infinitely Large Quantities

42 **Definition** We use the symbol $+\infty$ to denote a quantity that is larger than any positive real number.

Thus if $P \in \mathbb{R}$, no matter how large $P > 0$ might be, we always have

$$0 < P < +\infty.$$

43 **Definition** We use the symbol $-\infty$ to denote a quantity that is smaller than any negative real number.

Thus if $N \in \mathbb{R}$, $N < 0$, no matter how large $|N|$ might be, we always have

$$-\infty < N < 0.$$

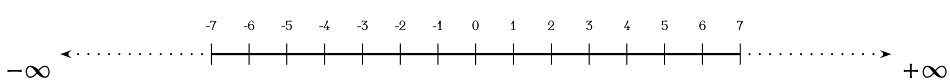


Figure 1.2: The Real Line.

Geometrically, each real number can be viewed as a point on a straight line. We make the convention that we orient the real line with 0 as the origin, the positive numbers increasing towards the right from 0 and the negative numbers decreasing towards the left of 0, as in figure 1.2. We append the object $+\infty$, which is larger than any real number, and the object $-\infty$, which is smaller than any real number. Letting $x \in \mathbb{R}$, we make the following conventions.

$$(+\infty) + (+\infty) = +\infty \tag{1.13}$$

$$(-\infty) + (-\infty) = -\infty \tag{1.14}$$

$$x + (+\infty) = +\infty \tag{1.15}$$

$$x + (-\infty) = -\infty \tag{1.16}$$

$$x(+\infty) = +\infty \quad \text{if } x > 0 \tag{1.17}$$

$$x(+\infty) = -\infty \quad \text{if } x < 0 \tag{1.18}$$

$$x(-\infty) = -\infty \quad \text{if } x > 0 \tag{1.19}$$

$$x(-\infty) = +\infty \quad \text{if } x < 0 \tag{1.20}$$

$$\frac{x}{\pm\infty} = 0 \tag{1.21}$$

Observe that we leave the following undefined:

$$\frac{\pm\infty}{\pm\infty}, \quad (+\infty) + (-\infty), \quad 0(\pm\infty).$$

44 Definition The notation $x \to +\infty$, read "x tends to plus infinity" means that increases without bound, eventually surpassing any preassigned large positive constant. The notation $x \to -\infty$, read "x tends to minus infinity" means that decreases without bound, eventually surpassing any preassigned large in magnitude negative constant.

45 Definition Let m and n be non-zero natural numbers with $m < n$. We say that x^n *dominates* x^m as $x \to \pm\infty$. We also say that x^n goes *faster* to $\pm\infty$ than x^m, or that x^m goes *slower* to $\pm\infty$ than x^n. We write this as $x^m = o\left(x^n\right)$. In general, if $\dfrac{f(x)}{g(x)} \to 0$, we write $f(x) = o\left(g(x)\right)$ as $x \to \pm\infty$ and if $a(x) - b(x) = o\left(c(x)\right)$, we write $a(x) = b(x) + o\left(c(x)\right)$. The term $o\left(c(x)\right)$ is called the *error term*.

The properties expounded in the preceding section for little oh as $x \to 0$ also hold for little oh as $x \to \pm\infty$, except for 29, which must be replaced by the following.

46 Axiom If $x \to \pm\infty$ and if $n \geq 1$, $n \in \mathbb{N}$, then $\dfrac{1}{x^n} = o\left(1\right)$.

We now change the definition of dominant term.

47 Definition Let $n_1 < n_2 < \cdots < n_k$ be natural numbers. The *dominant term* as $x \to \pm\infty$ of the polynomial

$$p(x) = a_{n_1}x^{n_1} + a_{n_2}x^{n_2} + \cdots + a_{n_k}x^{n_k}$$

is $a_{n_k}x^{n_k}$. We write this as $p(x) \sim a_{n_k}x^{n_k}$, read "p of x is asymptotic to a sub k times x to the power n sub k."

48 Example As $x \to +\infty$ we have $o\left(x^3\right) + o\left(x^4\right) = o\left(x^4\right)$, since the x^4 term dominates over the x^3 for large x.

49 Example Given that $a(x) = 2x^3 + x^2 + o\left(x^2\right)$ and $b(x) = x^4 - 4x + o(x)$, estimate the product $a(x)b(x)$ as $x \to +\infty$.

Solution: ▶ *We have*

$$\begin{aligned} a(x)b(x) &= \left(2x^3 + x^2 + o\left(x^2\right)\right)\left(x^4 - 4x + o(x)\right) \\ &= 2x^7 - 8x^4 + o\left(2x^4\right) + x^6 - 4x^3 + o\left(x^3\right) + o\left(x^6\right) + o\left(-4x^3\right) + o\left(x^3\right) \\ &= 2x^7 + x^6 + o\left(x^6\right), \end{aligned}$$

as $x \to +\infty$ since all other error terms are weaker than $o\left(x^6\right)$. ◀

50 Example As $x \to +\infty$, $3x^2 + 2x + x \sim 3x^2$.

51 Axiom (Product of Asymptotic Estimates) If $p(x)$ and $q(x)$ are polynomials, with $p(x) \sim ax^n$ and $q(x) \sim bx^m$ as $x \to 0$, then $p(x)q(x) \sim abx^{m+n}$ as $x \to \pm\infty$.

52 Example As $x \to +\infty$, $x(2x + 3x^2)(3 + 4x)(5 + 6x) \sim x \cdot 3x^2 \cdot 4x \cdot 6x = 72x^5$.

Homework

1.6.1 Problem Given $a(x) = x^3 + 2x^2 + o\left(x^2\right)$ and $b(x) = x + o(1)$ as $x \to +\infty$, estimate $a(x)b(x)$ as $x \to +\infty$.

1.7 Distance on the Plane and Lines

We now introduce some concepts from analytic geometry that will be used throughout these notes.

53 Theorem (Distance Between Two Points on the Plane) The distance between the points $A = (x_1, y_1)$, $B = (x_2, y_2)$ in $\mathbb{R}^2$ is given by

$$\boxed{AB = \mathrm{d}\langle(x_1, y_1), (x_2, y_2)\rangle := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.}$$

Proof: *Consider two points on the plane, as in figure 1.3. Constructing the segments CA and BC with $C = (x_2, y_1)$, we may find the length of the segment AB, that is, the distance from A to B, by utilising the Pythagorean Theorem:*

$$AB^2 = AC^2 + BC^2 \implies AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

❑

54 Example The length of the line segment joining the points $(1, 2)$ and $(-3, 4)$ on the plane is

$$\sqrt{(1 - (-3))^2 + (2 - 4)^2} = \sqrt{16 + 4} = 2\sqrt{5}.$$

55 Definition Let a and b be real number constants. A *vertical line* on the plane is a set of the form

$$\{(x, y) \in \mathbb{R}^2 : x = a\}.$$

Similarly, a *horizontal line* on the plane is a set of the form

$$\{(x, y) \in \mathbb{R}^2 : y = b\}.$$

Definition 55 is striking. It provides a link between Algebra and Geometry, and viceversa, between Geometry and Algebra. For example, if we are given the equation of a horizontal line (Algebra), then we can graph it (Geometry). Conversely, if we are given the picture of a horizontal line on the coordinate plane (Geometry), we may find its equation (Algebra).

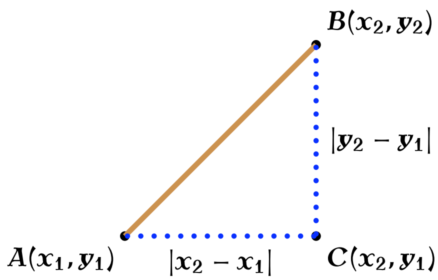


Figure 1.3: Distance between two points.

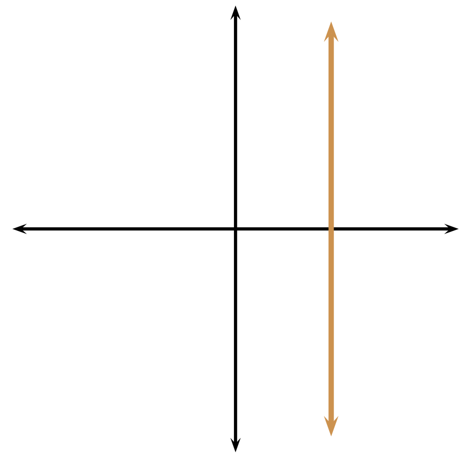


Figure 1.4: A vertical line.

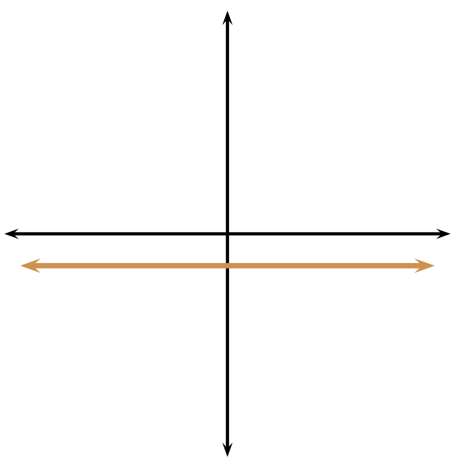


Figure 1.5: A horizontal line.

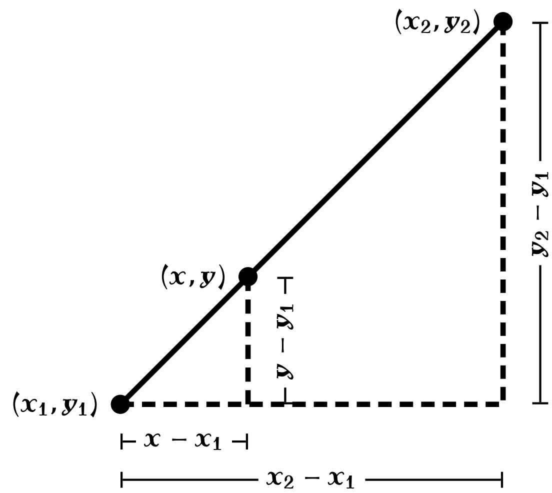


Figure 1.6: Theorem 56.

56 Theorem The equation of any non-vertical line on the plane can be written in the form $y = mx + k$, where m and k are real number constants. Conversely, any equation of the form $y = ax + b$, where a, b are fixed real numbers has as a line as a graph.

> **Proof:** *If the line is parallel to the x-axis, that is, if it is horizontal, then it is of the form $y = b$, where b is a constant and so we may take $m = 0$ and $k = b$. Consider now a line non-parallel to any of the axes, as in figure 1.6, and let (x, y), (x_1, y_1), (x_2, y_2) be three given points on the line. By similar triangles we have*
> $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1},$$
> *which, upon rearrangement, gives*
> $$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x - x_1 \left(\frac{y_2 - y_1}{x_2 - x_1}\right) + y_1,$$

and so we may take

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \qquad k = -x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right) + y_1.$$

Conversely, consider real numbers $x_1 < x_2 < x_3$*, and let* $P = (x_1, ax_1 + b)$*,* $Q = (x_2, ax_2 + b)$*, and* $R = (x_3, ax_3 + b)$ *be on the graph of the equation* $y = ax + b$*. We will shew that*

$$\mathrm{d}\langle P, Q\rangle + \mathrm{d}\langle Q, R\rangle = \mathrm{d}\langle P, R\rangle.$$

Since the points P, Q, R *are arbitrary, this means that any three points on the graph of the equation* $y = ax + b$ *are collinear, and so this graph is a line. Then*

$$\mathrm{d}\langle P, Q\rangle = \sqrt{(x_2 - x_1)^2 + (ax_2 - ax_1)^2} = |x_2 - x_1|\sqrt{1 + a^2} = (x_2 - x_1)\sqrt{1 + a^2},$$
$$\mathrm{d}\langle Q, R\rangle = \sqrt{(x_3 - x_2)^2 + (ax_3 - ax_2)^2} = |x_3 - x_2|\sqrt{1 + a^2} = (x_3 - x_2)\sqrt{1 + a^2},$$
$$\mathrm{d}\langle P, Q\rangle = \sqrt{(x_3 - x_1)^2 + (ax_3 - ax_1)^2} = |x_3 - x_1|\sqrt{1 + a^2} = (x_3 - x_1)\sqrt{1 + a^2},$$

from where

$$\mathrm{d}\langle P, Q\rangle + \mathrm{d}\langle Q, R\rangle = \mathrm{d}\langle P, R\rangle$$

follows. This means that the points $P, Q,$ *and* R *lie on a straight line, which finishes the proof of the theorem.* ❑

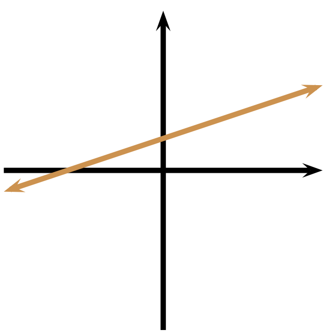


Figure 1.7: $m > 0$

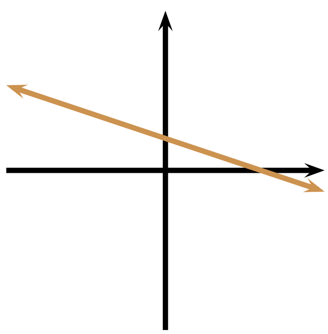


Figure 1.8: $m < 0$

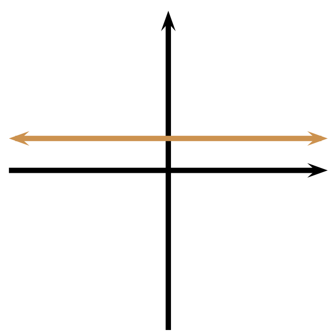


Figure 1.9: $m = 0$

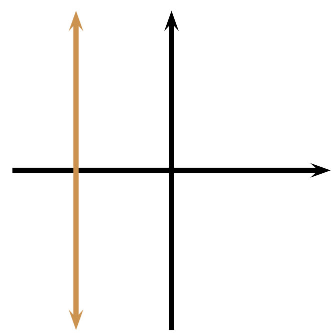


Figure 1.10: $m = \infty$

57 Definition The quantity $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ in Theorem 56 is the *slope or gradient of the line passing through* (x_1, y_1) *and* (x_2, y_2). Since $y = m(0) + k$, the point $(0, k)$ is the *y-intercept of the line joining* (x_1, y_1) *and* (x_2, y_2). Figures 1.7 through 1.10 shew how the various inclinations change with the sign of m.

58 Example By Theorem 56, the equation $y = x$ represents a line with slope 1 and passing through the origin. Since $y = x$, the line makes a $45°$ angle with the x-axis, and bisects quadrants I and III. See figure 1.11

59 Example A line passes through $(-3, 10)$ and $(6, -5)$. Find its equation and draw it.

Solution: ▶ *The equation is of the form* $y = mx + k$*. We must find the slope and the* y*-intercept. To find* m *we compute the ratio*

$$m = \frac{10 - (-5)}{-3 - 6} = -\frac{5}{3}.$$

Thus the equation is of the form $y = -\dfrac{5}{3}x + k$ *and we must now determine* k*. To do so, we substitute either point, say the first, into* $y = -\dfrac{5}{3}x + k$ *obtaining* $10 = -\dfrac{5}{3}(-3) + k$*, whence* $k = 5$*.*

The equation sought is thus $y = -\frac{5}{3}x + 5$*. To draw the graph, first locate the* y*-intercept (at* $(0, 5)$*). Since the slope is* $-\frac{5}{3}$*, move five units down (to* $(0, 0)$*) and three to the right (to* $(3, 0)$*). Connect now the points* $(0, 5)$ *and* $(3, 0)$*. The graph appears in figure 1.12.* ◀

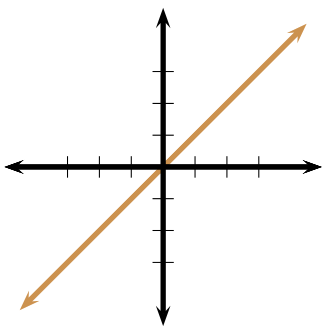


Figure 1.11: Example 58.

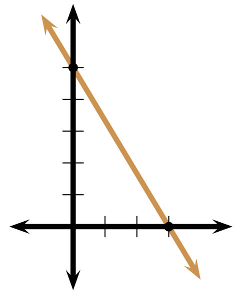


Figure 1.12: Example 59.

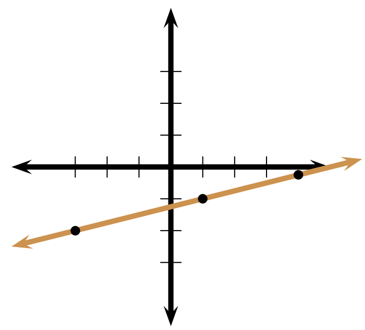


Figure 1.13: Example 60.

60 **Example** Three points $(4, u), (1, -1)$ and $(-3, -2)$ lie on the same line. Find u.

Solution: ▶ *Since the points lie on the same line, any choice of pairs of points used to compute the gradient must yield the same quantity. Therefore*

$$\frac{u - (-1)}{4 - 1} = \frac{-1 - (-2)}{1 - (-3)}$$

which simplifies to the equation

$$\frac{u + 1}{3} = \frac{1}{4}.$$

Solving for u *we obtain* $u = -\frac{1}{4}$*. See figure 1.13.*◀

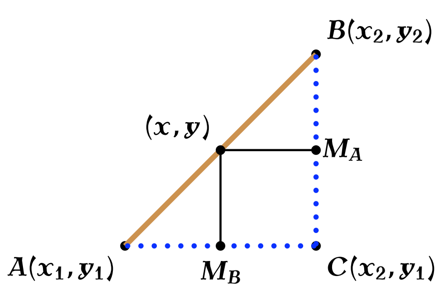


Figure 1.14: Midpoint of a line segment.

61 Theorem (Midpoint of a Line Segment) The point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ lies on the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$, and it is equidistant from both points.

Proof: *First observe that it is easy to find the midpoint of a vertical or horizontal line segment. The interval* $[a\ ; b]$ *has length* $b - a$*. Hence, its midpoint is at* $a + \frac{b - a}{2} = \frac{a + b}{2}$*.*

Let (x, y) be the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$. With $C(x_2, y_1)$, form the triangle $\triangle ABC$, right-angled at C. From (x, y), consider the projections of this point onto the line segments AC and BC. Notice that these projections are parallel to the legs of the triangle and so these projections pass through the midpoints of the legs. Since AC is a horizontal segment, its midpoint is at $M_B = (\frac{x_1+x_2}{2}, y_1)$. As BC is a horizontal segment, its midpoint is $M_A = (x_2, \frac{y_1+y_2}{2})$. The result is obtained on noting that (x, y) must have the same abscissa as M_B and the same ordinate as M_A. ❑

Homework

1.7.1 Problem Find the distance of the line segments having $(1, -2)$ and $(-3, 4)$ as endpoints.

1.7.2 Problem Find the equation of the line passing through $(1, -2)$ and $(-3, 4)$.

1.7.3 Problem What is the slope of the line with equation $\frac{x}{a} + \frac{y}{b} = 1$?

1.7.4 Problem Let $(a, b) \in \mathbb{R}^2$. Find the equation of the straight line joining (a, b) and (b, a).

1.7.5 Problem Which points on the line with equation $y = 6 - 2x$ are equidistant from the axes?

1.7.6 Problem In figure 1.15, point M has coordinates $(2, 2)$, points A, S are on the x-axis, point B is on the y-axis $\triangle SMA$ is isosceles at M, and the line segment SM has slope $\frac{1}{2}$. Find the coordinates of points A, B, S.

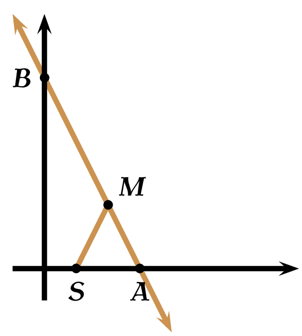


Figure 1.15: Problem 1.7.6.

2 An Informal View of Graphs of Functions

2.1 Graphs of Functions

62 Definition The graph of a function f is the set on the plane

$$\Gamma_f = \{(x, y) \in \mathbb{R}^2 : x \in \mathrm{Dom}(f), y = f(x)\}.$$

63 Example Consider the function

$$\mathbf{Id} : \begin{array}{rcl} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array},$$

called the *identity* function. By example 58, we know that its graph is the straight line that appears in figure 1.11.

64 Example Consider the function

$$f : \begin{array}{rcl} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & -\dfrac{5}{3}x + 5 \end{array}$$

By example 59, we know that its graph is the straight line that appears in figure 1.12.

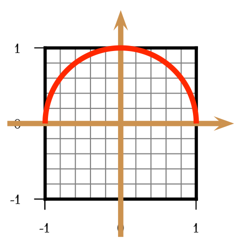


Figure 2.1: Example 65. $y = f(x)$

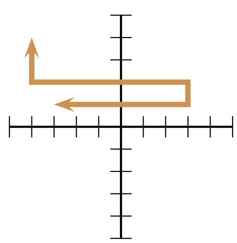


Figure 2.2: Fails the vertical line test. Not a function.

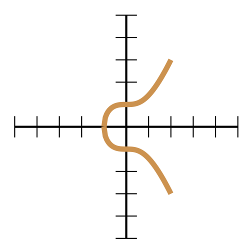


Figure 2.3: Fails the vertical line test. Not a function.

65 Example Demonstrate that the graph of the function

$$f : \begin{array}{rcl} [-1\,;1] & \to & \mathbb{R} \\ x & \mapsto & \sqrt{1 - x^2} \end{array}$$

is the upper semicircle in figure 65.

Solution: ► *If* $y = f(x) = \sqrt{1 - x^2}$*, then*

$$x^2 + y^2 = 1 \implies \sqrt{(x - 0)^2 + (y - 0)^2} = 1,$$

which means that any point (x, y) on the graph of the function is at distance 1 from the origin. Since $y \geq 0$, the points satisfying these requirement form the aforementioned semicircle. ◀

By the definition of the graph of a function, the x-axis contains the set of inputs and y-axis has the set of outputs. Since in the definition of a function every input goes to exactly one output, wee see that if a vertical line crosses two or more points of a graph, the graph does not represent a function. *We will call this the* vertical line test *for a function. See figures 2.2 and 2.3.*

Given a function f, it is generally difficult to know *a priori* what its graph looks like. Most of the material on this chapter will be dedicated to the investigation of tools for sketching graphs of functions. In the subsections below we will informally discuss some features of graphs of (algebraic) functions, which will be elaborated later on.

2.1.1 Symmetry

66 Definition A function f is *even* if for all x in its domain, $-x$ is also in its domain, and it is verified that $f(x) = f(-x)$, that is, if the portion of the graph for $x < 0$ is a mirror reflexion of the part of the graph for $x > 0$. This means that the graph of f is symmetric about the y-axis. A function g is *odd* if for all x, $-x$ is also in its domain, and it is verified that $g(-x) = -g(x)$, in other words, g is odd if it is symmetric about the origin. This implies that the portion of the graph appearing in quadrant I is a $180°$ rotation of the portion of the graph appearing in quadrant III, and the portion of the graph appearing in quadrant II is a $180°$ rotation of the portion of the graph appearing in quadrant IV.

67 Example The function whose graph appears in figure 2.4 is even. The function whose graph appears in figure 2.5 is odd.

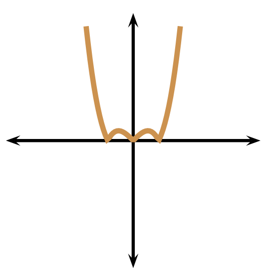


Figure 2.4: The graph of an even function.

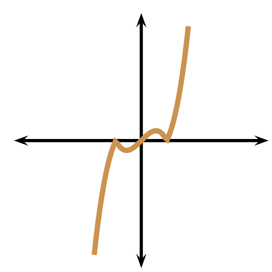


Figure 2.5: The graph of an odd function.

2.1.2 Continuity

Heuristically speaking, a *continuous function* at the point $x = a$ is one whose graph has no "breaks" at the point $x = a$.

What could possibly go wrong and make a function discontinuos? Let us consider the following three types of discontinuities. Let f be a function and $a \in \mathbb{R}$. Assume that f is defined in a neighbourhood of a, but not precisely at $x = a$. Which value can we reasonably assign to $f(a)$? Consider the situations depicted in figures 2.6 through 2.8. In figure 2.6 it seems reasonably to assign $a(0) = 0$. What value can we reasonably assign in figure 2.7? $b(0) = \dfrac{-1+1}{2} = 0$? In figure 2.8, what value would it be reasonable to assign? $c(0) = 0$?, $c(0) = +\infty$?, $c(0) = -\infty$? The situations presented here are typical, but not necessarily exhaustive. Thus of the three discontinuous functions in figures 2.6 through 2.8, only that of figure 2.6 has a chance of being continuous, and that happens only if one defines $a(0)$ as 0.

2.1.3 Intercepts

68 Definition Given a function f, the set

$$\{(x, f(x)) \in \mathbb{R}^2 : x \in \mathbf{Dom}(f), f(x) = 0\}$$

is called the set of *x-intercepts (zeroes or roots)* of f. If f is defined at 0, then the point $(0, f(0))$ is called the *y-intercept* of f. In other words, the intercepts are the places where the graph of a function touches the axes.

69 Example In figure 2.9, the points A, B, C are the x-intercepts of the function and the point L is the y-intercept.

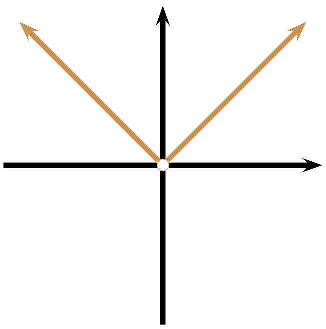


Figure 2.6: $y = a(x)$.

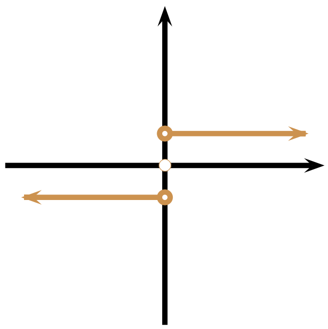


Figure 2.7: $y = b(x)$.

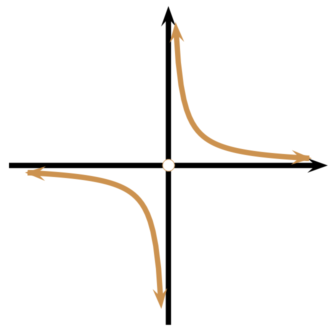


Figure 2.8: $y = c(x)$.

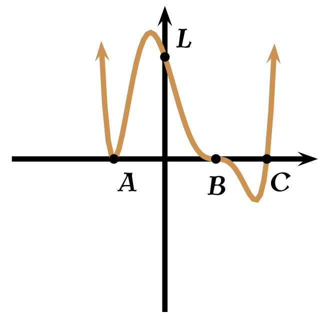


Figure 2.9: Intercepts

2.1.4 Asymptotes

70 Definition Given a function f we write $f(-\infty)$ for the value that f may eventually approach for large (in absolute value) and negative inputs and $f(+\infty)$ for the value that f may eventually approach for large (in absolute value) and positive input. The line $y = b$ is a (horizontal) *asymptote* for the function f if either

$$f(-\infty) = b \qquad \textbf{or} \qquad f(+\infty) = b.$$

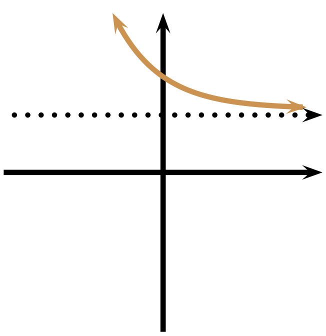


Figure 2.10: $f(x) \to b$ as $x \to +\infty$.

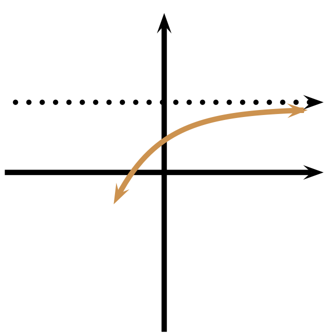


Figure 2.11: $f(x) \to b$ as $x \to +\infty$.

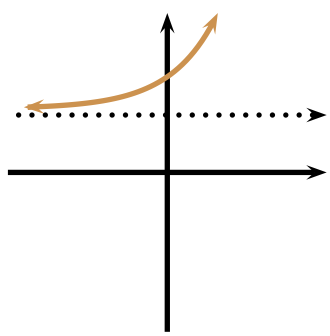


Figure 2.12: $f(x) \to b$ as $x \to -\infty$.

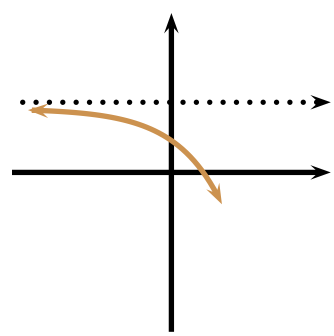


Figure 2.13: $f(x) \to b$ as $x \to -\infty$.

Heuristically, the line $y = b$ is an asymptote of the function, if eventually, the graph of the function gets closer and closer to it without actually touching it. Figures 2.10 through 2.13 shew different ways in which the line $y = b$ can be the asymptote of a function.

71 Example Find the asymptotes, if any, of the function

$$f: \begin{array}{ccc} \mathbb{R}\setminus\{-1,1\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{(2x+1)^2(1-3x)^2}{x^4-1} \end{array}.$$

Solution: ▶ *To do this, we find the dominant terms of the numerator and denominator as* $x \to \pm\infty$*:*

$$y = \frac{(2x+1)^2(1-3x)^2}{x^4-1} \sim \frac{(2x)^2(-3x)^2}{x^4} = 36,$$

which means that $y = 36$ *is its horizontal asymptote.* ◀

72 Example Find the asymptotes, if any, of the function

$$g: \begin{array}{ccc} \mathbb{R}\setminus\{-1,1\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{(2x+1)^3(1-3x)^2}{x^4-1} \end{array}.$$

Solution: ▶ *To do this, we find the dominant terms of the numerator and denominator as* $x \to \pm\infty$*:*

$$y = \frac{(2x+1)^3(1-3x)^2}{x^4-1} \sim \frac{(2x)^3(-3x)^2}{x^4} = 72x,$$

which tends to $\pm\infty$ *as* $x \pm \infty$*, and hence does not have a horizontal asymptote.*[1] ◀

73 Example Find the asymptotes, if any, of the function

$$h: \begin{array}{ccc} \mathbb{R}\setminus\{-1,1\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{(2x+1)(1-3x)}{x^4-1} \end{array}.$$

Solution: ▶ *To do this, we find the dominant terms of the numerator and denominator as* $x \to \pm\infty$*:*

$$y = \frac{(2x+1)(1-3x)}{x^4-1} \sim \frac{(2x)(-3x)}{x^4} = -\frac{6}{x^2} \to 0,$$

as $x \pm \infty$*, and hence* $y = 0$ *is its horizontal asymptote.* ◀

2.1.5 Poles

74 Definition Let $k > 0$ be an integer. A function f has a *pole of order* k at the point $x = a$ if $(x-a)^{k-1}f(x) \to \pm\infty$ as $x \to a$, but $(x-a)^k f(x)$ as $x \to a$ is finite. Some authors prefer to use the term *vertical asymptote*, rather than pole.

75 Example Find the poles, if any, of the function

$$h: \begin{array}{ccc} \mathbb{R}\setminus\{-1,1\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{(2x+1)(1-3x)}{x^4-1} \end{array}.$$

[1] The expert will note that this curve has a slanted asymptote, but we will discuss these later in the text.

Solution: ► *To do this, we find the real zeroes of the denominator:*

$$x^4 - 1 = 0 \implies (x-1)(x+1)(x^2+1) \implies x \in \{-1, 1\},$$

from where the lines $x = -1$ and $x = 1$ are poles. ◄

76 Example Find the poles, if any, of the function

$$r : \begin{array}{ccc} \mathbb{R} \setminus \{-1, 1\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{(2x+1)(1-3x)}{x^4+1} \end{array}.$$

Solution: ► *To do this, we find the real zeroes of the denominator. But since $1 + x^4 = 1 + (x^2)^2$ is 1 plus a real square, and real squares are always positive, the denominator is never 0, as it is, in fact, always ≥ 1. This means that r does not have any poles.* ◄

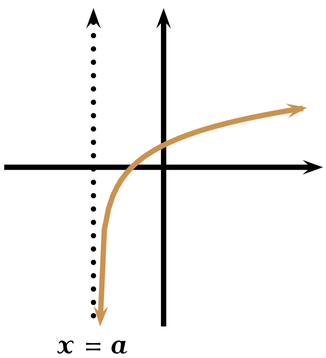


Figure 2.14: $f(x) \to -\infty$ as $x \to a+$.

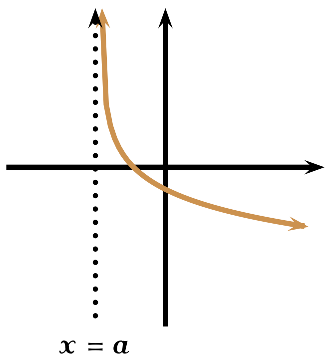


Figure 2.15: $f(x) \to +\infty$ as $x \to a+$.

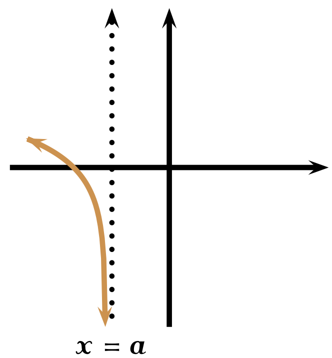


Figure 2.16: $f(x) \to -\infty$ as $x \to a-$.

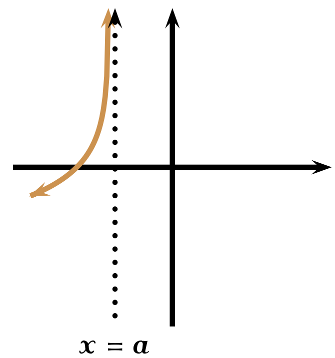


Figure 2.17: $f(x) \to +\infty$ as $x \to a-$.

Figures 2.14 through 2.17 exhibit the various ways a function may have a pole.

2.1.6 Monotonicity

77 Definition A function f is said to be *increasing* (respectively, *strictly increasing*) if $a < b \implies f(a) \leq f(b)$ (respectively, $a < b \implies f(a) < f(b)$). A function g is said to be *decreasing* (respectively, *strictly decreasing*) if $a < b \implies g(a) \leq g(b)$ (respectively, $a < b \implies g(a) < g(b)$). A function is *monotonic* if it is either (strictly) increasing or decreasing. By the *intervals of monotonicity of a function* we mean the intervals where the function might be (strictly) increasing or decreasing.

STOP *If the function f is (strictly) increasing, its opposite $-f$ is (strictly) decreasing, and viceversa.*

The following theorem is immediate.

78 Theorem A function f is (strictly) increasing if for all $a < b$ for which it is defined

$$\frac{f(b) - f(a)}{b - a} \geq 0 \qquad (\text{respectively, } \frac{f(b) - f(a)}{b - a} > 0).$$

Similarly, a function g is (strictly) decreasing if for all $a < b$ for which it is defined

$$\frac{g(b) - g(a)}{b - a} \leq 0 \qquad (\text{respectively, } \frac{g(b) - g(a)}{b - a} < 0).$$

79 Definition If there is a point a for which $f(x) \leq f(M)$ for all x in a neighbourhood centred at $x = M$ then we say that f has a *local maximum* at $x = M$. Similarly, if there is a point m for which $f(x) \geq f(m)$ for all x in a neighbourhood centred at $x = m$ then we say that f has a *local minimum* at $x = m$. The maxima and the minima of a function are called its *extrema*.

2.1.7 Convexity

80 Definition A function $f : \mathbf{Dom}(f) \to \mathbb{R}$ is *(mid-point) convex* in $\mathbf{Dom}(f)$ if for all a, b, in $\mathbf{Dom}(f)$ and

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}.$$

Similarly, a function $g : \mathbf{Dom}(g) \to \mathbb{R}$ is *concave* in $\mathbf{Dom}(g)$ if for all a, b, in $\mathbf{Dom}(g)$

$$g\left(\frac{a+b}{2}\right) \geq \frac{g(a)+g(b)}{2}.$$

By the *intervals of convexity (concavity) of a function* we mean the intervals where the function is convex (concave). An *inflexion point* is a point where a graph changes convexity.[2]

Consider figure 2.18, where

$$A = (a, f(a)), \qquad B = (b, f(b)), \qquad M = \left(\frac{a+b}{2}, \frac{f(a)+f(b)}{2}\right), \qquad L = \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right).$$

The definition of convexity has the following geometric interpretation: for any line segment joining two points on the curve, the midpoint of the line segment lines on top of the curve, and hence, the line segment itself lies on top of the curve. See figure 2.18. Observe that if f is convex, then $-f$ is concave, and so in a concave curve, any line segment joining two points on the curve, lies below the curve. See figure 2.19.

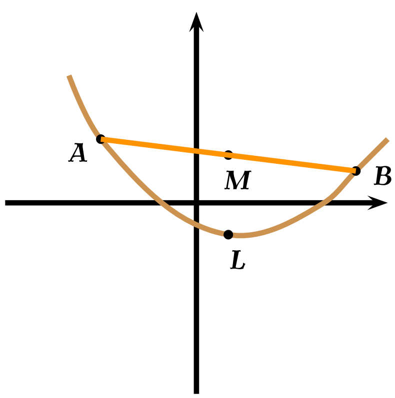


Figure 2.18: A convex curve.

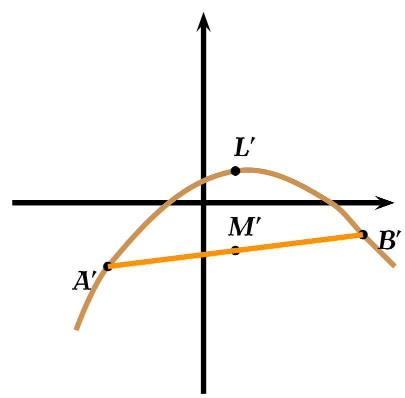


Figure 2.19: A concave curve.

Homework

2.1.1 Problem Draw the graph of a function f meeting the following conditions:

1. f is continuous everywhere;

[2]American books, bowing to *Newspeak*, have created such terms as "concave up" and concave down." Their goal is to reduce vocabulary, thereby stultifying students even more. We will not follow such practice here.

2. f has zeroes at $x = -3$, $x = 1$, and $x = 2$
3. $f(-\infty) = f(+\infty) = -\infty$
4. $f(-1) = 2$
5. $f(5) = -16$

2.1.2 Problem Draw the graph of a function f meeting the following conditions:

1. f is everywhere continuous;
2. $f(-2) = f(0) = f(2) = 0$;
3. $f(-\infty) = 0$, $f(+\infty) = +\infty$
4. $f(-4) = -1$, $f(-1) = 1$, $f(1) = -1$, $f(4) = 1$

2.1.3 Problem Draw the graph of a function f meeting the following conditions:

1. f is continuous in $\mathbb{R} \setminus \{0\}$;
2. f has a pole at $x = 0$, with $f(0-) = -\infty$ and $f(0+) = +\infty$;
3. $f(-\infty) = -\infty$, $f(+\infty) = +\infty$

2.1.4 Problem Consider the functions f and g in the figure below. You may assume that they are composed solely of straight line segments.

1. Draw $f + g$.
2. Draw $f \circ g$.
3. Draw $2f + g$.

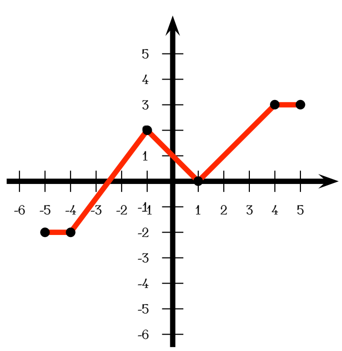


Figure 2.20: f

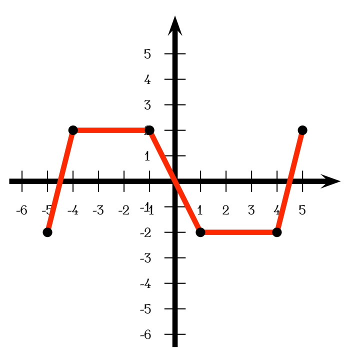


Figure 2.21: g

2.2 Transformations of Graphs of Functions

2.2.1 Translations

81 Theorem Let f be a function and let v and h be real numbers. If (x_0, y_0) is on the graph of f, then $(x_0, y_0 + v)$ is on the graph of g, where $g(x) = f(x) + v$, and if (x_1, y_1) is on the graph of f, then $(x_1 - h, y_1)$ is on the graph of j, where $j(x) = f(x + h)$.

Proof: *Let $\Gamma_f, \Gamma_g, \Gamma_j$ denote the graphs of f, g, j respectively.*

$$(x_0, y_0) \in \Gamma_f \iff y_0 = f(x_0) \iff y_0 + v = f(x_0) + v \iff y_0 + v = g(x_0) \iff (x_0, y_0 + v) \in \Gamma_g.$$

Similarly,

$$(x_1, y_1) \in \Gamma_f \iff y_1 = f(x_1) \iff y_1 = f(x_1 - h + h) \iff y_1 = j(x_1 - h) \iff (x_1 - h, y_1) \in \Gamma_j.$$

❑

82 Definition Let f be a function and let v and h be real numbers. We say that the curve $y = f(x) + v$ is a *vertical translation* of the curve $y = f(x)$. If $v > 0$ the translation is v up, and if $v < 0$, it is v units down. Similarly, we say that the curve $y = f(x + h)$ is a *horizontal translation* of the curve $y = f(x)$. If $h > 0$, the translation is h units left, and if $h < 0$, then the translation is h units right.

83 Example Figures 2.23 through 2.25 shew various translations of the graph of a function f.

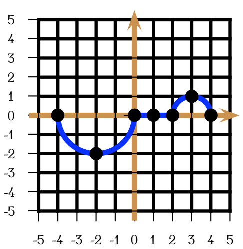


Figure 2.22: $y = f(x)$.

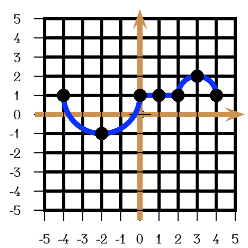


Figure 2.23: $y = f(x) + 1$.

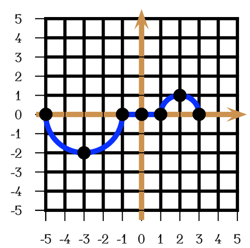


Figure 2.24: $y = f(x + 1)$.

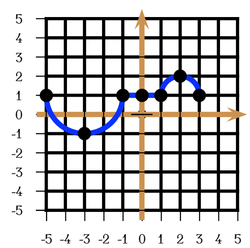


Figure 2.25: $y = f(x + 1) + 1$.

2.2.2 Reflexions

84 Theorem Let f be a function If (x_0, y_0) is on the graph of f, then $(x_0, -y_0)$ is on the graph of g, where $g(x) = -f(x)$, and if (x_1, y_1) is on the graph of f, then $(-x_1, y_1)$ is on the graph of j, where $j(x) = f(-x)$.

Proof: *Let $\Gamma_f, \Gamma_g, \Gamma_j$ denote the graphs of f, g, j respectively.*

$$(x_0, y_0) \in \Gamma_f \iff y_0 = f(x_0) \iff -y_0 = -f(x_0) \iff -y_0 = g(x_0) \iff (x_0, -y_0) \in \Gamma_g.$$

Similarly,

$$(x_1, y_1) \in \Gamma_f \iff y_1 = f(x_1) \iff y_1 = f(-(-x_1)) \iff y_1 = j(-x_1) \iff (-x_1, y_1) \in \Gamma_j.$$

❑

85 Definition Let f be a function. The curve $y = -f(x)$ is said to be the *reflexion of f about the x-axis* and the curve $y = f(-x)$ is said to be the *reflexion of f about the y-axis*.

86 Example Figure 2.26 shews the graph of the function f. Figure 2.27 shews the graph of its reflexion a about the x-axis, and figure 2.28 shews the graph of its reflexion b about the y-axis.

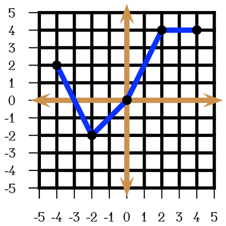


Figure 2.26: $y = f(x)$.

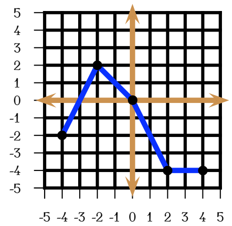


Figure 2.27: $y = -f(x)$.

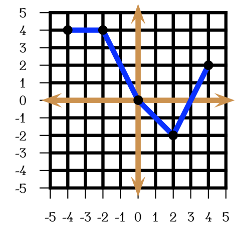


Figure 2.28: $y = f(-x)$.

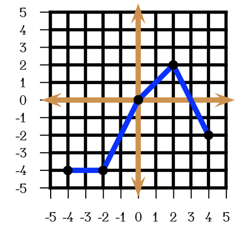


Figure 2.29: $y = -f(-x)$.

2.2.3 Distortions

87 Theorem Let f be a function and let $V \neq 0$ and $H \neq 0$ be real numbers. If (x_0, y_0) is on the graph of f, then (x_0, Vy_0) is on the graph of g, where $g(x) = Vf(x)$, and if (x_1, y_1) is on the graph of f, then $\left(\frac{x_1}{H}, y_1\right)$ is on the graph of j, where $j(x) = f(Hx)$.

Proof: *Let $\Gamma_f, \Gamma_g, \Gamma_j$ denote the graphs of f, g, j respectively.*

$$(x_0, y_0) \in \Gamma_f \iff y_0 = f(x_0) \iff Vy_0 = Vf(x_0) \iff Vy_0 = g(x_0) \iff (x_0, Vy_0) \in \Gamma_g.$$

Similarly,

$$(x_1, y_1) \in \Gamma_f \iff y_1 = f(x_1) \iff y_1 = f\left(\frac{x_1}{H} \cdot H\right) \iff y_1 = j\left(\frac{x_1}{H}\right) \iff \left(\frac{x_1}{H}, y_1\right) \in \Gamma_j.$$

❑

88 Definition Let $V > 0$, $H > 0$, and let f be a function. The curve $y = Vf(x)$ is called a *vertical distortion* of the curve $y = f(x)$. The graph of $y = Vf(x)$ is a *vertical dilatation* of the graph of $y = f(x)$ if $V > 1$ and a *vertical contraction* if $0 < V < 1$. The curve $y = f(Hx)$ is called a *horizontal distortion* of the curve $y = f(x)$ The graph of $y = f(Hx)$ is a *horizontal dilatation* of the graph of $y = f(x)$ if $0 < H < 1$ and a *horizontal contraction* if $H > 1$.

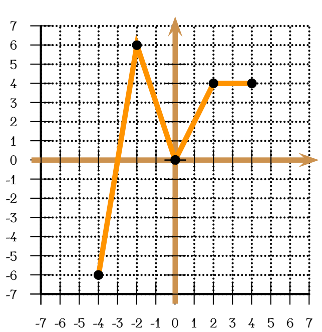


Figure 2.30: $y = f(x)$

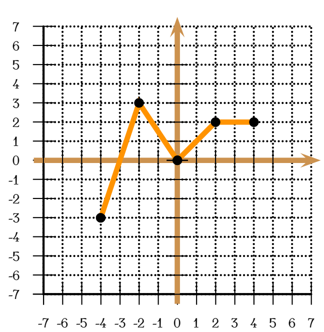


Figure 2.31: $y = \dfrac{f(x)}{2}$

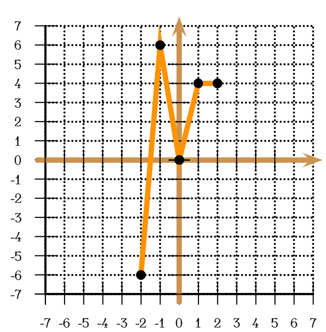


Figure 2.32: $y = f(2x)$

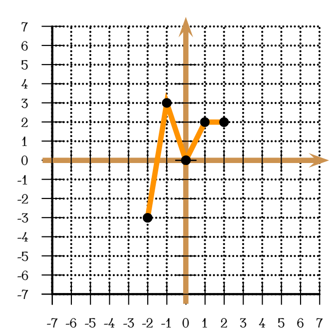


Figure 2.33: $y = \dfrac{f(2x)}{2}$

89 Example Consider the function f whose graph appears in figure 2.30.

In figure 2.31. we have the graph of the function a with $a(x) = \dfrac{f(x)}{2}$.

In figure 2.32 we have the graph of the function b with $b(x) = f(2x)$.

In figure 2.33 we have the graph of the function c with $c(x) = \dfrac{f(2x)}{2}$.

Homework

2.2.1 Problem For the functional curve $y = f(x)$ in figure 2.34, draw $y = f(x+1)$, $y = f(1-x)$ and $y = -f(1-x)$.

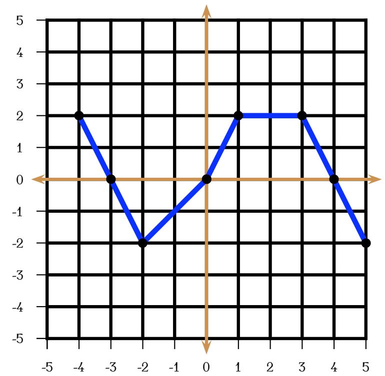


Figure 2.34: Problem 2.2.1.

3 Introduction to the Infinitesimal Calculus

3.1 Continuity

90 Definition A function f is *continuous* at $x = a$ if as $\varepsilon \to 0$,

$$f(a + \varepsilon) = f(a) + o(1).$$

This means, essentially, that if $x \to a$, then $f(x) \to f(a)$.

91 Example Let $K \in \mathbb{R}$. Demonstrate that the constant function

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & K \end{array}$$

is continuous.

Solution: ▶ *Let $a \in \mathbb{R}$ be fixed. Then*

$$f(a + \varepsilon) = K = K + 0 = K + o(1) = f(a) + o(1),$$

proving that a constant function is continuous. ◀

92 Example Demonstrate that the identity function

$$\mathrm{Id} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array}$$

is continuous.

Solution: ▶ *Let $a \in \mathbb{R}$ be fixed. Then*

$$\mathrm{Id}(a + \varepsilon) = a + \varepsilon.$$

Since $\varepsilon \to 0$ means the same as $\varepsilon = o(1)$, we have proved that

$$\mathrm{Id}(a + \varepsilon) = a + \varepsilon = \mathrm{Id}(a) + o(1),$$

and so the identity function is continuous. ◀

93 Example Demonstrate that the square function

$$\mathrm{Sq} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is continuous.

Solution: ► *Let* $a \in \mathbb{R}$ *be fixed. Then*

$$\mathrm{Sq}(a+\varepsilon) = a^2 + 2a\varepsilon + \varepsilon^2.$$

Since a *is fixed,* $\varepsilon \to 0$ *implies that* $2a\varepsilon = o(1)$*. Also,* $\varepsilon^2 = o(1)$*, and so we have proved that*

$$\mathrm{Sq}(a+\varepsilon) = a^2 + 2a\varepsilon + \varepsilon^2 = \mathrm{Sq}(a) + o(1) + o(1) = \mathrm{Sq}(a) + o(1),$$

and so the square function is continuous. ◄

94 Example If

$$f(x) = \begin{cases} \sqrt{x+2}+b & \text{if } -2 \le x < 0 \\ a + 2x & \text{if } 0 \le x < 1 \\ 1 - 2x^2 & \text{if } x \ge 1 \end{cases}$$

is continuous in its domain of definition, determine the value of (a, b).

Solution: ► *The conditions stipulate that*

$$f(0-) = f(0+) \implies \sqrt{2} + b = a, \qquad f(1-) = f(1+) \implies a + 2 = -1,$$

whence $a = -3$ *and* $b = -3 - \sqrt{2}$*.* ◄

95 Theorem Let f and g be continuous at $x = a$. Then $f+g$ and fg are also continuous at $x = a$.

Proof: *We are given that as* $\varepsilon \to 0$,

$$f(a+\varepsilon) = f(a) + o(1), \qquad g(a+\varepsilon) = g(a) + o(1).$$

Then

$$(f+g)(a+\varepsilon) = f(a+\varepsilon)+g(a+\varepsilon) = f(a)+o(1)+g(a)+o(1) = f(a)+g(a)+o(1) = (f+g)(a)+o(1),$$

proving that $f+g$ *is continuous at* $x = a$*.*

Similarly,

$$\begin{aligned} (fg)(a+\varepsilon) &= f(a+\varepsilon)\cdot g(a+\varepsilon) \\ &= (f(a)+o(1))\cdot(g(a)+o(1)) \\ &= f(a)g(a) + o(f(a)) + o(g(a)) + o(1) \\ &= (fg)(a) + o(1), \end{aligned}$$

since $f(a)$ *and* $g(a)$ *are constants. This proves that* fg *is continuous at* $x = a$*.* ❑

96 Theorem (Law of Inertia) If g is continuous at $x = a$ and $g(a) \neq 0$, then there exists a neighbourhood of $x = a$ where g keeps the same sign as $g(a)$.

Proof: *We are given that*

$$g(a+\varepsilon) = g(a) + o(1).$$

The quantity $o(1)$ *will eventually (as* $\varepsilon \to 0$*) be smaller in absolute value than, say* $\frac{|g(a)|}{2}$*. But then this means that*

$$g(a+\varepsilon) \approx g(a) + \frac{|g(a)|}{2} \implies g(a) - \frac{g(a)}{2} < g(a+\varepsilon) < g(a) + \frac{g(a)}{2} \implies \frac{g(a)}{2} < g(a+\varepsilon) < \frac{3g(a)}{2},$$

and so $g(a+\varepsilon)$ *keeps the same sign as* $g(a)$ *for a sufficiently small neighbourhood of* a*.* ❑

97 **Lemma** As $\varepsilon \to 0$,

$$\frac{1}{1+o(1)} = 1 + o(1).$$

Proof: *For if* $|t| < 1$ *then*

$$1 - t^2 < 1 \implies (1+t)(1-t) < 1 \implies 1 - t < \frac{1}{1+t}.$$

Now, if $0 < t < 1$ *then* $1 < 1 + t$ *and so* $\frac{1}{1+t} < 1$*, obtaining*

$$1 - t < \frac{1}{1+t} < 1, \qquad 0 < t < 1,$$

which gives the lemma in this case.

Let us assume now that $-\frac{1}{2} < t < 0$*. We claim that*

$$\frac{1}{1+t} < 1 - t + 2t^2.$$

To prove this, observe that

$$-\frac{1}{2} < t \implies 1 + 2t > 0 \implies 1 + t > -t > 0,$$

and so

$$\begin{aligned} 0 < t^2(1+2t) &\implies 1 < 1 + t^2 + 2t^3 \\ &\implies 1 < 1 - t^2 + 2t^2 + 2t^3 \\ &\implies 1 < (1+t)(1-t) + (1+t)2t^2 \\ &\implies \frac{1}{1+t} < 1 - t + 2t^2, \end{aligned}$$

as was to be shewn. Hence, for $-\frac{1}{2} < t < 0$*, we have*

$$1 - t < \frac{1}{1+t} < 1 - t + 2t^2,$$

which gives the lemma in the remaining case. ❑

98 **Theorem** If g is continuous at $x = a$ and $g(a) \neq 0$, then $\frac{1}{g}$ is also continuous at $x = a$.

Proof: *By Theorem 96, for sufficiently ε close 0, $g(a+\varepsilon) \neq 0$. Hence*

$$\begin{aligned}
\left(\frac{1}{g}\right)(a+\varepsilon) &= \frac{1}{g(a+\varepsilon)} \\
&= \frac{1}{g(a)+o(1)} \\
&= \frac{1}{g(a)} \cdot \frac{1}{1+o\left(\frac{1}{g(a)}\right)} \\
&= \frac{1}{g(a)} \cdot \frac{1}{1+o(1)} \\
&= \frac{1}{g(a)} \cdot (1+o(1)) \\
&= \frac{1}{g(a)} + o\left(\frac{1}{g(a)}\right) \\
&= \frac{1}{g(a)} + o(1),
\end{aligned}$$

by Lemma 97. This proves that $\frac{1}{g}$ is continuous at $x = a$.

❑

99 **Corollary** If the functions f and g are continuous at $x = a$ and if $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at $x = a$.

Proof: *Immediate from theorems 95 and 98.* ❑

100 **Corollary (Polynomials are continuous functions)** Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial with real coefficients. Then p is continuous at every $a \in \mathbb{R}$.

Proof: *The proof is by induction on the degree on the degree of $\mathbf{p}$. If $\deg \mathbf{p} = 0$, then $\mathbf{p}$ is a constant function, which is continuous by example 91.*

If $\deg \mathbf{p} = 1$ then $\mathbf{p}(x) = ax + \mathbf{b}$, for some constants $a, \mathbf{b}$. Since the identity function is continuous by example 92, the function $x \mapsto ax$ is continuous being the product of two continuous functions $x \mapsto x$ and $x \mapsto a$. We then see that $\mathbf{p}$ is the sum of the continuous functions $x \mapsto ax$ and $x \mapsto \mathbf{b}$.

Assume now that any polynomial of degree $\mathbf{n} - 1$ is continuous at $x = a$. Then if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = x(a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1) + a_0 := xq(x) + a_0,$$

where $\deg q(x) = \mathbf{n} - 1$. By the induction hypothesis, $q(x)$ is continuous at $x = a$. By example 92, the identity functions is continuous, and so $x \mapsto xq(x)$ is the product of two continuous functions, and hence continuous. Since the constant function $x \mapsto a_0$ is continuous, $\mathbf{p}$ is then the sum of two continuous functions and hence continuous.

❑

101 Corollary (Rational Functions are Continuous) Let $p(x)$ and $q(x)$ be two polynomials with real coefficients. Let $Z = \{x \in \mathbb{R} : q(x) = 0\}$. Then the rational function

$$r : \begin{array}{ccc} \mathbb{R} \setminus Z & \to & \mathbb{R} \\ x & \mapsto & \dfrac{p(x)}{q(x)} \end{array}$$

is continuous.

Proof: *This is immediate from Corollary 100 and Theorem 98.* ❑

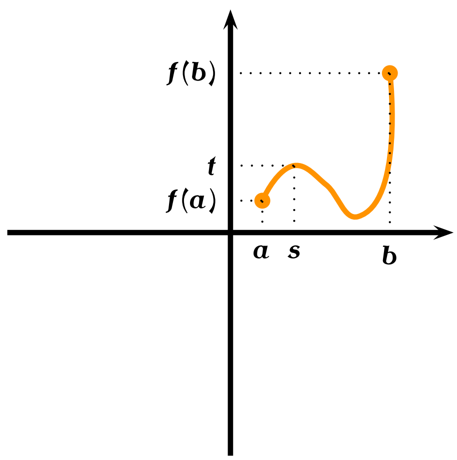


Figure 3.1: Intermediate Value Theorem.

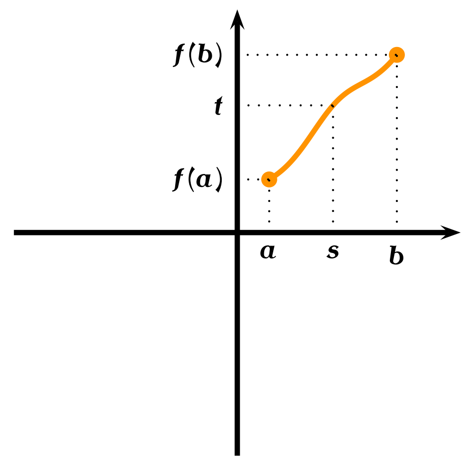


Figure 3.2: Intermediate Value Theorem.

102 Theorem Let g be continuous at $x = a$ and f continuous at $x = g(a)$. Then $f \circ g$ is continuous at $x = a$.

Proof: *We are given that as* $\varepsilon \to 0$, *then*

$$f(g(a) + \varepsilon) = f(g(a)) + o(1), \qquad g(a + \varepsilon) = g(a) + o(1).$$

We must shew that

$$(f \circ g)(a + \varepsilon) = (f \circ g)(a) + o(1).$$

We have

$$(f \circ g)(a + \varepsilon) = f(g(a + \varepsilon)) = f(g(a) + o(1)) = f(g(a)) + o(1) = (f \circ g)(a) + o(1),$$

as was to be shewn. ❑

103 Definition Let $f : \mathrm{Dom}(f) \to \mathbb{R}$ and assume $]a\,;b[\subseteq \mathrm{Dom}(f)$. We say that f is *continuous* in $]a\,;b[$ if f is continuous at every point $x \in]a\,;b[$.

104 Definition Let $f : \mathrm{Dom}(f) \to \mathbb{R}$ and assume $[a\,;b] \subseteq \mathrm{Dom}(f)$. We say that f is continuous in $[a\,;b]$ if f is continuous at every point $x \in]a\,;b[$ and if $f(a+) = f(a)$ and $f(b-) = f(b)$.

105 Theorem (Darboux's Intermediate Value Theorem) Let $f : [a\,;b] \to \mathbb{R}$ be continuous in $[a\,;b]$. Let $m = \min(f(a), f(b))$ and $M = \max(f(a), f(b))$. Then, for every $t \in [m\,;M]$, there exists $s \in [a\,;b]$ such that $f(s) = t$.

Proof: *We shall not give a rigorous proof of this assertion, but refer the reader to figures 3.1 and 3.2. Since there are no breaks in the graph, the graph will not "jump over" the horizontal line* $y = t$. ❑

106 Corollary A continuous function defined on an interval maps that interval into an interval.

Proof: *This follows at once from the Intermediate Value Theorem and the definition of an interval.* ❑

107 Theorem (Bolzano's Theorem) If $f : [u\ ; v] \to \mathbb{R}$ is continuous and $f(u)f(v) < 0$, then there is a $w \in]u\ ; v[$ such that $f(w) = 0$.

Proof: *This follows at once from the Intermediate Value Theorem by putting* $a = \min(f(u), f(v)) < 0$ *and* $b = \max(f(u), f(v)) > 0$. ❑

108 Corollary Every polynomial of odd degree and real coefficients has at least one real root.

Proof: *Since* $p(-\infty)$ *and* $p(+\infty)$ *have opposite sign, this follows by the Intermediate Value Theorem.* ❑

109 Example Demonstrate that the polynomial $p(x) = x^5 - x - 1$ has a root α in $[1\ ; 2]$. Further, find an interval of length $\frac{1}{10}$ or smaller that contains this root.

Solution: ► *Observe that* $p(1) = -1 < 0$ *and* $p(2) = 29 > 0$, *and so, since the polynomial changes sign in* $[1\ ; 2]$, *by Bolzano's Theorem, the polynomial has a root* $\alpha \in [1\ ; 2]$. *In fact, we find*

$$p(1.1) \approx -0.5, \qquad p(1.2) \approx 0.3,$$

and so $\alpha \in [1.1\ ; 1.2]$, *which is an interval of length* $\frac{1}{10}$ *containing the zero. Still*

$$p(1.15) \approx -.14, \qquad p(1.17) \approx 0.02,$$

meaning that $\alpha \in [1.15\ ; 1.17]$. *This interval has length* $1.17 - 1.15 = 0.02 = \frac{1}{50}$, *so we have done better than the required* $\frac{1}{10}$.

◄

110 Theorem (Weierstrass Theorem) A continuous function $f : [a\ ; b] \to \mathbb{R}$ attains a maximum and a minimum on $[a\ ; b]$.

Proof: *By Darboux's Theorem and by Bolzano's Theorem,* $f([a\ ; b])$ *is a closed finite interval. The minimum sought is the left endpoint of this interval and the maximum sought is the right endpoint of this interval.* ❑

Homework

3.1.1 Problem Given that the equation $x^7 - x^6 - 2x^4 + x^3 + 1 = 0$ has exactly three different real roots, find intervals, of length 1 or shorter, containing each root.

3.1.2 Problem Let $p(x), q(x)$ be polynomials with real coefficients such that

$$p(x^2 + x + 1) = p(x)q(x).$$

Prove that p must have even degree.

3.1.3 Problem A function f defined over all real numbers is continuous and for all real x satisfies

$$(f(x)) \cdot ((f \circ f)(x)) = 1.$$

Given that $f(1000) = 999$, find $f(500)$.

3.1.4 Problem Suppose that $f : [0\,;1] \to [0\,;1]$ is continuous. Prove that there is a number c in $[0\,;1]$ such that $f(c) = 1 - c$.

3.1.5 Problem (Universal Chord Theorem) Suppose that f is a continuous function of $[0\,;1]$ and that $f(0) = f(1)$. Let n be a strictly positive integer. Prove that there is some number $x \in [0\,;1]$ such that $f(x) = f(x+1/n)$.

3.2 Graphical Differentiation

In this section we will take a very informal approach to differentiation. A more formal approach will be given in the next section.

111 Definition Let f be a continuous function and let $A(a, f(a))$ be a point on the graph of the function. We say that f is *smooth at* A if upon imagining a particle travelling at some steady speed along the curve, then the particle does not experience an abrupt change of direction.

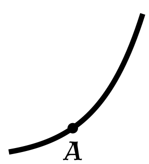


Figure 3.3: Smooth curve at A.

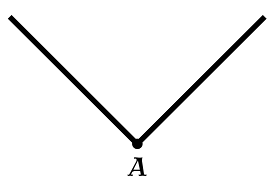


Figure 3.4: Corner at A.

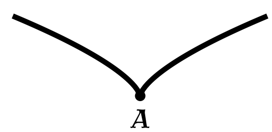


Figure 3.5: Cusp at A.

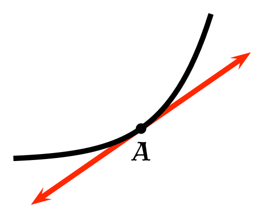


Figure 3.6: Tangent line at $x = a$.

112 Example Figure 3.3 gives an example of a smooth curve. The curve in figure 3.4 has a "sharp corner" at A and it is not smooth. The curve in figure 3.5 is a "cusp."

113 Definition If f is an affine function, we define the *tangent line* at any point $A(a, f(a))$ to be the line that is the graph of f. Otherwise, if f is a smooth function at the point $A(a, f(a))$, then *tangent line at* A is the unique line with the following properties:

1. for a sufficiently small neighbourhood of A, the line just touches the curve at A.
2. for this sufficiently small neighbourhood of A, the portion of the curve inside the neighbourhood is on only one side of the line.

The *derivative* of f at $x = a$, denoted by $f'(a)$, is the slope of the tangent line at $x = a$.

114 Example Figure 3.6 gives an example of the tangent line to a curve.

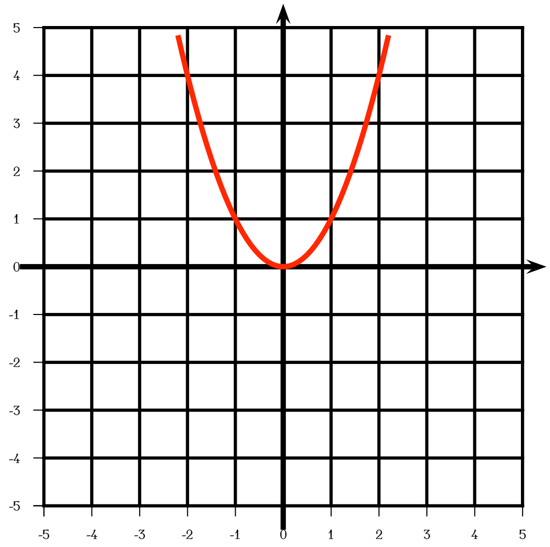


Figure 3.7: Example 115.

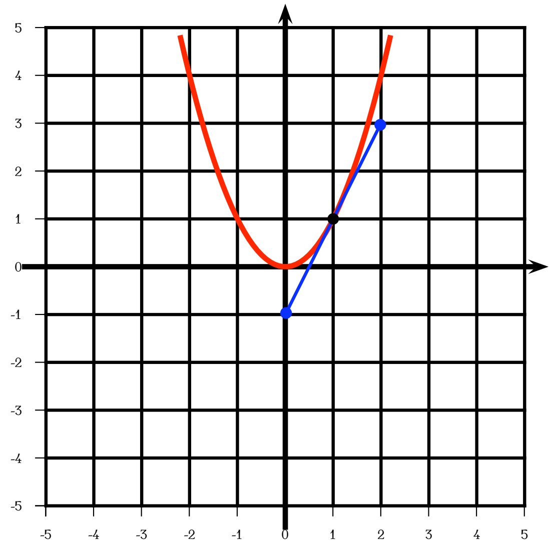


Figure 3.8: Example 115.

115 Example For the curve in figure, find, approximately, the value of $f'(1)$.

Solution: ▶ *We draw a line through the point $(1, f(1))$ that just grazes the curve, as in figure 3.8. We compute the slope of this line:*

$$\frac{3-(-1)}{2-0} = 2.$$

Hence we find $f'(1) = 2$.

◀

116 Definition If a function f is differentiable at every point x of its domain, then the *derivative* function of f, denoted by f', is the function with assignment rule $x \mapsto f'(x)$.

Given the graph of a smooth curve, we can approximately obtain the graph of its derivative by taking the following steps.

1. We divide up the domain of f into intervals of the same length.
2. For each endpoint x of an interval above, we look at the point $(x, f(x))$ on the graph of f.
3. We place a ruler so that it is tangent to the curve at $(x, f(x))$.
4. We find the slope of the ruler. Recall that any two points on the tangent line (the ruler) can be used to find the slope.
5. We tabulate the slopes obtained and we plot these values, obtaining thereby an approximate graph of f'.

To further aid our graphing of the derivative, we make the following observations.

1. If the function increases, then the slope of the tangent is positive.

2. If the function decreases, then the slope of the tangent is negative.
3. If the curve is convex, then the slope of the tangent increases.
4. If the curve is concave, then the slope of the tangent decreases.

See figures 3.9 through 3.12 for several examples.

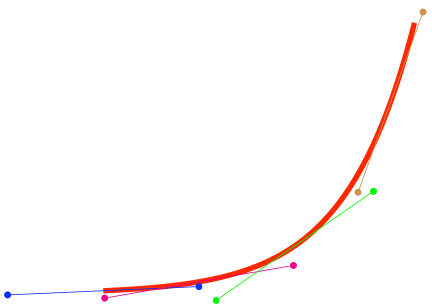


Figure 3.9: Increasing and convex.

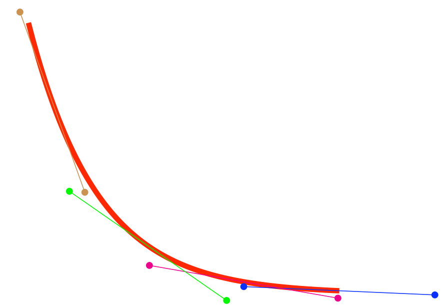


Figure 3.10: Decreasing and convex.

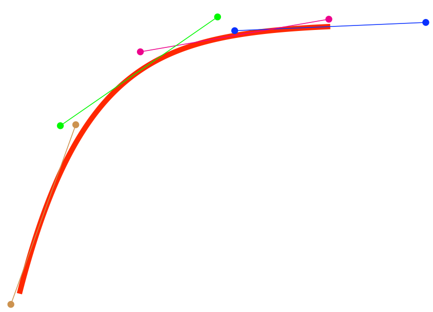


Figure 3.11: Increasing and concave.

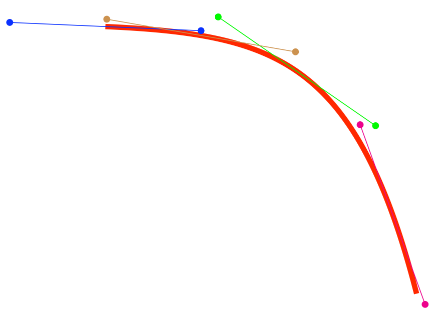


Figure 3.12: Decreasing and concave.

117 Example Find an approximate graph for the derivative of f given in figure 3.13.

Solution: ▶ *Observe that from the remarks following figure 3.10, we expect f' to be positive in $[-1.4;-0.6]$, since f increases there. We expect f' to be 0 at $x = -0.6$, since f appears to have a (local) maximum there. We expect f' to be negative in $[-0.6;0.6]$ since f decreases there. We expect f' to be 0 at $x = 0.6$, since f appears to have a (local) minimum there. Finally we expect f' to be positive for $[0.6;1.4]$ since f is increasing there.*

In our case we obtain the following (approximate) values for $f'(x)$.

x	-1.4	-1.2	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1	1.2	1.4
$f'(x)$	4.88	3.32	2	0.92	0.08	-0.52	-0.88	-1	-0.88	-0.52	0.08	0.92	2	3.32	4.88

An approximate graph of the strong derivative appears in figure 3.14. ◀

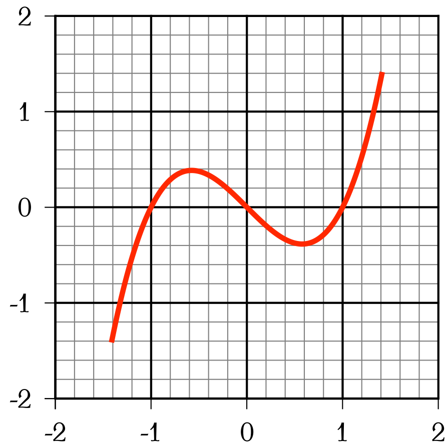


Figure 3.13: Example 117. $y = f(x)$

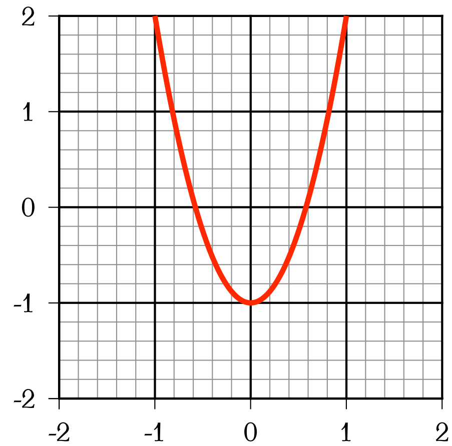


Figure 3.14: Example 117. $y = f'(x)$

118 Example Figure 3.16 gives an approximate graph of the derivative of the graph appearing in figure 3.15. We see that the curve is increasing and so the derivative is positive. As the curve travels the interval $]-1\,;0[$, the slope of the tangent is going from almost horizontal (slope 0) to almost vertical (infinite slope). At $x = 0$ the tangent is vertical, and so the derivative has a pole there. As the curve travels the interval $]0\,;+1[$, the slope of the tangent going from almost vertical to almost 0.

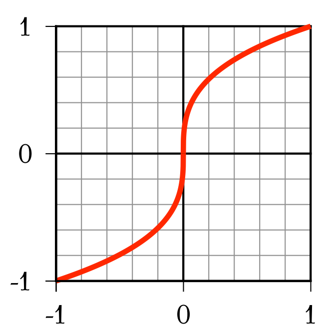


Figure 3.15: Example 118. $y = f(x)$

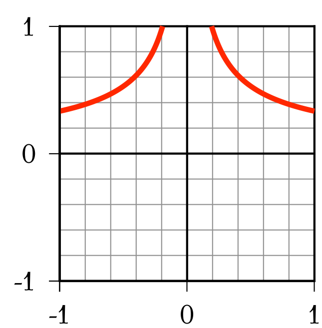


Figure 3.16: Example 118. $y = f'(x)$

To obtain the graph of the derivative above, we have served ourselves of the table below.

x	-1.0	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8	1.0
$f'(x)$	0.3	0.4	0.5	0.6	1.0	$+\infty$	1.0	0.6	0.5	0.4	0.3

Homework

3.2.1 Problem Draw an example of a curve that is everywhere continuous, but fails to be differentiable for $x \in \{-1, 0, 1\}$.

3.2.2 Problem Draw an example of a curve f that is everywhere continuous, differentiable in $\mathbb{R} \setminus \{-2, 2\}$, and such that $f'(x) = -1$ for $x < -2$, $f'(x) = 0$ for $-2 < x < 2$ and $f'(x) = 1$ for $x > 2$.

3.2.3 Problem Draw an example of a curve f that is continuous and differentiable in $\mathbb{R} \setminus \{0\}$, and such that $f(0+) = +\infty$, $f'(0+) = +\infty$, $f(0-) = -\infty$, $f'(0-) = -\infty$, $f'(-\infty) = 0$, $f'(+\infty) = 0$, $f(1) = 1$ and $f(-1) = -1$.

3.2.4 Problem Draw an example of an everywhere differentiable curve $y = f(x)$ with

$$f(-2) = f(0) = f(2) = 0, \qquad f(-1) = 4, \quad f(1) = -2, \qquad f'(-1) = f'(0) = f'(1) = f'(2) = 0,$$

$$f(+\infty) = 0, \quad f(-\infty) = -2, \quad f''(x) > 0 \text{ for } x \in \left]-\infty\,;-2\right[, \quad f''(x) < 0 \text{ for } x \in \left]+\frac{5}{2}\,;+\infty\right[$$

or explain why such a curve does not exist.

3.2.5 Problem Use the graph in figure 3.17 below to complete the following table for a function f. Also, make a rough sketch of f'.

x	-1	-0.4	-0.2	0	0.2	0.4	1
$f(x)$							
$f'(x)$							

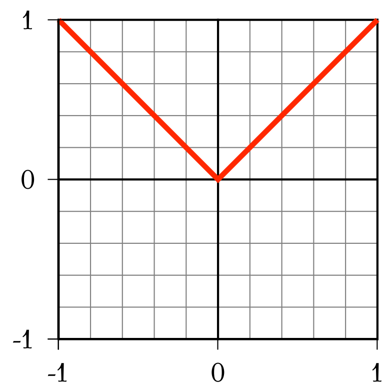


Figure 3.17: Problem 3.2.5.

3.2.6 Problem Use the graph in figure 3.18 below to complete the following table for a function f. Also, make a rough sketch of f'.

x	-1	-0.4	-0.2	0	0.2	0.4	1
$f(x)$							
$f'(x)$							

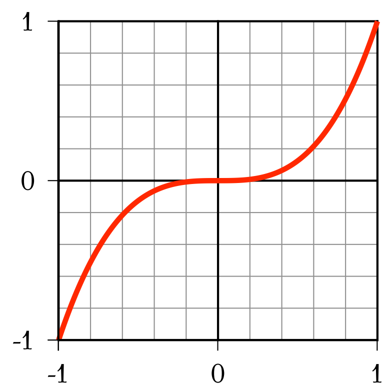


Figure 3.18: Problem 3.2.6.

3.3 The Strong Derivative

Given a finite number of points, we can find infinitely many curves passing through them. See for example figure 3.19, where we see three very different curves f, g, h each simultaneously passing through the points A, B, C. Thus plotting a few points of the graph of a function can give a misleading picture.

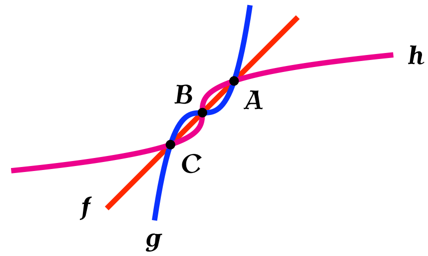


Figure 3.19: A few points do not a graph determine.

By the same token, given a formula, the plotting of a few points does not give the salient features of a graph. For example, let us say that we wanted to graph $y = 4x - x^3$. In figures 3.20 through 3.23 we have chosen a few selected points on the curve and interpolated between them through lines. But relying on this method does not give proof that the graph will not have more turns or bends, say, or that it will grow indefinitely for values of x of large magnitude.

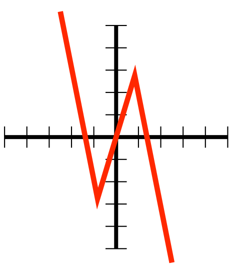


Figure 3.20: Four plot points.

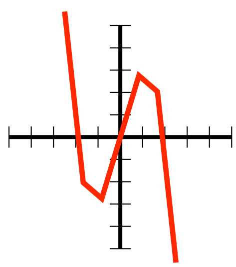


Figure 3.21: Seven plot points.

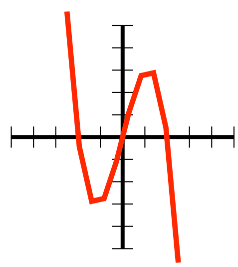


Figure 3.22: Ten plot points.

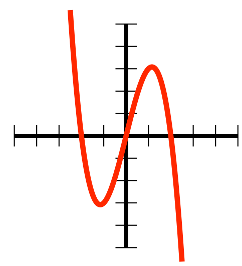


Figure 3.23: One thousand plot points.

But for all its faults, the progression of shapes in figures 3.20 through 3.23 suggests that a “reasonable” graph can be approximated by a series of straight lines. By a “reasonable” graph we mean one that does not have many sharp turns, does not oscillate wildly, does not have many jumps or many asymptotes, and that it is mostly continuous and “smooth.” Admittedly, these concepts are vague, but we will gain more insight into them as we progress.

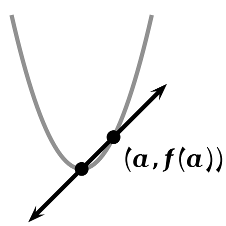


Figure 3.24: Left secant through $(a, f(a))$.

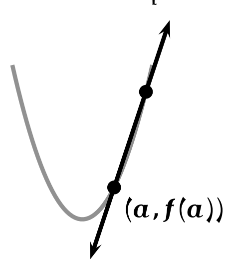


Figure 3.25: Right secant through $(a, f(a))$.

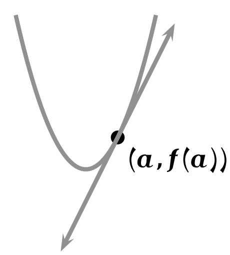


Figure 3.26: Line grazing $(a, f(a))$.

How do we choose the lines to approximate a given “reasonable” curve? Given a function f consider the point $(a, f(a))$ on the graph of the function. What happens around this point? If we approached a through values $x < a$ and joined the line with endpoints $(x, f(x))$ and $(a, f(a))$, we would obtain a secant line like that of figure 3.24. If we approached a through values $x > a$ and joined the line with endpoints $(a, f(a))$ and $(x, f(x))$, we would obtain a secant line like that of figure 3.25. Eventually, on getting closer to $(a, f(a))$ we obtain a line just barely grazing the curve—that is, "tangent" to the curve—at the point $(a, f(a))$, as in figure 3.26.

In the simplest of cases, if our curve is the line $L : y = mx + k$, then in a neighbourhood of the point $x = a$ the tangent line to L should be itself! Given now the formula for a function f and a point $(a, f(a))$ on the graph of f, how do we determine the tangent line to f at $(a, f(a))$? Recall that if $\epsilon \to 0$, then $a + \epsilon$ is in a neighbourhood of a. The slope of the secant line joining $(a + \epsilon, f(a + \epsilon))$ and $(a, f(a))$ is

$$\frac{f(a+\epsilon) - f(a)}{a + \epsilon - \epsilon} = \frac{f(a+\epsilon) - f(a)}{\epsilon} \tag{3.1}$$

We denote the value—if there is one—of 3.1— as $x \to \epsilon$ by $f'(a)$. Hence for fixed but small ϵ we have

$$\frac{f(a+\epsilon) - f(a)}{\epsilon} \approx f'(a) \implies f(a+\epsilon) \approx f(a) + f'(a)\epsilon.$$

There is, generally, an error in taking the dextral side as an approximation for the sinistral side on the above formula. We will settle for having an error of the order of $o(\epsilon)$, which will normally will be a good compromise for most of the formulæ we will encounter. This prepares the ground for our main definition of this section.

119 Definition Let f be a function and let $a \in \mathbf{Dom}(f)$. When there is a number $f'(a)$ such that

$$f(a+\epsilon) = f(a) + f'(a)\epsilon + o(\epsilon) \qquad \text{as} \qquad \epsilon \to 0$$

then we say that the function f has a *strong derivative* $f'(a)$ or that f is *strongly differentiable* at $x = a$. If we consider the set $\{x \in \mathbf{Dom}(f) : f'(x) \ \mathbf{exists}\}$ then we may form the function $x \mapsto f'(x)$ with domain $\mathbf{Dom}(f') = \{x \in \mathbf{Dom}(f) : f'(x) \ \mathbf{exists}\}$. We call the function f' the *strong derivative* of f. We will also often use the symbol $\dfrac{\mathrm{d}}{\mathrm{d}x} f(x)$ to denote the function $x \mapsto f'(x)$.

120 Definition If f' is itself differentiable, then the function $(f')' = f''$ is the *second derivative* of f. It is also denoted by $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} f(x)$. We similarly define the third, fourth, etc., derivatives. It is customary to denote the first three derivatives of a function with primes, as in f', f'', f''', and any higher derivative with either roman numbers or with the order of the derivative enclosed in parenthesis, as in f^{iv}, f^{v} or $f^{(4)}, f^{(5)}$, etc.

We will give but two very simple examples of the computation of derivatives. These will be needed in this chapter. More substantial examples will appear in subsequent chapters.

121 Lemma The strong derivative of a constant function is the 0 function. In symbols, if f is a function with assignment rule $f(x) = k$, constant, then for all x, $f'(x) = 0$.

Proof: *We have*

$$f(x+\epsilon) = k = k + 0 \cdot \epsilon + 0\epsilon^2 = k + 0 \cdot \epsilon + o(\epsilon),$$

which proves the assertion. ❑

STOP *For ellipsis we will write* $(k)' = 0$ *or* $\dfrac{\mathrm{d}}{\mathrm{d}x} k = 0$.

122 Lemma The strong derivative of the identity function $x \mapsto x$ is the constant function $x \mapsto 1$.

Proof: *If $f(x) = x$ we have*

$$f(x+\epsilon) = x + \epsilon = x + 1 \cdot \epsilon + 0\epsilon^2 = x + 1 \cdot \epsilon + o(\epsilon),$$

and so $f'(x) = 1$. ❑

For ellipsis we will write $(x)' = 1$ or $\frac{d}{dx} x = 1$.

3.4 Derivatives and Graphs

In this section we prove the remarks preceding figures 3.9 through 3.12, which will help us with the graphing in subsequent chapters.

123 Theorem If f is strongly differentiable at x then f is continuous at x.

Proof: *We have $f(x+\epsilon) = f(x) + f'(x)\epsilon + o(\epsilon)$. If $\epsilon > 0$ and $\epsilon \to 0$ then $f(x+\epsilon) = f(x+)$ and similarly if $\epsilon < 0$ and $\epsilon \to 0$ $f(x+\epsilon) = f(x-)$. Hence we have $f(x+) = f(x) = f(x-)$, and f is continuous at x.* ❑

124 Theorem Let f be strongly differentiable at x. If $f'(x) > 0$ then f is increasing in a neighbourhood of x, if $f'(x) < 0$ then f is decreasing in a neighbourhood of x.

Proof: *We have $f(x+\epsilon) - f(x) = f'(x)\epsilon + o(\epsilon)$. For ϵ very small, this means that*

$$f(x+\epsilon) - f(x) \approx f'(x)\epsilon,$$

that is, the sign of $f(x+\epsilon) - f(x)$ is the same as the sign of $f'(x)\epsilon$. Thus if $\epsilon > 0$ and $f'(x) > 0$, then $f(x+\epsilon) > f(x)$, that is, f is increasing. If $\epsilon > 0$ and $f'(x) < 0$, then $f(x+\epsilon) < f(x)$, that is, f is decreasing. Similar conclusions are reached when considering $\epsilon < 0$ and the theorem is proved. ❑

125 Definition If f is strongly differentiable at x and $f'(x) = 0$, then we say that x is a *stationary point of f*.

126 Definition If there is a point a for which $f(x) \leq f(a)$ for all x in a neighbourhood centred at $x = a$ then we say that f has a *local maximum* at $x = a$. Similarly, if there is a point b for which $f(x) \geq f(b)$ for all x in a neighbourhood centred at $x = b$ then we say that f has a *local minimum* at $x = b$.

127 Theorem If f is strongly differentiable at $x = a$, $f'(a) = 0$, and f' changes from $+$ to $-$ in a neighbourhood of a then $x = a$ is a local maximum. If f is strongly differentiable at $x = b$, $f'(b) = 0$, and f' changes from $-$ to $+$ in a neighbourhood of b then $x = b$ is a local minimum.

Proof: *By Theorem 124, when f' changes from $+$ to $-$, f is increasing and then decreasing in a neighbourhood of $x = a$. By Weierstrass's Theorem (Theorem 110), f assumes a maximum on a closed neighbourhood containing a. It cannot be to the left of a since the function is increasing there, and it cannot be to the right of a since the function is decreasing there. Hence the maximum must be at $x = a$ and so $f(x) \leq f(a)$ for x in a neighbourhood of a. The result just obtained applied to $-f$ yields the second half of the theorem.* ❑

128 Lemma If f' increases in a neighbourhood of x, then f is convex in a neighbourhood of x. Similarly, if f' decreases in a neighbourhood of x, then f is concave in a neighbourhood of x.

Proof: *Let $\epsilon > 0$. Then*

$$f(x) = f(x - \epsilon + \epsilon) = f(x - \epsilon) + f'(x - \epsilon)\epsilon + o(\epsilon),$$

$$f(x) = f(x + \epsilon - \epsilon) = f(x + \epsilon) - f'(x + \epsilon)\epsilon + o(\epsilon).$$

Adding,

$$2f(x) = f(x + \epsilon) + f(x - \epsilon) + \left(f'(x - \epsilon) - f'(x + \epsilon)\right)\epsilon + o(\epsilon).$$

For ϵ very small we then have

$$2f(x) \approx f(x + \epsilon) + f(x - \epsilon) + \left(f'(x - \epsilon) - f'(x + \epsilon)\right)\epsilon.$$

If f' is increasing then $f'(x - \epsilon) - f'(x + \epsilon) < 0$. Since $\epsilon > 0$ this implies that

$$2f(x) < f(x + \epsilon) + f(x - \epsilon),$$

which means that f is convex in a neighbourhood of x. This result now applied to $-f$ gives the second half of the theorem. ❑

129 Theorem A twice strongly differentiable -function f is convex in a neighbourhood of $x = a$ if $f''(a) > 0$. It is concave in a neighbourhood of $x = b$ if $f''(b) < 0$.

Proof: *This follows from Lemma 128 and Theorem 124.* ❑

130 Example The graph of the strong derivative f' of a function f is given in figure 3.27. Then according to Theorem 127 f has a local minimum at $x = -2$ and $x = 2$, and a local maximum at $x = 0$ and $x = 4$.

131 Example Figure 3.28 below shews the graph of a function f. At the point A, $f' > 0$ and $f'' < 0$. At B, $f' = 0$, $f'' < 0$. At C, $f' < 0$, $f'' < 0$. At D, $f' = 0$ and $f'' > 0$. At E, $f' > 0$, $f'' > 0$.

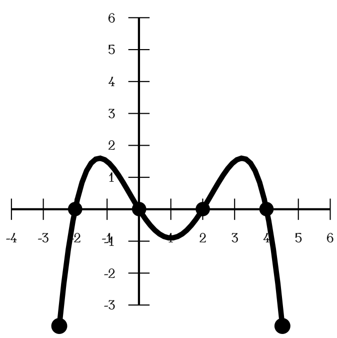


Figure 3.27: Example 130.

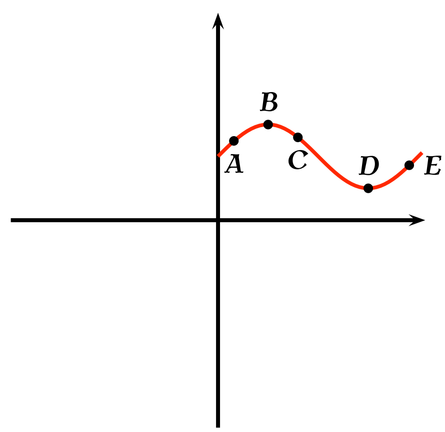


Figure 3.28: Example 131.

Homework

3.4.1 Problem Draw an example of a curve p satisfying the following criteria:

1. p is everywhere differentiable,
2. $p(-3) = p(0) = p(2) = 0$,

3. $p(x) < 0$ if $x \in]-\infty\,;-3[\,\cup\,]0\,;2[$,

4. $p(x) > 0$ if $x \in]-3\,;0[\,\cup\,]2\,;+\infty[$,

5. $-2 < p(x) < \dfrac{3}{2}$ for $x \in [-3\,;2]$,

6. p has an inflexion point near $x = -\dfrac{1}{3}$,

7. $p(-\infty) = -\infty$, $p(+\infty) = +\infty$.

3.4.2 Problem Draw an example of a curve f satisfying the following criteria:

1. f is everywhere differentiable,

2. $f'(x) > 0$ for all $x \in \mathbb{R}$,

3. $f(-\infty) = -1$, $f(+\infty) = 1$,

4. $f''(x) > 0$ for $x < 0$, $f''(x) < 0$ for $x > 0$,

5. $f(0) = f''(0) = 0$.

3.4.3 Problem In figure 3.29 below the graph of f' is given for a certain function $f : [0\,;6] \to \mathbb{R}$. Find the x-coordinate of the local maximum of f. How many inflexion points does the graph of f have?

3.4.4 Problem The graph of the *derivative* f' of a certain function f defined on $[-3;2]$ is given in figure 3.30 below.

Find the x-coordinate of the local minimum of f.

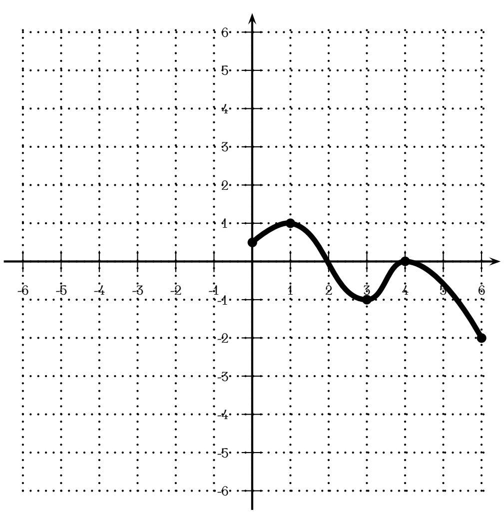


Figure 3.29: Problem 3.4.3.

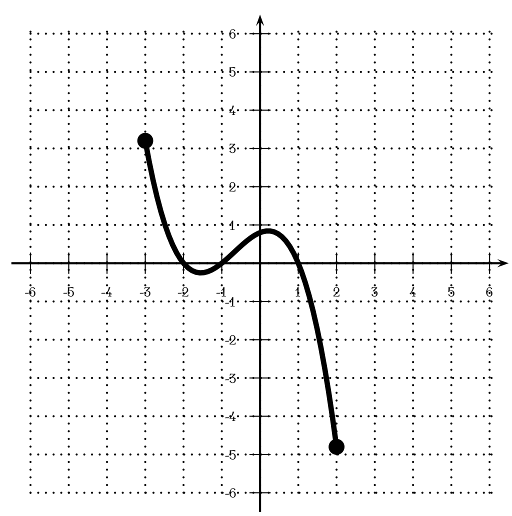


Figure 3.30: Problem 3.4.4.

3.4.5 Problem Suppose $f : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable and that $f(+\infty) = K$, a constant. Must $f'(+\infty) = 0$?

3.5 Graphical Integration

We now consider the following problem: Given a sufficiently "decent curve"[1], what is the area bounded by the curve and the x-axis in a given finite interval?

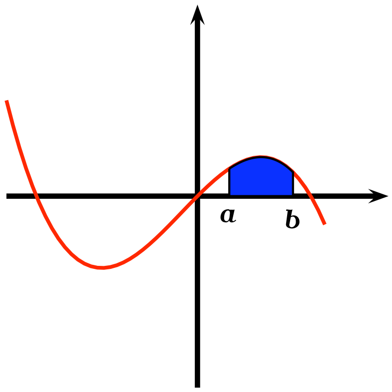


Figure 3.31: $\int_a^b f \geq 0$

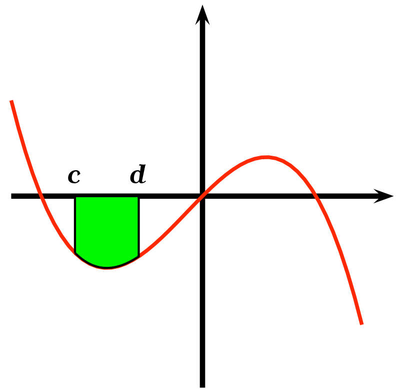


Figure 3.32: $\int_c^d f \leq 0$

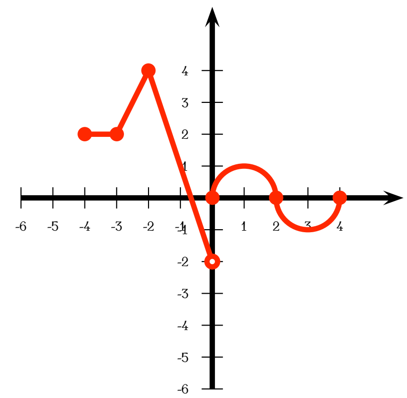


Figure 3.33: 133

132 Definition Let f be a function. We denote the signed area between f and the x-axis between $x = a$ and $x = b$ by $\int_a^b f$ or $\int_a^b f(x)\mathrm{d}x$. The symbol $\int_a^b f$ is read "the (definite) integral of f from a to b. See figures 3.31 and 3.32.

We will normally use the notation $\int_a^b f$ in the cases when we are given a picture of f, and we will relegate the notation $\int_a^b f(x)\mathrm{d}x$ when we are given a formula for f. Also, there is nothing holy about which variable of integration to use. Thus $\int_a^b f(x)\mathrm{d}x = \int_a^b f(t)\mathrm{d}t = \int_a^b f(u)\mathrm{d}u$, etc.

133 Example Figure 3.33 is composed of lines and semicircles. One finds

1. $\int_{-4}^{-3} f = 2(1)$ since the figure between the curve and the x-axis is a rectangle of base 1 and height 2.

2. $\int_{-3}^{-2} f = \left(\dfrac{2+4}{2}\right) 1 = 3$ since the figure between the curve and the x-axis is a trapezoid with bases of length 2 and 4 and height 1.

3. $\int_{-4}^{-2} f = 2 + 3 = 5$ by adding the areas of the rectangle and the trapezoid.

4. $\int_{-2}^{0} f = 2$. The figure consists of two triangles, one with height 4 above the x-axis (and hence having positive area, and another one with height 2 below the x-axis (thus having negative area). To find the area we enclose both triangles in a rectangle $ABCD$ with vertices at $A = (-2, 4)$, $B = (0, 4)$, $C = (0, -2)$,

[1] A curve that pays its taxes, observes the Torah, and goes to synagogue every Shabath....

$D = (-2,-2)$. Put $E = (-2,0)$ and $F = (0,0)$. Observe that the area of $ABCD$ is the area of rectangle $ABFE$ minus the area of rectangle $CDEF$ hence it is $2(4) - 2(2) = 4$. The area of the triangles is half this, that is, 2. Another approach is the following. Let a be the length of base of the triangle with positive area and let b the length of the base of the triangle with negative area. Then $a + b = 2$. Also, by similar triangles, $\frac{4}{a} = \frac{2}{b}$, and hence $a = 2b$. Thus $2b + b = 2 \implies b = \frac{2}{3}$ and $a = \frac{4}{3}$. Whence the area of the two triangles is $\frac{1}{2}(\frac{4}{3})(4) - \frac{1}{2}(\frac{2}{3})(2) = 2$, as before.

5. $\int_{-1}^{0} f = -\frac{5}{9}$. Observe that the area sought is composed of a triangle of base $1 - \frac{2}{3} = \frac{1}{3}$ (with positive area) and the triangle of the preceding part of base $\frac{2}{3}$ and height 2 (with negative area). Again, by similar triangles, if h is the height of the triangle of base $\frac{1}{3}$ then $\frac{h}{\frac{1}{3}} = \frac{2}{\frac{2}{3}}$, thus $h = 1$. Thus the area sought is $\frac{1}{2}(\frac{1}{3})(1) - \frac{1}{2}(\frac{2}{3})(2) = -\frac{1}{2}$.

6. $\int_{0}^{2} f = \frac{\pi}{2}$ since the figure between the curve and the x-axis is a semicircle of radius 1.

7. $\int_{0}^{1} f = \frac{\pi}{4}$ since the figure between the curve and the x-axis is a quarter circle of radius 1.

8. $\int_{0}^{3} f = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ by adding the (positive) area of the semicircle to the (negative) area of the quarter circle.

9. $\int_{0}^{4} f = 0$ since the areas of the semicircles cancel each other.

In the particular case when the function is a straight line, it is easy to compute the integral.

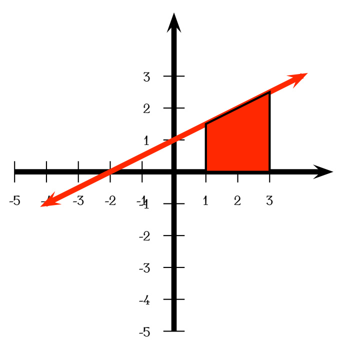


Figure 3.34: Example 134.

134 Example Find the area of the plane region

$$\{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 3, 0 \leq y \leq \frac{x}{2} + 1\}.$$

Solution: ► *This is the area under the line $y = \frac{x}{2} + 1$ and above the x-axis, bounded by the vertical lines $x = 1$ and $x = 3$, as depicted in figure 3.34. Since this is a trapezoid with height 2 and parallel bases of length $\frac{3}{2}$ and $\frac{5}{2}$, we obtain*

$$\int_{1}^{3} \left(\frac{x}{2} + 1\right) dx = \frac{2}{2}\left(\frac{3}{2} + \frac{5}{2}\right) = 4.$$

◀

Homework

3.5.1 Problem The graph of a function f defined on $[-3;4]$ is given in figure 3.35 below. It is composed of a semicircle and lines.

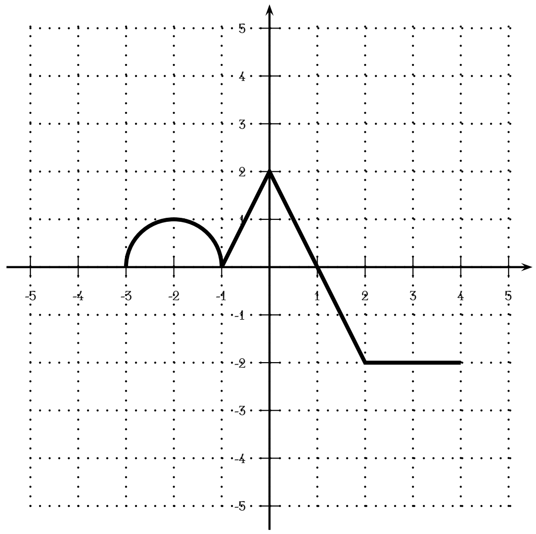


Figure 3.35: Problem 3.5.1.

If $g(x) = \int_{-3}^{x} f(t)\mathrm{d}t$, find $g(2)$. Where does g attain its absolute maximum value?

3.5.2 Problem Give a geometric argument to justify that $\int_0^t x\mathrm{d}x = \frac{t^2}{2}$.

3.5.3 Problem Give a geometric argument to justify that $\int_{-2}^{3} (2x + 1)\mathrm{d}x = 10$.

3.5.4 Problem Give a geometric argument to justify that $\int_{-1}^{1} \sqrt{1 - x^2}\mathrm{d}x = \frac{\pi}{2}$.

3.6 The Fundamental Theorem of Calculus

In this section we seek to establish a link between differentiation and integration.

135 Definition Let f be a function. Then $t \mapsto \int_a^t f$ is the function that for each value of t gives the area between the graph of f and the x-axis bounded by vertical lines at $x = a$ and $x = t$. We prefer to write, in brief, $\int_a f$ rather than $t \mapsto \int_a^t f$.

From the additivity of area, one has

136 Theorem (Chasles' Rule) $\displaystyle\int_a^b f + \int_b^c f = \int_a^c f.$

Letting $a = b = c$ in Chasles' Rule 136 one obtains

$$\int_a^a f + \int_a^a f = \int_a^a f \implies \int_a^a f = \int_a^a f - \int_a^a f$$

hence

137 Theorem $\displaystyle\int_a^a f = 0.$

Letting $a = c$ and using Rule 137 one obtains

$$\int_a^b f + \int_b^a f = \int_a^a f \implies \int_a^a f = \int_a^a f - \int_b^a f = -\int_b^a f,$$

whence

138 Theorem $\displaystyle\int_a^b f = -\int_b^a f.$

Given constants a and b how are $\displaystyle\int_a f$ and $\displaystyle\int_b f$ related? We have

$$\int_a^b f = \int_a^t + \int_t^b f = \int_a^t - \int_b^t f \implies \int_a^t = \int_a^b f + \int_b^t f,$$

hence $\displaystyle\int_a f$ and $\displaystyle\int_b f$ differ by the constant $\displaystyle\int_a^b f$.

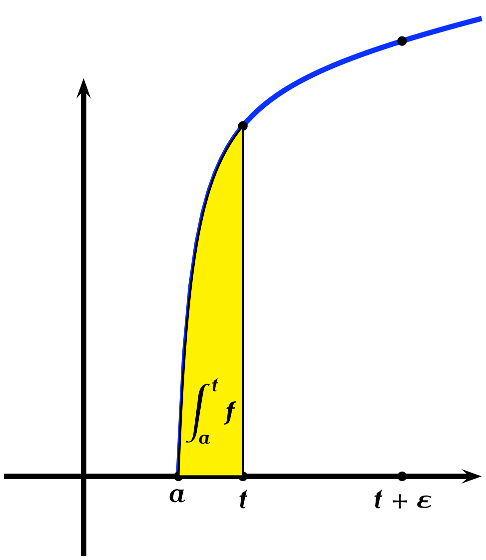


Figure 3.36: Fundamental Theorem of Calculus.

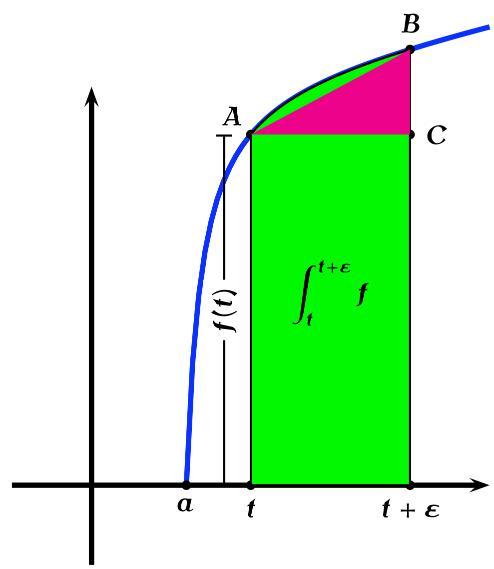


Figure 3.37: Fundamental Theorem of Calculus.

The area in diagram 3.37 below is given exactly by

$$\int_t^{t+\varepsilon} f = \int_a^{t+\varepsilon} f - \int_a^t f.$$

If ε is very small, the region is nearly trapezoidal, composed of a rectangle of height $f(t)$ and width ε and $\triangle ABC$ of height $f(t+\varepsilon) - f(t)$ and width ε. Hence, for ε very small

$$\int_a^{t+\varepsilon} f - \int_a^t f \approx f(t)\varepsilon + \frac{\varepsilon}{2}(f(t+\varepsilon) - f(t)).$$

If, besides, f is continuous, then, as $\varepsilon \to 0$, $f(t+\varepsilon) - f(t) = o(1)$. Whence we deduce,

$$\int_a^{t+\varepsilon} f - \int_a^t f \approx f(t)\varepsilon + \frac{\varepsilon}{2}(f(t+\varepsilon) - f(t)) = f(t)h + o(\varepsilon).$$

If we put $I(t) = \int_a^t f$, what we are saying is that as $\varepsilon \to 0$,

$$I(t+\varepsilon) \approx I(t) + f(t)\varepsilon + o(\varepsilon) \implies I'(t) \approx f(t).$$

Upon replacing the approximation symbol with true equality we obtain the following link between derivatives and integrals.

139 Theorem (First Form of the Fundamental Theorem of Calculus) If f is continuous on $[a\,;b]$ and $x \in [a\,;b]$ then

$$I'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f = f(t).$$

140 Definition A *primitive* (or *antiderivative*) of a function f is a function F such that $F' = f$.

By the First Form of the Fundamental Theorem of Calculus, $I(t) = \int_a^t f(x)\mathrm{d}x$ is a primitive of $f(x)$. If c is a constant and $F(t) = \int_c^t f(x)\mathrm{d}x$, then

$$I(t) - F(t) = \int_a^t f(x)\mathrm{d}x - \int_c^t f(x)\mathrm{d}x = \int_a^t f(x)\mathrm{d}x + \int_t^c f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x,$$

whence $I(t)$ and $F(t)$ differ by a constant. Since by Lemma 121, the derivative of a constant function is 0, we have

$$(I(t) - F(t))' = 0.$$

Assuming for the moment that the derivative of a sum is the sum of the derivatives (a fact that will be proved in Theorem 153), we have

$$I'(t) - F'(t) = (I(t) - F(t))' = 0 \implies I'(t) = F'(t),$$

which means that any two primitives of f only differ by a constant. This means that we may write

$$F(t) = \int_a^t f(x)\mathrm{d}x + k,$$

for some constant k. This in turn gives

$$F(b) - F(a) = \left(\int_a^b f(x)\mathrm{d}x + k\right) - \left(\int_a^a f(x)\mathrm{d}x + k\right) = \int_a^b f(x)\mathrm{d}x,$$

since by Theorem 137, $\int_a^a f(x)\mathrm{d}x = 0$. Furthermore, since

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x) = f(x),$$

let us write then

$$\mathrm{d}F(x) = f(x)\mathrm{d}x. \tag{3.2}$$

We have then demonstrated the following theorem.

141 Theorem (Second Form of the Fundamental Theorem of Calculus) Let the function f be continuous on $[a\,;b]$ and let F be a primitive of f. Then

$$\int_a^b f(x)\mathrm{d}x = \int_a^b \mathrm{d}F(x) = F(x)\Big|_a^b = F(b) - F(a).$$

142 Example Since $x \mapsto \dfrac{x+1}{x^8+1}$ is continuous everywhere, the First Form of Fundamental Theorem of Calculus asserts that

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{t+1}{t^8+1}\mathrm{d}t = \frac{x+1}{x^8+1}.$$

Notice that at this stage, we have no idea how to obtain $\displaystyle\int_a^x \frac{t+1}{t^8+1}\mathrm{d}t$.

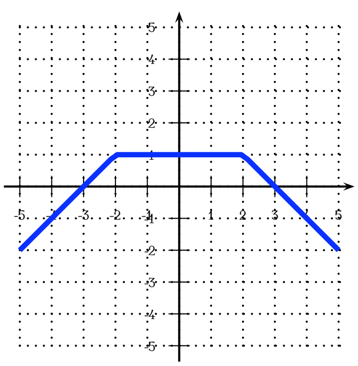


Figure 3.38: $x \mapsto f(x)$.

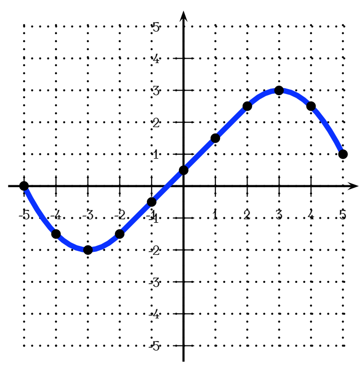


Figure 3.39: $t \mapsto \displaystyle\int_{-5}^t f$.

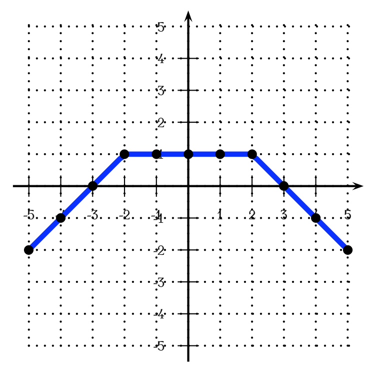


Figure 3.40: $t \mapsto \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_{-5}^t f$.

143 Example Given the curve $f : [-5\,;5] \to \mathbb{R}$ in figure 3.38 we have the graph of a function f. We tabulate approximate values of $\displaystyle\int_{-5} f$ below, and plot these in the graph 3.39.

t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$I(t) = \int_{-5}^t f$	0	$-\frac{3}{2}$	-2	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	3	$\frac{5}{2}$	1

We then tabulate approximate values of $I'(t)$, and plot them in the graph 3.40.

t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$I'(t)$	-2	-1	0	1	1	1	1	1	0	-1	-2

It seems that the graphs of $x \mapsto f(x)$ and $x \mapsto I'(x)$ resemble one another, giving empirical evidence of the First Form Fundamental Theorem of Calculus.

Homework

3.6.1 Problem For the function $f : [-1\,;1] \to \mathbb{R}$ in figure 3.41 below, graph $F : x \mapsto \int_{-1}^{x} f$, $x \in [-1\,;1]$ and $F' : x \mapsto \frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{x} f$, $x \in [-1\,;1]$.

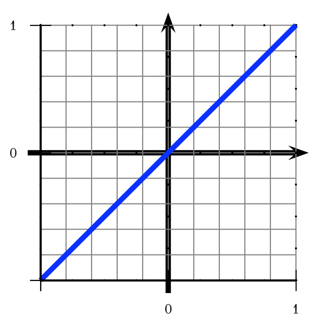


Figure 3.41: Problem 3.6.1.

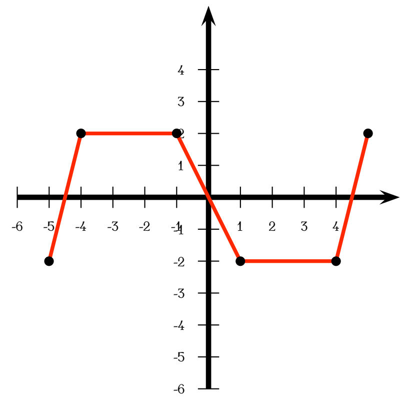


Figure 3.42: f for problem 3.6.2.

3.6.2 Problem Consider the function f in figure 3.42 below. You may assume it is composed solely of straight line segments.

1. Draw f', the derivative of f.
2. Complete the following table.

$\int_{-5}^{-5} f =$	$\int_{-5}^{-4} f =$	$\int_{-5}^{-3} f =$	$\int_{-5}^{-2} f =$	$\int_{-5}^{-1} f =$
$\int_{-5}^{0} f =$	$\int_{-5}^{1} f =$	$\int_{-5}^{2} f =$	$\int_{-5}^{3} f =$	$\int_{-5}^{4} f =$

3. Complete the following table.
4. Using the table in part 3 or otherwise, draw the graph of $\int_{-5} f'$.

$\int_{-5}^{-5} f' =$	$\int_{-5}^{-4} f' =$	$\int_{-5}^{-3} f' =$	$\int_{-5}^{-2} f' =$	$\int_{-5}^{-1} f' =$
$\int_{-5}^{0} f' =$	$\int_{-5}^{1} f' =$	$\int_{-5}^{2} f' =$	$\int_{-5}^{3} f' =$	$\int_{-5}^{4} f' =$

4 Polynomial Functions

In this chapter we study polynomials and their graphs. In order to do the latter, we demonstrate the Power Rule, the Sum Rule, the Product Rule, and the Chain Rule for derivatives. We also study the integration of polynomial functions and some algebraic topics related to the roots of a polynomial.

4.1 Differentiating Power Functions

By a *power function* we mean a function of the form $x \mapsto x^\alpha$, where $\alpha \in \mathbb{R}$. In this chapter we will only study the case when α is a positive integer.

If n is a positive integer, we are interested in how to graph $x \mapsto x^n$. For $n = 0$, the function $x \mapsto 1$ is a constant function, whose graph is the straight line $y = 1$ parallel to the x-axis. For $n = 1$, the function $x \mapsto x$ is the identity function, whose graph is the straight line $y = x$, which bisects the first and third quadrant. We reproduce their graphs below in figures 4.1 through 4.2 for easy reference.

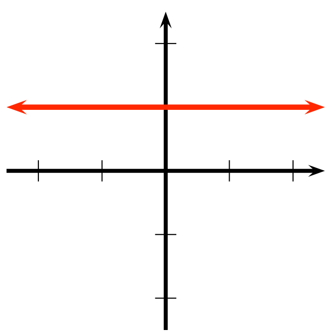


Figure 4.1: $x \mapsto 1$.

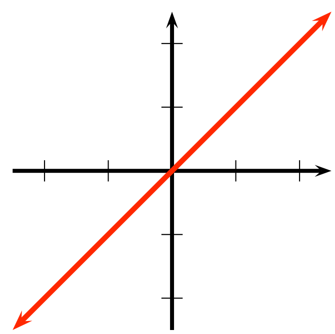


Figure 4.2: $x \mapsto x$.

By the groundwork from the preceding chapter, we know we can gather information about the monotonicity and convexity of the function $x \mapsto x^n$ by studying its first and second derivatives. For that we first establish a series of lemmata.

144 Lemma The strong derivative of the square function $x \mapsto x^2$ is the function $x \mapsto 2x$.

Proof: *If* $f(x) = x^2$ *we have*

$$f(x+\epsilon) = (x+\epsilon)^2 = x^2 + 2x\epsilon + \epsilon^2 = f(x) + 2x\epsilon + o(\epsilon),$$

proving the assertion. For ellipsis we will write $(x^2)' = 2x$ *or* $\dfrac{d}{dx}x^2 = 2x$. ❑

145 Lemma The strong derivative of the cubic function $x \mapsto x^3$ is the function $x \mapsto 3x^2$.

Proof: *If $f(x) = x^3$ we have, using Lemma 144,*

$$\begin{aligned}(x+\epsilon)^3 &= (x+\epsilon)(x^2+2x\epsilon+o(\epsilon)) \\ &= x^3+2x^2\epsilon+O(x\epsilon^2)+x^2\epsilon+2x\epsilon^2+O(\epsilon^3) \\ &= x^3+3x^2\epsilon+o(\epsilon),\end{aligned}$$

as $\epsilon \to 0$, and so $(x^3)' = 3x^2$ (or $\frac{d}{dx}x^3 = 3x^2$). ❑

We will now see that the pattern

$$\frac{d}{dx}x^0 = 0, \qquad \frac{d}{dx}x^1 = 1, \qquad \frac{d}{dx}x^2 = 2x, \qquad \frac{d}{dx}x^3 = 3x^2,$$

is preserved for higher powers of the exponent. Arguing as in Lemma 145, we obtain the following theorem.

146 Theorem (Power Rule) If n is a positive integer, $\frac{d}{dx}x^n = nx^{n-1}$.

Proof: *If the strong derivative of $\frac{d}{dx}x^n = d_n(x)$, we have*

$$\begin{aligned}(x+\epsilon)^{n+1} &= (x+\epsilon)(x^n+d_n(x)\epsilon+o(\epsilon)) \\ &= x^{n+1}+(xd_n(x)+x^n)\epsilon+o(\epsilon)\end{aligned}$$

from where the strong derivative of x^{n+1} is $xd_n(x)+x^n$. Since $d_1(x) = 1$, we have by recurrence,

$$\begin{aligned}d_2(x) &= xd_1(x)+x^1 &&= x\cdot 1+x &&= 2x \\ d_3(x) &= xd_2(x)+x^2 &&= x\cdot(2x)+x^2 &&= 3x^2 \\ d_4(x) &= xd_3(x)+x^3 &&= x\cdot(3x^2)+x^3 &&= 4x^3 \\ d_5(x) &= xd_4(x)+x^4 &&= x\cdot(4x^3)+x^4 &&= 5x^4 \\ d_6(x) &= xd_5(x)+x^5 &&= x\cdot(5x^4)+x^5 &&= 6x^5 \\ d_7(x) &= xd_6(x)+x^6 &&= x\cdot(6x^5)+x^6 &&= 7x^6,\end{aligned}$$

and so, by recursion, $d_n(x) = nx^{n-1}$. ❑

We may express the above result in the "differential" form

$$dx^n = nx^{n-1}dx. \tag{4.1}$$

147 Example We have

$$\frac{d}{dx}x^3 = 3x^2, \qquad \frac{d}{dx}x^7 = 7x^6, \qquad \frac{d}{dx}x^{1000} = 1000x^{999},$$

etc.

4.2 Graphs of Power Functions

In this section we discuss how to graph functions with assignment rules of the form $x \mapsto x^n$, where $n \in \mathbb{N}$. Observe that the natural domain of definition for such rules is $\mathbb{R}$.

We already know that the graphs for the cases $n = 0$ and $n = 1$ are straight lines. Hence we will focus on the cases when $n \geq 2$.

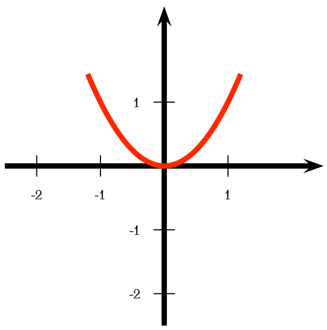


Figure 4.3: $y = x^2$.

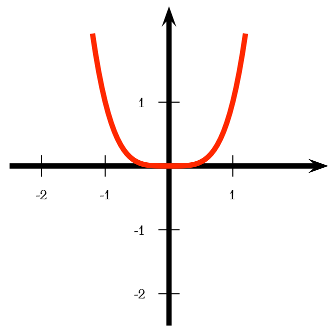


Figure 4.4: $y = x^4$.

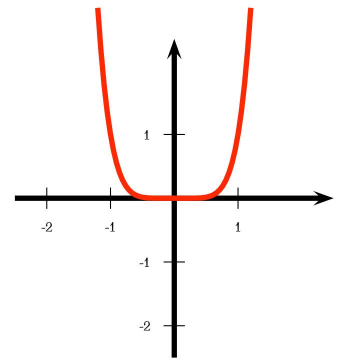


Figure 4.5: $y = x^6$.

We will see in Theorem 148 that for even $n \geq 2$, the graphs of $x \mapsto x^n$ all resemble one another, and look like those in figures 4.3 through 4.5. For $-1 \leq x \leq 1$, the higher the exponent, the flatter the graph (closer to the x-axis) will be, since

$$|x| < 1 \implies \cdots < x^6 < x^4 < x^2 < 1.$$

For $|x| \geq 1$, the higher the exponent, the steeper the graph will be since

$$|x| > 1 \implies \cdots > x^6 > x^4 > x^2 > 1.$$

We will also see that for odd $n \geq 3$, the graphs of $x \mapsto x^n$ all resemble one another, and look like those in figures 4.6 through 4.8.

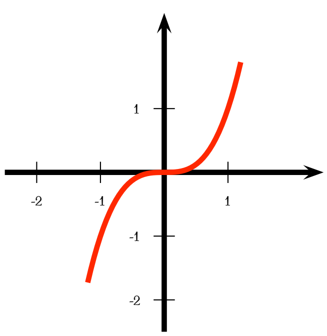


Figure 4.6: $y = x^3$.

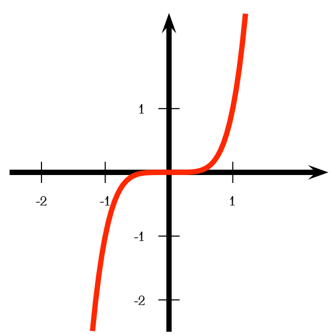


Figure 4.7: $y = x^5$.

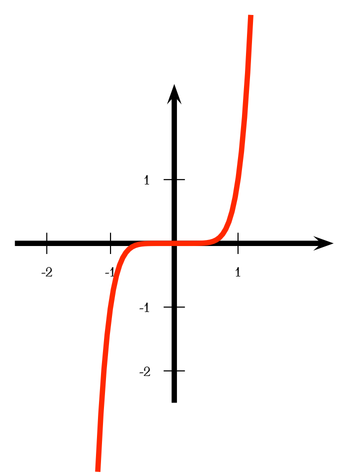


Figure 4.8: $y = x^7$.

We now address the problem of how to graph $x \mapsto x^n$.

148 Theorem Let $n \geq 2$ be an integer and $f(x) = x^n$. Then

- if n is even, f is convex, f is decreasing for $x < 0$, and f is increasing for $x > 0$. Also, $f(-\infty) = f(+\infty) = +\infty$.
- if n is odd, f is increasing, f is concave for $x < 0$, and f is convex for $x > 0$. Also, $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$.

Proof: *If $n \geq 2$ is even, $n-1 \geq 1$ is odd, and $n-2$ is even. Now $f'(x) = nx^{n-1}$ and $f''(x) = n(n-1)x^{n-2}$. Since $x^{n-2} > 0$ for all $x \neq 0$, $f''(x) > 0$ for $x \neq 0$ and so it is convex. Since $x < 0 \implies x^{n-1} < 0 \implies f'(x) < 0$, f is decreasing for $x < 0$. A similar argument shews that f is increasing for $x > 0$. It is clear that $f(-\infty) = f(+\infty) = +\infty$.*

If $n \geq 3$ is odd, $n-1 \geq 1$ is even, and $n-2$ is odd. Now $f'(x) = nx^{n-1}$ and $f''(x) = n(n-1)x^{n-2}$. Since $x^{n-1} > 0$ for all $x \neq 0$, $f'(x) > 0$ for $x \neq 0$ and so it is increasing. Since $x < 0 \implies x^{n-2} < 0 \implies f''(x) < 0$, f is concave for $x < 0$. A similar argument shews that f is convex for $x > 0$. It is clear that $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$. ❑

Homework

4.2.1 Problem Graph the following transformations of power functions.

1. $x \mapsto (x+1)^2$
2. $x \mapsto x^2 + 1$
3. $x \mapsto (x+1)^2 + 1$
4. $x \mapsto -x^3$
5. $x \mapsto 1 - x^3$
6. $x \mapsto 1 - (x-1)^3$

4.3 Integrals of Power Functions

Let $n \in \mathbb{N}$ and

$$F(x) = \frac{x^{n+1}}{n+1}.$$

Since

$$F'(x) = x^n,$$

F is a primitive of $f : x \mapsto x^n$ and so by the Second Form of the Fundamental Theorem of Calculus (Theorem 141), we deduce the following result.

149 Theorem Let $n \in \mathbb{N}$. Then

$$\int_a^b x^n dx = \int_a^b d\frac{x^{n+1}}{n+1} = \left.\frac{x^{n+1}}{n+1}\right|_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}.$$

Theorem 149 allows us to find areas of "curvy" regions.

150 Example Find the area of the plane region

$$\{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2, 0 \leq y \leq x^2\}.$$

Solution: ► *This is the area under the curve $y = x^2$ and above the x-axis, bounded by the vertical lines $x = 1$ and $x = 2$, as depicted in figure 4.9. By Theorem 149, this is*

$$\int_1^2 x^2 dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

◄

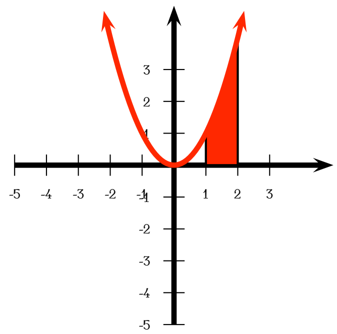


Figure 4.9: Example 150.

Homework

4.3.1 Problem Let $a \in \mathbb{R}$ be a constant and let $n \in \mathbb{N}$. Prove that if n is even then

$$\int_{-a}^{a} x^n dx = \frac{2}{n+1} a^{n+1}$$

and that if n is odd then

$$\int_{-a}^{a} x^n dx = 0.$$

4.4 Differentiating Polynomials

In this section we prove some standard differentiation rules.

151 Theorem (Constant-Times-Function Rule) If $k \in \mathbb{R}$ is a real number constant, and f is a strongly differentiable function at x, then kf is strongly differentiable at x and $(kf)'(x) = kf'(x)$. Alternatively, in differential notation,

$$dkf(x) = kdf(x).$$

Proof: *We have*

$$\begin{aligned} (kf)(x+\epsilon) &= k(f(x+\epsilon)) \\ &= k\left(f(x) + f'(x)\epsilon + o(\epsilon)\right) \\ &= kf(x) + kf'(x)\epsilon + o(\epsilon), \end{aligned}$$

from where the theorem follows. ❑

152 Example We have proved that $(x^2)' = 2x$. Hence $(-3x^2)' = -3(x^2)' = -3(2x) = -6x$.

153 Theorem (Sum Rule) If f, g are strongly differentiable functions at x, then $f + g$ is strongly differentiable at x and

$$(f + g)'(x) = f'(x) + g'(x).$$

Alternatively, in differential notation,

$$\mathrm{d}(f(x) + g(x)) = \mathrm{d}f(x) + \mathrm{d}g(x).$$

Proof: *We have*

$$\begin{aligned} f(x + \epsilon) + g(x + \epsilon) &= (f(x) + f'(x)\epsilon + o(\varepsilon)) + (g(x) + g'(x)\epsilon + o(\varepsilon)) \\ &= (f(x) + g(x)) + (f'(x) + g'(x))\epsilon + o(\varepsilon), \end{aligned}$$

from where the theorem follows. ❑

154 Example We have proved that $(x^2)' = 2x$ and that $(x)' = 1$. Hence $(x^2 + x)' = (x^2)' + (x)' = 2x + 1$.

155 Example Let $f(x) = 2x^3 - x^2 + 5x - 1$. Find $f'(1)$ and $f''(-1)$.

Solution: ► *We have*

$$f'(x) = 2(3x^2) - 2x + 5 = 6x^2 - 2x + 5, \qquad f''(x) = 2(6x) - 2 = 12x - 2.$$

Hence $f'(1) = 9$ *and* $f''(-1) = -14$. ◄

We now develop tools for differentiating more complex formulæ.

156 Theorem (Product Rule) If f, g are are strongly differentiable functions at x, then fg is strongly differentiable at x and then $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$. Alternatively, in differential notation,

$$\mathrm{d}f(x)g(x) = f(x)\mathrm{d}g(x) + g(x)\mathrm{d}f(x).$$

Proof: *We have*

$$\begin{aligned} f(x + \epsilon)g(x + \epsilon) &= (f(x) + f'(x)\epsilon + o(\varepsilon))\,(g(x) + g'(x)\epsilon + o(\varepsilon)) \\ &= f(x)g(x) + (f'(x)g(x) + f(x)g'(x))\,\epsilon + o(\varepsilon), \end{aligned}$$

from where the theorem follows. ❑

It is **not true** *in general that* $(fg)' = f'g'$.

157 Example Let $f(x) = x^3$ and $g(x) = x^4$. Then from the Product Rule

$$(x^7)' = (x^3 \cdot x^4)' = x^4(x^3)' + x^3(x^4)' = x^4(3x^2) + x^4(4x^2) = 3x^6 + 4x^6 = 7x^6,$$

which is what we expect from the Power Rule.

By recurrence we can apply the product rule to more than two functions.

158 Example

$$\begin{aligned}(x(x+x^2)(1+x+x^2))' &= (x)'(x+x^2)(1+x+x^2)+x(x+x^2)'(1+x+x^2)+x(x+x^2)(1+x+x^2)' \\ &= 1(x+x^2)(1+x+x^2)+x(1+2x)(1+x+x^2)+x(x+x^2)(1+2x) \\ &= (x+2x^2+2x^3+x^4)+(x+3x^2+3x^3+2x^4)+(x^2+3x^3+2x^4) \\ &= 2x+6x^2+8x^3+5x^4.\end{aligned}$$

159 Theorem (Chain Rule) If g is strongly differentiable at x and f is strongly differentiable at $g(x)$, then $f\circ g$ is strongly differentiable at x and $(f\circ g)'(x)=f'(g(x))\,g'(x)$. Alternatively, in differential notation,

$$\mathrm{d}(f(g(x)))=f'(g(x))\mathrm{d}g(x).$$

Proof: *We have, putting* $\epsilon_1=g'(x)\epsilon+o(\varepsilon)$,

$$\begin{aligned}f(g(x+\epsilon)) &= f(g(x)+g'(x)\epsilon+o(\varepsilon)) \\ &= f(g(x)+\epsilon_1) \\ &= f(g(x))+f'(g(x))\epsilon_1+O(\epsilon_1^2) \\ &= f(g(x))+f'(g(x))(g'(x)\epsilon+o(\varepsilon))+O((g'(x)\epsilon+o(\varepsilon))^2) \\ &= f(g(x))+f'(g(x))\,g'(x)\epsilon+o(\varepsilon)\end{aligned}$$

and the theorem follows. ❑

160 Example Consider $h(x)=(x+1)^2$. Then $h(x)=(f\circ g)(x)$ with $f(x)=x^2$ and $g(x)=x+1$. Hence

$$((x+1)^2)'=(f\circ g)'(x)=f'(g(x))g'(x)=2(x+1)^1(1)=2x+2.$$

161 Example

$$((x^2+x)^3)'=3(x^2+x)^2(2x+1).$$

162 Example Using the Product Rule and the Chain Rule,

$$(x(x+a)^2)'=(x+a)^2(x)'+x((x+a)^2)'=(x+a)^2+2x(x+a)=(x+a)(3x+a).$$

163 Example Using the Product Rule and the Chain Rule,

$$\begin{aligned}(x(x+a)^2(x+b)^3)' &= (x+a)^2(x+b)^3(x)'+x(x+b)^3((x+a)^2)'+x(x+a)^2((x+b)^3)' \\ &= (x+a)^2(x+b)^3+2x(x+a)(x+b)^3+3x(x+a)^2(x+b)^2 \\ &= (x+a)(x+b)^2((x+a)(x+b)+2x(x+b)+3x(x+a)) \\ &= (x+a)(x+b)^2(6x^2+x(4a+3b)+ab).\end{aligned}$$

164 Example Let f be strongly differentiable with $f(4)=a$ and $f'(4)=b$. If $g(x)=x^2f(x^2)$, find $g'(2)$.

Solution: ► *Using both the Product Rule and the Chain Rule*

$$g'(x) = 2xf(x^2) + x^2f'(x^2)(2x) = 2xf(x^2) + 2x^3f'(x^2).$$

Hence $g'(2) = 2(2)f(4) + 2(8)f'(4) = 4a + 16b$. ◄

In some situations we may apply the Fundamental Theorem of Calculus.

165 Example Find $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^{x^2} f(t)\mathrm{d}t$.

Solution: ► *Put* $F(x) = \displaystyle\int_0^{x} f(t)\mathrm{d}t$. *By Fundamental Theorem of Calculus,* $F'(x) = f(x)$. *By the Chain Rule,*

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^{x^2} f(t)\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x} F(x^2) = 2xF'(x^2) = 2xf(x^2).$$

◄

166 Example Let $g(x) = x^2 \displaystyle\int_0^{x^2} f(t)\mathrm{d}t$. Find $g'(x)$.

Solution: ► *We use the product rule and example 165*

$$\begin{aligned} g'(x) &= 2x\int_0^{x^2} f(t)\mathrm{d}t + x^2\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} f(t)\mathrm{d}t \\ &= 2x\int_0^{x^2} f(t)\mathrm{d}t + 2x^3f(x^2). \end{aligned}$$

◄

Homework

4.4.1 Problem Find the following derivatives.

1. $f'(x)$ if $f(x) = x(x-1)(x+1)$
2. $g'(x)$ if $g(x) = (ax + (bx + (cx + 1)^2)^3)^4$
3. $h'(x)$ if $h(x) = (x^3 + 2)^2(x^2 + 2x + 2)^3$.
4. $u'(x)$ if $u(x) = x(x-1)^2(x+2)^3$
5. $v'(x)$ if $v(x) = 1 - (1 - (1 - (1 - (1 - x^2)^2)^2)^2)^2$

4.4.2 Problem Find $a'(x)$ if $a(x) = (1+x)(1+x+x^2)(1+x+x^2+x^4)$

4.4.3 Problem Let p be the polynomial of degree 100

$$a_0x^{100} + a_1x^{99} + a_2x^{98} + \cdots + a_{99}x + a_{100} = (x-2)^{50}(2x-1)^{50}.$$

Find $100a_0 + 99a_1 + 98a_2 + \cdots + 2a_{98} + a_{99}$.

4.4.4 Problem Let f be an everywhere differentiable function and let $g = f \circ f$. It is known that

$$f(1) = 2; \quad f(2) = 3; \quad f'(1) = 1; \quad f'(2) = 1, \quad f''(1) = -1, \quad f''(2) = -1.$$

Find $g'(1)$ and $g''(1)$.

4.4.5 Problem Consider an *even* function $f:\mathbb{R}\to\mathbb{R}$ whose derivative f' exists everywhere and which satisfies

$$f(1)=2;\quad f(-2)=1;\quad f'(-1)=1;\quad f'(2)=2,\quad f''(-1)=0;\quad f''(2)=3.$$

If $a(x)=xf(x^2)$, find $a''(-1)$.

4.4.6 Problem Determine $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_x^{x^2}\sqrt{1+t^2}\mathrm{d}t$.

4.5 Integrating Polynomials

Each of the differentiation rules given in the preceding section has a corresponding integration equivalent.

We recall here that the Second Form of the Fundamental Theorem of Calculus asserts that if f is continuous in $[a\,;b]$ and $F'=f$, then

$$\int_a^b f(x)\mathrm{d}x=\int_a^b \mathrm{d}F(x)=F(b)-F(a). \tag{4.2}$$

Most of the integration problems we will see in this course can be solved by reducing them to the form in 4.2. The trick is to recognise the integrand as the derivative of some function, most times perhaps, as the result of a differentiation through the Chain Rule.

167 Example Let $f:[1\,;2]\to\mathbb{R}$ be such that f' is continuous and suppose $f(2)=a$ and $f(5)=b$. Determine $\displaystyle\int_1^2 xf'(x^2+1)\mathrm{d}x$.

Solution: ► *We recognise the integrand* $xf'(x^2+1)$ *as de derivative of* $\dfrac{f(x^2+1)}{2}$. *Hence*

$$\int_1^2 xf'(x^2+1)\mathrm{d}x=\int_1^2 \mathrm{d}\left(\frac{f(x^2+1)}{2}\right)=\left.\frac{f(x^2+1)}{2}\right|_1^2=\frac{f(5)}{2}-\frac{f(2)}{2}=\frac{b-a}{2}.$$

◄

168 Example We have

$$\int_0^1(2x+1)(x^2+1)^{2008}\mathrm{d}x=\int_0^1\mathrm{d}\left(\frac{(x^2+1)^{2009}}{2009}\right)=\left.\frac{(x^2+1)^{2009}}{2009}\right|_0^1=\frac{2^{2009}}{2009}-\frac{1^{2009}}{2009}=\frac{2^{2009}-1}{2009}.$$

169 Example We have

$$\begin{aligned}\int_0^t(2008x^{2007}+1004x^{1003}+502x^{501})(x^{2008}+x^{1004}+x^{502})^{10}\mathrm{d}x &= \int_0^t\mathrm{d}\left(\frac{(x^{2008}+x^{1004}+x^{502})^{11}}{11}\right)\\ &= \left.\frac{(x^{2008}+x^{1004}+x^{502})^{11}}{11}\right|_0^t\\ &= \frac{(t^{2008}+t^{1004}+t^{502})^{11}}{11}.\end{aligned}$$

170 Theorem (Product of Function Times a Constant Rule) Let f be continuous on $[a\,;b]$ and let $k\in\mathbb{R}$ be a constant. Then

$$\int_a^b(kf(x))\mathrm{d}x=k\int_a^b f(x)\mathrm{d}x.$$

Proof: *Let* $\mathrm{d}F(x) = f(x)\mathrm{d}x$ *and* $\mathrm{d}G(x) = kf(x)\mathrm{d}x$. *By Theorem 151,*

$$\mathrm{d}G(x) = \mathrm{d}(kF(x)).$$

Now, by the Second Form of the Fundamental Theorem of Calculus

$$\int_a^b kf(x)\mathrm{d}x = \int_a^b \mathrm{d}G(x) = \int_a^b \mathrm{d}(kF(x)) = kF(b) - kF(a) = k(F(b) - F(a)) = k\int_a^b f(x)\mathrm{d}x.$$

❑

171 Theorem (Integral of a Sum Rule) Let f and g be continuous on $[a\,;b]$ Then

$$\int_a^b (f(x) + g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x.$$

Proof: *Let* $\mathrm{d}F(x) = f(x)\mathrm{d}x$ *and* $\mathrm{d}G(x) = g(x)\mathrm{d}x$. *Now, by the Second Form of the Fundamental Theorem of Calculus*

$$\begin{aligned}
\int_a^b (f(x) + g(x))\mathrm{d}x &= \int_a^b \mathrm{d}(F(x) + G(x)) \\
&= (F(x) + G(x))\Big|_a^b \\
&= (F(b) + G(b)) - (F(a) + G(a)) \\
&= F(b) - F(a) + G(b) - G(a) \\
&= \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x.
\end{aligned}$$

❑

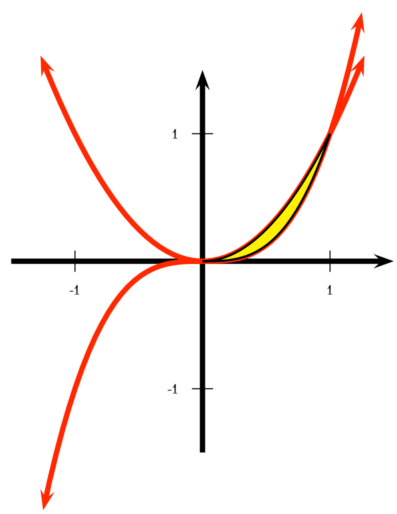


Figure 4.10: Example 172.

172 Example Find the area of the plane region between the curves $y = x^2$ and $y = x^3$.

Solution: ► *Observe that* $x^2 = x^3 \implies x^2(1 - x) = 0 \implies x \in \{0, 1\}$, *and so the curves meet at* $x = 0$ *and* $x = 1$. *In the interval* $0 \le x \le 1$, $y = x^2$ *lies atop of* $y = x^3$, *as depicted in figure*

4.10. By Theorems 149 and 171, this is

$$\begin{aligned}\int_0^1 (x^2 - x^3)dx &= \int_0^1 x^2dx - \int_0^1 x^3dx \\ &= \int_0^1 d\frac{x^3}{3} - \int_0^1 d\frac{x^4}{4} \\ &= \frac{x^3}{3}\Big|_0^1 - \frac{x^4}{4}\Big|_0^1 \\ &= \frac{1}{3} - \frac{1}{4} \\ &= \frac{1}{12}.\end{aligned}$$

◀

We will know demonstrate one of the most powerful integration techniques, that of Integration by Parts.

173 Theorem (Integration by Parts) Let f' and g' be continuous on $[a\ ;b]$ Then

$$\int_a^b f(x)dg(x) = f(x)g(x)\Big|_a^b - \int_a^b g(x)df(x).$$

Proof: *By the Product Rule for derivatives,*

$$d(f(x)g(x)) = f(x)dg(x) + g(x)df(x).$$

Integrating on both sides and using the Sum Rule for Integrals (Theorem 171),

$$\begin{aligned}d(f(x)g(x)) = f(x)dg(x) + g(x)df(x) &\implies \int_a^b d(f(x)g(x)) = \int_a^b f(x)dg(x) + \int_a^b g(x)df(x) \\ &\implies f(x)g(x)\Big|_a^b = \int_a^b f(x)dg(x) + \int_a^b g(x)df(x),\end{aligned}$$

which readily gives the result.

❑

174 Example Let f be a continuously differentiable function such that $f(0) = 0$, $f(1) = 1$, and $\int_0^1 f(x)dx = -\frac{1}{2}$. Determine $\int_0^1 xf'(x)dx$.

Solution: ► *Integrating by parts,*

$$\int_0^1 xf'(x)dx = \int_0^1 xdf(x) = xf(x)\Big|_0^1 - \int_0^1 f(x)dx = f(1) - \int_0^1 f(x)dx = 1 - \left(-\frac{1}{2}\right) = \frac{3}{2}.$$

◀

175 Theorem (Integration by Substitution Rule) Let f' and g' be continuous on $[a\ ;b]$ Then

$$\int_a^b f(g(x))dg(x) = \int_{g(a)}^{g(b)} f(x)dx.$$

Proof: *Let $P'(x) = f(g(x))g'(x)dx =$ and $G'(x) = kf(x)$. Then*

$$kF'(x) = G'(x)$$

❑

Homework

4.5.1 Problem Evaluate $\int_0^a (x+1)^{100}\,\mathrm{d}x$, where a is a real constant.

4.5.2 Problem Let $f : [0\,;4] \to \mathbb{R}$ be continuous. If $\int_0^4 f(x)\,\mathrm{d}x = 9$, find $\int_0^2 xf(x^2)\,\mathrm{d}x$.

4.5.3 Problem Consider a continuously twice differentiable function f such that $f(0) = a$, $f(1) = b$, $\int_0^1 f(x)\,\mathrm{d}x = c$, $f'(0) = 2b$, and $f'(1) = 2a$. Find $\int_0^1 xf'(x)\,\mathrm{d}x$ and $\int_0^1 xf''(x)\,\mathrm{d}x$.

4.6 Affine Functions

176 Definition Let m, k be real number constants. A function of the form $x \mapsto mx + k$ is called an *affine function*. In the particular case that $m = 0$, we call $x \mapsto k$ a *constant function*. If, however, $k = 0$, then we call the function $x \mapsto mx$ a *linear function*.

By virtue of Theorem 56, the graph of the function $x \mapsto mx + k$ is a straight line. Since the derivative of $x \mapsto mx + k$ is $(mx + k)' = m$, we see that $x \mapsto mx + k$ is strictly increasing if $m > 0$ and strictly decreasing if $m < 0$ in view of Theorem 124. If $m \neq 0$ then $mx + k = 0 \implies x = -\frac{k}{m}$, meaning that $x \mapsto mx + k$ has a unique zero (crosses the x-axis) at $x = -\frac{k}{m}$. This information is summarised in the following tables.

x	$-\infty$		$-\frac{k}{m}$		$+\infty$
$f(x) = mx + k$		$\nearrow$	0	$\nearrow$	

Table 4.1: $x \mapsto mx + k$, with $m > 0$.

x	$-\infty$		$-\frac{k}{m}$		$+\infty$
$f(x) = mx + k$		$\searrow$	0	$\searrow$	

Table 4.2: $x \mapsto mx + k$, with $m < 0$.

4.7 Quadratic Functions

177 Definition Let a, b, c be real numbers, with $a \neq 0$. A function of the form

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & ax^2 + bx + c \end{array}$$

is called a *quadratic function*.

178 Theorem Let $a \neq 0, b, c$ be real numbers and let $x \mapsto ax^2 + bx + c$ be a quadratic function. Then its graph is a parabola. If $a > 0$ the parabola has a local minimum at $x = -\dfrac{b}{2a}$ and it is convex. If $a < 0$ the parabola has a local maximum at $x = -\dfrac{b}{2a}$ and it is concave.

Proof: *Put* $f(x) = ax^2 + bx + c$. *Completing squares,*

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}, \end{aligned}$$

and hence this is a horizontal translation $-\dfrac{b}{2a}$ *units and a vertical translation* $\dfrac{4ac-b^2}{4a}$ *units of the square function* $x \mapsto x^2$ *and so it follows from example* **??** *and Theorems 81 and 87, that the graph of* f *is a parabola.*

We have $f'(x) = 2ax + b$. *Assume first that* $a > 0$. *Then*

$$f'(x) > 0 \iff x > -\frac{b}{2a}, \qquad f'(x) < 0 \iff x < -\frac{b}{2a}.$$

Thus the function decreases for values $< -\dfrac{b}{2a}$ *and increases otherwise. Hence in view of Theorem 127, it must have a minimum at* $x = -\dfrac{b}{2a}$. *Since* $f''(x) = 2a > 0$, f *is convex by virtue of Theorem 129. The case when* $a < 0$ *can be similarly treated.* ❑

The information of Theorem 178 is summarised in the following tables.

x	$-\infty$	$-\frac{b}{2a}$	$+\infty$
$f(x) = ax^2 + bx + c$	$\searrow$	0	$\nearrow$

Table 4.3: $x \mapsto ax^2 + bx + c$, with $a > 0$.

x	$-\infty$	$-\frac{b}{2a}$	$+\infty$
$f(x) = ax^2 + bx + c$	$\nearrow$	0	$\searrow$

Table 4.4: $x \mapsto ax^2 + bx + c$, with $a < 0$.

179 Definition The point $\left(-\dfrac{b}{2a}, \dfrac{4ac-b^2}{4a}\right)$ lies on the parabola and it is called the *vertex* of the parabola $y = ax^2 + bx + c$. The quantity $b^2 - 4ac$ is called the *discriminant* of $ax^2 + bx + c$. The equation

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

is called the *canonical equation of the parabola* $y = ax^2 + bx + c$.

The parabola $x \mapsto ax^2 + bx + c$ is symmetric about the vertical line $x = -\dfrac{b}{2a}$ passing through its vertex.

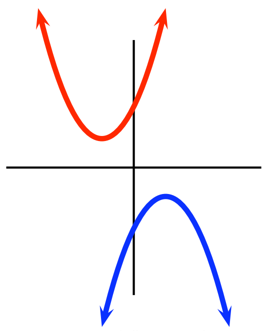


Figure 4.11: No real zeroes.

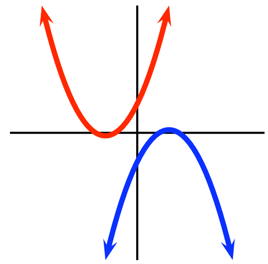


Figure 4.12: One real zero.

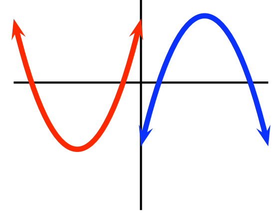


Figure 4.13: Two real zeros.

180 Corollary (Quadratic Formula) The roots of the equation $ax^2 + bx + c = 0$ are given by the formula

$$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{4.3}$$

If $a \neq 0, b, c$ are real numbers and $b^2 - 4ac = 0$, the parabola $x \mapsto ax^2 + bx + c$ is tangent to the x-axis and has one (repeated) real root. If $b^2 - 4ac > 0$ then the parabola has two distinct real roots. Finally, if $b^2 - 4ac < 0$ the parabola has two complex roots.

Proof: *By Theorem 178 we have*

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a},$$

and so

$$\begin{aligned} ax^2 + bx + c = 0 \quad &\iff \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \\ &\iff \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2|a|} \\ &\iff \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \end{aligned}$$

where we have dropped the absolute values on the last line because the only effect of having $a < 0$ is to change from $\pm$ to $\mp$.

If $b^2 - 4ac = 0$ then the vertex of the parabola is at $\left(-\dfrac{b}{2a}, 0\right)$ on the x-axis, and so the parabola is tangent there. Also, $x = -\dfrac{b}{2a}$ would be the only root of this equation. This is illustrated in figure 4.12.

If $b^2 - 4ac > 0$, then $\sqrt{b^2 - 4ac}$ is a real number $\neq 0$ and so $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ are distinct numbers. This is illustrated in figure 4.13.

If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is a complex number $\neq 0$ and so $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ are distinct complex numbers. This is illustrated in figure 4.11. ❑

If a quadratic has real roots, then the vertex lies on a line crossing the midpoint between the roots.

181 Example Consider the quadratic function $f(x) = x^2 - 5x + 3$.

❶ Find $f'(x)$. Solve $f'(x) = 0$ and hence find the vertex of f. Determine the intervals of monotonicity of f.

❷ Write this parabola in canonical form.

❸ Determine $f''(x)$ and find the convexity intervals of f.

❹ Find the x-intercepts and y-intercepts of f.

❺ Graph $y = f(x)$, $y = |f(x)|$, and $y = f(|x|)$.

❻ Determine the set of real numbers x for which $f(x) > 0$.

Solution:

❶ We have $f'(x) = 2x - 5$. Now, $2x - 5 = 0 \implies x = \frac{5}{2}$. At $x = \frac{5}{2}$ we have $f\left(\frac{5}{2}\right) = -\frac{13}{4}$, whence the vertex is at $\left(\frac{5}{2}, -\frac{13}{4}\right)$. Also,

$$f'(x) > 0 \implies 2x - 5 > 0 \implies x > \frac{5}{2},$$

and f will be increasing for $x > \frac{5}{2}$. It will be decreasing for $x < \frac{5}{2}$.

❷ Completing squares

$$y = x^2 - 5x + 3 = \left(x - \frac{5}{2}\right)^2 - \frac{13}{4}.$$

❸ We have $f''(x) = (2x)' = 2$. Since $f''(x) = 2 > 0$ for all real values x, f is concave for all real values of x.

❹ For $x = 0$, $f(0) = 0^2 - 5 \cdot 0 + 3 = 3$, and hence $y = f(0) = 3$ is the y-intercept. By the quadratic formula,

$$f(x) = 0 \iff x^2 - 5x + 3 = 0 \iff x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)} = \frac{5 \pm \sqrt{13}}{2}.$$

Observe that $\frac{5 - \sqrt{13}}{2} \approx 0.697224362$ and $\frac{5 + \sqrt{13}}{2} \approx 4.302775638$.

❺ The graphs appear in figures 4.14 through 4.16.

❻ From the graph in figure 4.14, $x^2 - 5x + 3 > 0$ for values $x \in \left]-\infty\,;\frac{5 - \sqrt{13}}{2}\right[$ or $x \in \left]\frac{5 + \sqrt{13}}{2}\,;+\infty\right[$.

182 Corollary If $a \neq 0, b, c$ are real numbers and if $b^2 - 4ac < 0$, then $ax^2 + bx + c$ has the same sign as a.

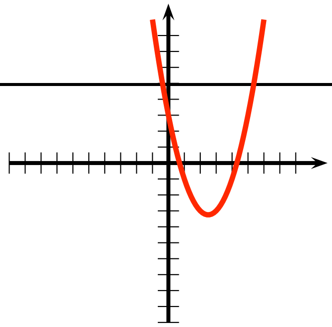


Figure 4.14: $y = x^2 - 5x + 3$

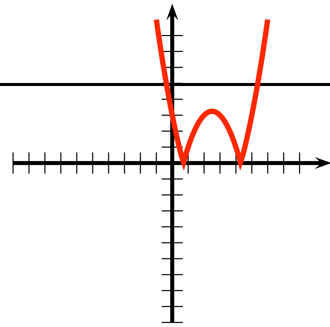


Figure 4.15: $y = |x^2 - 5x + 3|$

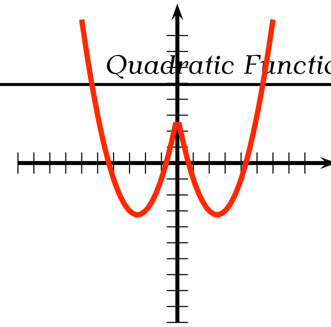


Figure 4.16: $y = |x|^2 - 5|x| + 3$

Proof: *Since*

$$ax^2 + bx + c = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right),$$

and $4ac - b^2 > 0$, $\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right) > 0$ *and so* $ax^2 + bx + c$ *has the same sign as* a.

❑

183 Example Prove that the quantity $q(x) = 2x^2 + x + 1$ is positive regardless of the value of x.

Solution: The discriminant is $1^2 - 4(2)(1) = -7 < 0$, hence the roots are complex. By Corollary 182, since its leading coefficient is $2 > 0$, $q(x) > 0$ regardless of the value of x. Another way of seeing this is to complete squares and notice the inequality

$$2x^2 + x + 1 = 2\left(x + \frac{1}{4}\right)^2 + \frac{7}{8} \geq \frac{7}{8},$$

since $\left(x + \frac{1}{4}\right)^2$ being the square of a real number, is ≥ 0.

By Corollary 180, if $a \neq 0, b, c$ are real numbers and if $b^2 - 4ac \neq 0$ then the numbers

$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

are distinct solutions of the equation $ax^2 + bx + c = 0$. Since

$$r_1 + r_2 = -\frac{b}{a}, \quad \text{and} \quad r_1 r_2 = \frac{c}{a},$$

any quadratic can be written in the form

$$ax^2 + bx + c = a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right) = a\left(x^2 - (r_1 + r_2)x + r_1 r_2\right) = a(x - r_1)(x - r_2).$$

We call $a(x - r_1)(x - r_2)$ a *factorisation* of the quadratic $ax^2 + bx + c$.

184 Example A quadratic polynomial p has $1 \pm \sqrt{5}$ as roots and it satisfies $p(1) = 2$. Find its equation.

Solution: Observe that the sum of the roots is

$$r_1 + r_2 = 1 - \sqrt{5} + 1 + \sqrt{5} = 2$$

and the product of the roots is

$$r_1 r_2 = (1-\sqrt{5})(1+\sqrt{5}) = 1-(\sqrt{5})^2 = 1-5 = -4.$$[1]

Hence p has the form

$$p(x) = a\left(x^2-(r_1+r_2)x+r_1r_2\right) = a(x^2-2x-4).$$

Since

$$2 = p(1) \implies 2 = a(1^2-2(1)-4) \implies a = -\frac{2}{5},$$

the polynomial sought is

$$p(x) = -\frac{2}{5}\left(x^2-2x-4\right).$$

4.8 Polynomials

4.8.1 Roots

In sections 4.6 and 4.7 we learned how to find the roots of equations (in the unknown x) of the type $ax+b=0$ and $ax^2+bx+c=0$, respectively. We would like to see what can be done for equations where the power of x is higher than 2. We recall that

185 Definition A *polynomial* $p(x)$ of degree $n \in \mathbb{N}$ is an expression of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0, \qquad a_k \in \mathbb{R},$$

where the a_k are constants. If the a_k are all integers then we say that p has integer coefficients, and we write $p(x) \in \mathbb{Z}[x]$; if the a_k are real numbers then we say that p has real coefficients and we write $p(x) \in \mathbb{R}[x]$; etc. The degree of the polynomial p is denoted by $\deg p$. The coefficient a_n is called the *leading coefficient* of $p(x)$. A *root* of p is a solution to the equation $p(x) = 0$.

186 Example Here are a few examples of polynomials.

- $a(x) = 2x+1 \in \mathbb{Z}[x]$, is a polynomial of degree 1, and leading coefficient 2. It has $x = -\frac{1}{2}$ as its only root. A polynomial of degree 1 is also known as an *affine function*.
- $b(x) = \pi x^2 + x - \sqrt{3} \in \mathbb{R}[x]$, is a polynomial of degree 2 and leading coefficient π. By the quadratic formula b has the two roots

  $$x = \frac{-1+\sqrt{1+4\pi\sqrt{3}}}{2\pi} \qquad \text{and} \qquad x = \frac{-1-\sqrt{1+4\pi\sqrt{3}}}{2\pi}.$$

  A polynomial of degree 2 is also called a *quadratic polynomial* or *quadratic function*.
- $C(x) = 1 \equiv 1 \cdot x^0$[2], is a constant polynomial, of degree 0. It has no roots, since it is never zero.

187 Theorem The degree of the product of two polynomials is the sum of their degrees. In symbols, if p, q are polynomials, $\deg pq = \deg p + \deg q$.

Proof: *If* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, *and* $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, *with* $a_n \neq 0$ *and* $b_m \neq 0$ *then upon multiplication,*

$$p(x)q(x) = (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)(b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0) = a_n b_m x^{m+n} + \cdots +,$$

with non-vanishing leading coefficient $a_n b_m$. ❑

[1] As a shortcut for this multiplication you may wish to recall the *difference of squares identity*: $(a-b)(a+b) = a^2-b^2$.

[2] The symbol $\equiv$ is read "identically equal to" and it means that both expressions are always the same, regardless of the value of the input parameter.

188 Example The polynomial $p(x) = (1 + 2x + 3x^3)^4(1 - 2x^2)^5$ has leading coefficient $3^4(-2)^5 = -2592$ and degree $3 \cdot 4 + 2 \cdot 5 = 22$.

189 Example What is the degree of the polynomial identically equal to 0? Put $p(x) \equiv 0$ and, say, $q(x) = x + 1$. Then by Theorem 187 we must have $\deg pq = \deg p + \deg q = \deg p + 1$. But pq is identically 0, and hence $\deg pq = \deg p$. But if $\deg p$ were finite then

$$\deg p = \deg pq = \deg p + 1 \implies 0 = 1^{3},$$

nonsense. Thus the 0-polynomial does not have any finite degree. We attach to it, by convention, degree $-\infty$.

190 Definition If all the roots of a polynomial are in $\mathbb{Z}$ (integer roots), then we say that the *polynomial splits or factors over* $\mathbb{Z}$. If all the roots of a polynomial are in $\mathbb{Q}$ (rational roots), then we say that the *polynomial splits or factors over* $\mathbb{Q}$. If all the roots of a polynomial are in $\mathbb{C}$ (complex roots), then we say that the *polynomial splits (factors) over* $\mathbb{C}$.

STOP *Since $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, any polynomial splitting on a smaller set immediately splits over a larger set.*

191 Example The polynomial $l(x) = x^2 - 1 = (x - 1)(x + 1)$ splits over $\mathbb{Z}$. The polynomial $p(x) = 4x^2 - 1 = (2x - 1)(2x + 1)$ splits over $\mathbb{Q}$ but not over $\mathbb{Z}$. The polynomial $q(x) = x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$ splits over $\mathbb{R}$ but not over $\mathbb{Q}$. The polynomial $r(x) = x^2 + 1 = (x - i)(x + i)$ splits over $\mathbb{C}$ but not over $\mathbb{R}$. Here $i = \sqrt{-1}$ is the imaginary unit.

4.8.2 Taylor Polynomials

In order to motivate the following theorem, let us consider the next example.

192 Example Write x^2 as a sum of powers of $x - 1$.

Solution: Observe that $x = x - 1 + 1$ and use the identity $(a + b)^2 = a^2 + 2ab + b^2$ to obtain

$$x^2 = (x - 1 + 1)^2 = (x - 1)^2 + 2(x - 1) + 1.$$

If such an identity is not known, one can proceed as follows, giving an idea of a general procedure. Put

$$x^2 = a + b(x - 1) + c(x - 1)^2,$$

where we stop at the second power since x^2 has degree 2. Let $x = 1$. Then $1 = a$. Differentiate to obtain

$$2x = b + 2c(x - 1).$$

Let again $x = 1$. This gives $2 = b$. Differentiate a second time to obtain

$$2 = 2c,$$

whence $c = 1$. Hence we have $a = 1, b = 2, c = 1$ and so

$$x^2 = a + b(x - 1) + c(x - 1)^2 = 1 + 2(x - 1) + (x - 1)^2,$$

as before.

[3] Much to the chagrin of our Vice-President for Academic Affairs—who claims that $1 = 2$—it is not true that $0 = 1$.

193 Theorem (Taylor Polynomials) Let $a \in \mathbb{R}$. Then any polynomial $p(x)$ of degree n can be written as

$$p(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n,$$

for some constants b_k.

Proof: *First observe that we stop at $(x-a)^n$ since p has degree n. Differentiating k times we obtain*

$$p^{(k)}(x) = b_k k! \quad + \quad (k+1)(k)\cdots(2)b_{k+1}(x-a) \quad + \quad \cdots \quad + \quad (n)(n-1)\cdots(n-k+1)b_n(x-a)^{n-k}.$$ [4]

Letting $x = a$ we obtain

$$b_k = \frac{p^{(k)}(a)}{k!},$$

proving the theorem. ❑

194 Definition The expansion

$$p(x) = p(a) + p'(a)(x-a) + \frac{p''(a)}{2!}(x-a)^2 + \cdots + \frac{p^{(n)}(a)}{n!}(x-a)^n \tag{4.4}$$

is known as the *Taylor polynomial expansion* about $x = a$ of p.

195 Example Find the Taylor polynomial expansion about $x = -2$ of $p(x) = x^3 + 2x + 1$.

Solution: We have

$$p'(x) = 3x^2 + 2, \qquad p''(x) = 6x, \qquad p'''(x) = 6.$$

Hence

$$p(-2) = -11, \qquad p'(-2) = 14, \qquad p''(-2) = -12, \qquad p'''(-2) = 6,$$

and

$$x^3 + 2x + 1 = -11 + 14(x+2) + \frac{-12}{2}(x+2)^2 + \frac{6}{6}(x+2)^3 = -11 + 14(x+2) - 6(x+2)^2 + (x+2)^3.$$

4.8.3 Ruffini's Factor Theorem

196 Theorem (Ruffini's Factor Theorem) The polynomial $p(x)$ is divisible by $x - a$ if and only if $p(a) = 0$. Thus if p is a polynomial of degree n, then $p(a) = 0$ if and only if $p(x) = (x-a)q(x)$ for some polynomial q of degree $n - 1$.

Proof: *The Taylor expansion of p about $x = a$ is*

$$p(x) = p(a) + (x-a)\left(p'(a) + \frac{p''(a)}{2!}(x-a) + \cdots + \frac{p^{(n)}(a)}{n!}(x-a)^{n-1}\right),$$

from where the result quickly follows. ❑

197 Example Find the value of a so that the polynomial

$$t(x) = x^3 - 3ax^2 + 2$$

be divisible by $x + 1$.

[4]The symbol $k!$—read "k factorial"—is the product $1\cdot 2\cdots k$. Thus for example $5! = 1\cdot2\cdot3\cdot4\cdot5 = 120$. We define $0! = 1$.

Solution: By Ruffini's Theorem 196, we must have

$$0 = t(-1) = (-1)^3 - 3a(-1)^2 + 2 \implies a = \frac{1}{3}.$$

198 Definition Let a be a root of a polynomial p. We say that a is a root of *multiplicity* m if $p(x)$ is divisible by $(x-a)^m$ but not by $(x-a)^{m+1}$. This means that p can be written in the form $p(x) = (x-a)^m q(x)$ for some polynomial q with $q(a) \neq 0$.

199 Corollary The number a is a root of multiplicity m if an only if

$$p(a) = p'(a) = p''(a) = \cdots = p^{(m-1)}(a) = 0, \qquad p^{(m)}(a) \neq 0.$$

> **Proof:** *This follows immediately by considering the Taylor expansion of p about $x = a$.* ❑

200 Example Factor the polynomial $p(x) = x^5 - 5x^4 + 11x^3 - 13x^2 + 8x - 2$ over $\mathbb{Z}[x]$.

Solution: We see that $p(1) = 0, p'(1) = 0, p''(1) = 0, p'''(1) \neq 0$. Hence $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ divides p. By long division

$$\begin{array}{r|rrrrrr}
 & & & & x^2 & -2x & +2 \\ \hline
x^3 - 3x^2 + 3x - 1 & x^5 & -5x^4 & +11x^3 & -13x^2 & +8x & -2 \\
 & -x^5 & +3x^4 & -3x^3 & +x^2 & & \\ \cline{2-5}
 & & -2x^4 & +8x^3 & -12x^2 & +8x & \\
 & & 2x^4 & -6x^3 & +6x^2 & -2x & \\ \cline{3-6}
 & & & 2x^3 & -6x^2 & +6x & -2 \\
 & & & -2x^3 & +6x^2 & -6x & +2 \\ \cline{4-7}
 & & & & & & 0
\end{array}$$

and so

$$x^5 - 5x^4 + 11x^3 - 13x^2 + 8x - 2 = (x-1)^3(x^2 - 2x + 2).$$

Observe that $x^2 - 2x + 2$ does not factor over $\mathbb{Z}[x]$ and hence we are finished.

201 Corollary If a polynomial of degree n had any roots at all, then it has at most n roots.

> **Proof:** *If it had at least $n+1$ roots then it would have at least $n+1$ factors of degree 1 and hence degree $n+1$ at least, a contradiction.* ❑

Notice that the above theorem only says that if a polynomial has any roots, then it must have at most its degree number of roots. It does not say that a polynomial must possess a root. That all polynomials have at least one root is much more difficult to prove. We will quote the theorem, without a proof.

202 Theorem (Fundamental Theorem of Algebra) A polynomial of degree at least one with complex number coefficients has at least one complex root.

> STOP *The Fundamental Theorem of Algebra implies then that a polynomial of degree n has exactly n roots (counting multiplicity).*

A more useful form of Ruffini's Theorem is given in the following corollary.

203 Corollary If the polynomial p with integer coefficients,

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

has a rational root $\frac{s}{t} \in \mathbb{Q}$ (here $\frac{s}{t}$ is assumed to be in lowest terms), then s divides a_0 and t divides a_n.

Proof: *We are given that*

$$0 = p\left(\frac{s}{t}\right) = a_n\left(\frac{s^n}{t^n}\right) + a_{n-1}\left(\frac{s^{n-1}}{t^{n-1}}\right) + \cdots + a_1\left(\frac{s}{t}\right) + a_0.$$

Clearing denominators,

$$0 = a_n s^n + a_{n-1}s^{n-1}t + \cdots + a_1 st^{n-1} + a_0 t^n.$$

This last equality implies that

$$-a_0 t^n = s(a_n s^{n-1} + a_{n-1}s^{n-2}t + \cdots + a_1 t^{n-1}).$$

Since both sides are integers, and since s and t have no factors in common, then s must divide a_0. We also gather that

$$-a_n s^n = t(a_{n-1}s^{n-1} + \cdots + a_1 st^{n-2} + a_0 t^{n-1}),$$

from where we deduce that t divides a_n, concluding the proof. ❑

204 Example Factorise $a(x) = x^3 - 3x - 5x^2 + 15$ over $\mathbb{Z}[x]$ and over $\mathbb{R}[x]$.

Solution: By Corollary 203, if $a(x)$ has integer roots then they must be in the set $\{-1, 1, -3, 3, -5, 5\}$. We test $a(\pm 1), a(\pm 3), a(\pm 5)$ to see which ones vanish. We find that $a(5) = 0$. By the Factor Theorem, $x - 5$ divides $a(x)$. Using long division,

$$\begin{array}{r|l} & x^2 \qquad\quad -3 \\ \hline x-5 & x^3 - 5x^2 - 3x + 15 \\ & -x^3 + 5x^2 \\ \cline{2-2} & \qquad\qquad -3x + 15 \\ & \qquad\qquad 3x - 15 \\ \cline{2-2} & \qquad\qquad\qquad 0 \end{array}$$

we find

$$a(x) = x^3 - 3x - 5x^2 + 15 = (x-5)(x^2 - 3),$$

which is the required factorisation over $\mathbb{Z}[x]$. The factorisation over $\mathbb{R}[x]$ is then

$$a(x) = x^3 - 3x - 5x^2 + 15 = (x-5)(x - \sqrt{3})(x + \sqrt{3}).$$

205 Example Factorise $b(x) = x^5 - x^4 - 4x + 4$ over $\mathbb{Z}[x]$ and over $\mathbb{R}[x]$.

Solution: By Corollary 203, if $b(x)$ has integer roots then they must be in the set $\{-1, 1, -2, 2, -4, 4\}$. We quickly see that $b(1) = 0$, and so, by the Factor Theorem, $x - 1$ divides $b(x)$. By long division

$$\begin{array}{r|l} & x^4 \qquad\quad -4 \\ \hline x-1 & x^5 - x^4 - 4x + 4 \\ & -x^5 + x^4 \\ \cline{2-2} & \qquad\qquad -4x + 4 \\ & \qquad\qquad 4x - 4 \\ \cline{2-2} & \qquad\qquad\qquad 0 \end{array}$$

we see that

$$b(x) = (x-1)(x^4-4) = (x-1)(x^2-2)(x^2+2),$$

which is the desired factorisation over $\mathbb{Z}[x]$. The factorisation over $\mathbb{R}$ is seen to be

$$b(x) = (x-1)(x-\sqrt{2})(x+\sqrt{2})(x^2+2).$$

Since the discriminant of x^2+2 is $-8 < 0$, x^2+2 does not split over $\mathbb{R}$.

206 Lemma Complex roots of a polynomial with real coefficients occur in conjugate pairs, that is, if p is a polynomial with real coefficients and if $u+vi$ is a root of p, then its conjugate $u-vi$ is also a root for p. Here $i = \sqrt{-1}$ is the imaginary unit.

Proof: *Assume*

$$p(x) = a_0 + a_1x + \cdots + a_nx^n$$

and that $p(u+vi) = 0$. *Since the conjugate of a real number is itself, and conjugation is multiplicative (Theorem ??), we have*

$$\begin{aligned} 0 &= \overline{0} \\ &= \overline{p(u+vi)} \\ &= \overline{a_0 + a_1(u+vi) + \cdots + a_n(u+vi)^n} \\ &= \overline{a_0} + \overline{a_1(u+vi)} + \cdots + \overline{a_n(u+vi)^n} \\ &= a_0 + a_1(u-vi) + \cdots + a_n(u-vi)^n \\ &= p(u-vi), \end{aligned}$$

whence $u-vi$ *is also a root.* ❑

Since the complex pair root $u \pm vi$ would give the polynomial with real coefficients

$$(x-u-vi)(x-u+vi) = x^2 - 2ux + (u^2+v^2),$$

we deduce the following theorem.

207 Theorem Any polynomial with real coefficients can be factored in the form

$$A(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x-r_k)^{m_k}(x^2+a_1x+b_1)^{n_1}(x^2+a_2x+b_2)^{n_2}\cdots(x^2+a_lx+b_l)^{n_l},$$

where each factor is distinct, the m_i, l_k are positive integers and A, r_i, a_i, b_i are real numbers.

4.9 Graphs of Polynomials

We start with the following theorem, which we will state without proof.

208 Theorem A polynomial function $x \mapsto p(x)$ is an everywhere continuous function.

209 Theorem Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad a_n \neq 0$, be a polynomial with real number coefficients. Then

$$p(-\infty) = (\mathbf{signum}\,(a_n))(-1)^n\infty, \qquad p(+\infty) = (\mathbf{signum}\,(a_n))\infty.$$

Thus a polynomial of odd degree will have opposite signs for values of large magnitude and different sign, and a polynomial of even degree will have the same sign for values of large magnitude and different sign.

Proof: *If $x \neq 0$ then*

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = a_nx^n\left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \sim a_nx^n,$$

since as $x \to \pm\infty$, the quantity in parenthesis tends to 1 and so the eventual sign of $p(x)$ is determined by a_nx^n, which gives the result. ❑

210 Corollary A polynomial of odd degree with real number coefficients always has a real root.

Proof: *Since a polynomial of odd degree eventually changes sign, since it is continuous, the corollary follows from Bolzano's Intermediate Value Theorem ??.* ❑

211 Example Consider the polynomial $p(x) = x^3 + 4x^2 + x - 6$.

1. Prove that p splits over $\mathbb{Z}$ and find its factorisation. Also, determine its y-intercept.
2. Determine $p(-\infty)$ and $p(+\infty)$
3. Find p' and determine the intervals of monotonicity of p.
4. Determine any local extrema of p.
5. Find p'' and determine the inflexion points of p and its convexity intervals.
6. Obtain an approximate graph of p.

Solution:

1. By Corollary 203, if there are integral roots of p they must divide -6. A quick inspection shews that $p(1) = 0$ and so $x - 1$ divides $p(x)$. By long division

$$\begin{array}{r|l} & \quad\;\; x^2 + 5x + 6 \\ \hline x - 1 & x^3 + 4x^2 + x - 6 \\ & -x^3 + x^2 \\ \hline & \quad\; 5x^2 + x \\ & \quad -5x^2 + 5x \\ \hline & \qquad\; 6x - 6 \\ & \qquad -6x + 6 \\ \hline & \qquad\quad 0 \end{array}$$

   whence
   $$p(x) = (x-1)(x^2 + 5x + 6) = (x-1)(x+2)(x+3).$$
   This means that p crosses the x-axis at $x = -3$, $x = -2$, and $x = 1$. Its y-intercept is $(0, p(0)) = (0, -6)$.

2. Since the leading coefficient of p is $1 > 0$ and since p has odd degree, by Theorem 209, $p(x) \sim (x)(x)(x) = x^3$, as $x \to +\infty$ and so $p(-\infty) = -\infty$ and $p(+\infty) = +\infty$.

3. $p'(x) = 3x^2 + 8x + 1$, whose graph is a convex parabola. Using the Quadratic Formula
   $$3x^2 + 8x + 1 = 0 \iff x = \frac{-4 - \sqrt{13}}{3} \quad \text{or} \quad x = \frac{-4 + \sqrt{13}}{3}$$
   and so $x \approx -2.54$ or $x \approx -0.13$. Since p' is a convex parabola this means that
   $$p'(x) > 0 \iff x \in \left]-\infty\,;\frac{-4 - \sqrt{13}}{3}\right[\cup \left]\frac{-4 + \sqrt{13}}{3}\,;+\infty\right[,$$
   and so p is increasing (approximately) in the intervals $]-\infty\,;-2.54[$ and $]-0.13\,;+\infty[$.

4. Since at $x = -2.54$ p' changes sign from $+$ to $-$, p has a local maximum there by virtue of Theorem 127, which is $p(-2.54) \approx 0.88$. Also, p' changes sign from $-$ to $+$ at $x = -0.13$ and so p has a local minimum there, which is $p(-0.13) \approx -6.06$.

5. We find $p''(x) = 6x + 8$. Now, $p''(x) = 0 \implies x = -\frac{4}{3} \approx -1.33$ and $p(-1.33) \approx -2.61$. Hence p changes convexity (approximately) at $(-1.33, -2.61)$.

6. The graph of p can be found in figure 4.17.

212 Example Consider the polynomial $p(x) = x^3 + x + 1$.

1. Prove that p is strictly increasing.
2. Prove that p has no positive roots.
3. Determine $p(-\infty)$ and $p(+\infty)$
4. Prove that p has a unique real root and find an interval $[a; b]$ of length $< \frac{1}{4}$ containing this root.
5. Find p'' and determine the inflexion points of p and its convexity intervals.
6. Obtain an approximate graph of p.

Solution:

1. We have $p'(x) = 3x^2 + 1 \geq 1 > 0$ since x^2 is always positive.[5] Since the derivative of p is always strictly positive, p is always strictly increasing.

2. Since p is strictly increasing, $p(x) > p(0) = 1$ for $x > 0$. Hence values $x > 0$ can never make p zero.

3. By Theorem 209, $p(-\infty) = -\infty$ and $p(+\infty) = +\infty$.

4. Since p changes sign, it must have a root. Since p is strictly increasing, it can cross the x-axis only once. Now, observe that
$$p(0) = 1, \qquad p(-1) = -1$$
so the root must lie in $[-1; 0]$. We bisect this interval and find $p(-0.5) \approx 0.375$, so the root must lie in $[-1; -0.5]$. We again bisect this interval and find that $p(-0.75) \approx -0.171875$, so the root must lie in $[-0.75; -0.5]$. Again, we bisect this interval and find that $p(-0.625) \approx 0.13$, so the root must lie in $[-0.75; -0.625]$. We now stop since we have reached an interval of within the desired length.

5. $p''(x) = 6x$ and so p is convex for $x > 0$ and concave for $x < 0$.

6. An approximate graph is shewn in figure 4.18

We now consider polynomials with real number coefficients and that split in $\mathbb{R}$. Such polynomials have the form
$$p(x) = a(x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_k)^{m_k},$$
where $a \neq 0$ and the r_i are real numbers and the $m_i \geq 1$ are integers. Graphing such polynomials will be achieved by referring to the following theorem.

[5]Another way of seeing that $3x^2 + 1 > 0$ always is by checking its discriminant.

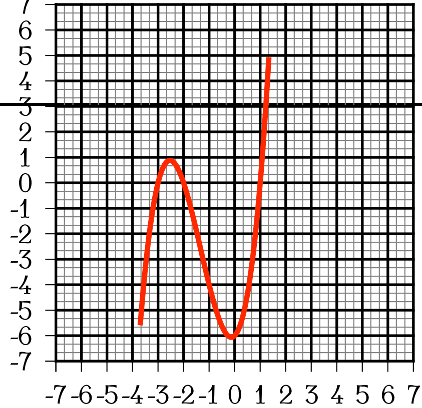


Figure 4.17: Example 211.

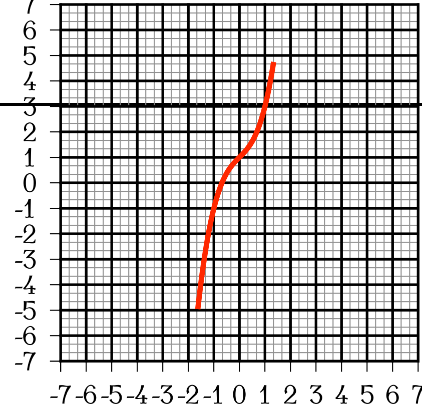


Figure 4.18: Example 212.

213 Theorem Let $a \neq 0$ and the r_i are real numbers and the m_i be positive integers. Then the graph of the polynomial

$$p(x) = a(x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_k)^{m_k},$$

- crosses the x-axis at $x = r_i$ if m_i is odd.
- is tangent to the x-axis at $x = r_i$ if m_i is even.
- has a convexity change at $x = r_i$ if $m_i \geq 3$ and m_i is odd.

Proof: *Since the local behaviour of $p(x)$ is that of $c(x - r_i)^{m_i}$ (where c is a real number constant) near r_i, the theorem follows at once from Theorem 148.* ❑

214 Example Make a rough sketch of the graph of $y = (x + 2)x(x - 1)$. Determine where it achieves its local extrema and their values. Determine where it changes convexity.

Solution: We have $p(x) = (x + 2)x(x - 1) \sim (x) \cdot x(x) = x^3$, as $x \to +\infty$. Hence $p(-\infty) = (-\infty)^3 = -\infty$ and $p(+\infty) = (+\infty)^3 = +\infty$. This means that for large negative values of x the graph will be on the negative side of the y-axis and that for large positive values of x the graph will be on the positive side of the y-axis. By Theorem 213, the graph crosses the x-axis at $x = -2$, $x = 0$, and $x = 1$.

Now, by the Product Rule,

$$\begin{aligned} p'(x) &= x(x - 1) + (x + 2)(x - 1) + (x + 2)x \\ &= 3x^2 + 2x - 2. \end{aligned}$$

Using the quadratic formula,

$$3x^2 + 2x - 2 = 0 \implies x = -\frac{1}{3} - \frac{\sqrt{7}}{3} \approx -1.22; \quad x = -\frac{1}{3} + \frac{\sqrt{7}}{3} \approx 0.55.$$

From geometrical considerations, $x \approx -1.22$ will be the x-coordinate of a local maximum, with y-coordinate $p(-1.22) \approx 2.11$ and $x \approx 0.55$ will be the x-coordinate of a local minimum, with y-coordinate $p(0.55) \approx -0.63$.

Also

$$p''(x) = 6x + 2,$$

so $p''(x) > 0$ for $x > -\frac{1}{3}$ and $p''(x) < 0$ for $x < -\frac{1}{3}$. This means that p is convex for $x > -\frac{1}{3}$ and concave for $x < -\frac{1}{3}$. The graph is shewn in figure 4.19.

215 Example Make a rough sketch of the graph of $y = (x + 2)^3x^2(1 - 2x)$.

Solution: We have $(x+2)^3x^2(1-2x) \sim x^3 \cdot x^2(-2x) = -2x^6$. Hence if $p(x) = (x+2)^3x^2(1-2x)$ then $p(-\infty) = -2(-\infty)^6 = -\infty$ and $p(+\infty) = -2(+\infty)^6 = -\infty$, which means that for both large positive and negative values of x the graph will be on the negative side of the y-axis. By Theorem 213, in a neighbourhood of $x = -2$, $p(x) \sim 20(x+2)^3$, so the graph crosses the x-axis changing convexity at $x = -2$. In a neighbourhood of 0, $p(x) \sim 8x^2$ and the graph is tangent to the x-axis at $x = 0$. In a neighbourhood of $x = \frac{1}{2}$, $p(x) \sim \frac{25}{16}(1-2x)$, and so the graph crosses the x-axis at $x = \frac{1}{2}$.

Now,

$$\begin{aligned} p'(x) &= 3(x+2)^2x^2(1-2x) + 2(x+2)^3x(1-2x) - 2(x+2)^3x^2 \\ &= x(x+2)^2(3x(1-2x) + 2(x+2)(1-2x) - 2(x+2)x) \\ &= -x(x+2)^2(12x^2+7x-4), \end{aligned}$$

and $p'(x) = 0$ when $x = 0, -2, -\frac{7}{24} + \frac{\sqrt{241}}{24} \approx 0.36, -\frac{7}{24} - \frac{\sqrt{241}}{24} \approx -0.94$. From geometrical considerations, $x = 0$ and $x = -2$ are local minima, both with y-coordinate $y = 0$, and both $x \approx 0.36$ (with corresponding $y = p(0.36) \approx 0.48$) and $x \approx -0.94$ (with corresponding y-coordinate $y = p(-0.94) \approx 3.03$) are local maxima. The graph is shewn in figure 4.20.

216 Example Make a rough sketch of the graph of $y = (x+2)^2x(1-x)^2$.

Solution: The dominant term of $(x+2)^2x(1-x)^2$ is $x^2 \cdot x(-x)^2 = x^5$. Hence if $p(x) = (x+2)^2x(1-x)^2$ then $p(-\infty) = (-\infty)^5 = -\infty$ and $p(+\infty) = (+\infty)^5 = +\infty$, which means that for large negative values of x the graph will be on the negative side of the y-axis and for large positive values of x the graph will be on the positive side of the y-axis. By Theorem 213, the graph crosses the x-axis changing convexity at $x = -2$, it is tangent to the x-axis at $x = 0$ and it crosses the x-axis at $x = \frac{1}{2}$. The graph is shewn in figure 4.21.

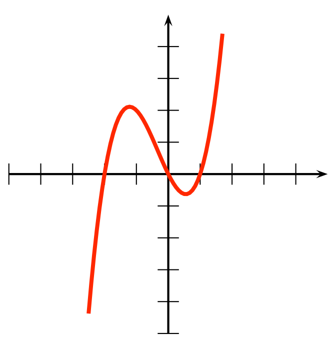


Figure 4.19: $y = (x+2)x(x-1)$.

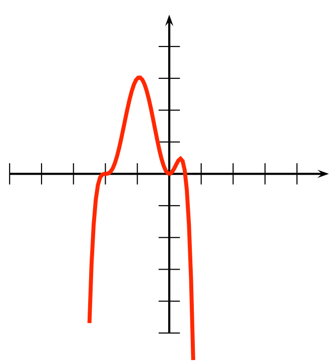


Figure 4.20: $y = (x+2)^3x^2(1-2x)$.

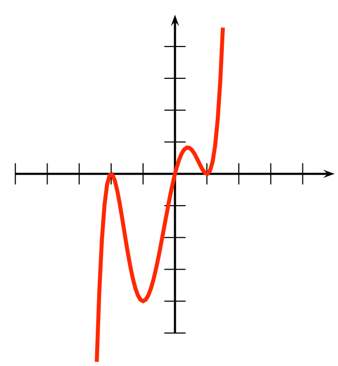


Figure 4.21: $y = (x+2)^2x(1-x)^2$.

5 Rational Functions and Algebraic Functions

5.1 Inverse Power Functions

We now proceed to investigate the behaviour of functions of the type $x \mapsto \frac{1}{x^n}$, where $n > 0$ is an integer.

217 Theorem The strong derivative of the reciprocal function $x \mapsto \frac{1}{x}$ is the function $x \mapsto -\frac{1}{x^2}$.

Proof: *Put* $f(x) = \frac{1}{x}$ *and* $g(x) = x$. *Observe that* $g'(x) = 1$ *and* $f(x)g(x) = 1$. *Hence by the product rule*

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} 1 = f'(x)g(x) + f(x)g'(x) = xf'(x) + \frac{1}{x},$$

and solving for $f'(x)$ *we obtain* $f'(x) = -\frac{1}{x^2}$.

An alternate proof from the definition proceeds as follows. Let $f(x) = \frac{1}{x}$ *and* $x \neq 0$. *First observe the algebraic identity*

$$\frac{1}{1+t} = 1 - t + \frac{t^2}{1+t}.$$

Hence, if $x \neq 0$ *is fixed,*

$$\frac{1}{x+\epsilon} = \frac{1}{x} \cdot \frac{1}{1+\epsilon/x} = \frac{1}{x}\left(1 - \frac{\epsilon}{x} + \frac{\epsilon^2}{x^2} \cdot \frac{1}{(1+\epsilon/x)}\right).$$

Now, since $\epsilon \to 0$ *we will have, eventually,* $|\epsilon| < \frac{|x|}{2}$. *Hence* $\frac{2}{3} < \frac{1}{(1+\epsilon/x)} < 2$. *This means that*

$$\frac{\epsilon^2}{x^2} \cdot \frac{1}{(1+\epsilon/x)} = o(\epsilon),$$

where the implied constant depends on (the fixed value of) x, *and so*

$$\frac{1}{x+\epsilon} = \frac{1}{x}\left(1 - \frac{\epsilon}{x} + o(\epsilon)\right) = \frac{1}{x} - \frac{\epsilon}{x^2} + o(\epsilon),$$

from where the assertion follows. ❑

218 Theorem If $n > 0$ is an integer and $x \neq 0$,

$$\left(\frac{1}{x^n}\right)' = (x^{-n})' = -nx^{-n-1}.$$

Proof: *Let* $f(x) = x^n$, $g(x) = \frac{1}{x}$, $h(x) = \frac{1}{x^n}$. *Then* $h = f \circ g$. *By the Chain Rule (Theorem 159) and Theorem 217,*

$$h'(x) = f'(g(x))g'(x) = n\left(\frac{1}{x}\right)^{n-1}\left(-\frac{1}{x^2}\right) = -\frac{n}{x^{n+1}} = -nx^{-n-1},$$

as it was to be demonstrated. ❑

Theorems 146 and 218 say that if α is an integer, then $(x^{\alpha})' = \alpha x^{\alpha-1}$.

With the derivatives of reciprocal powers determined, we can now address how to graph them.

219 Theorem Let $n > 0$ be an integer. Then

- if n is even, $x \mapsto \frac{1}{x^n}$ is increasing for $x < 0$, decreasing for $x > 0$ and convex for all $x \neq 0$.
- if n is odd, $x \mapsto \frac{1}{x^n}$ is decreasing for all $x \neq 0$, concave for $x < 0$, and convex for $x > 0$.

Thus $x \mapsto \frac{1}{x^n}$ has a pole of order n at $x = 0$ and a horizontal asymptote at $y = 0$.

Proof: *Let $h(x) = \frac{1}{x^n}$. By Theorem 218, $h'(x) = -\frac{n}{x^{n+1}}$ and $h''(x) = \frac{n(n+1)}{x^{n+2}}$. If n is odd, then $n+1$ is even and $n+2$ is odd. Hence $h'(x) > 0$ for $x \neq 0$, proving that h is increasing and $h''(x)$ has the same sign as x, proving that h is concave for $x < 0$ and convex for $x > 0$. A similar argument is used for when n is even, completing the proof.* ❑

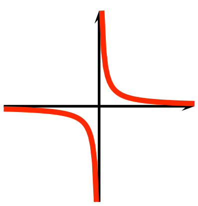


Figure 5.1: $x \mapsto \frac{1}{x}$

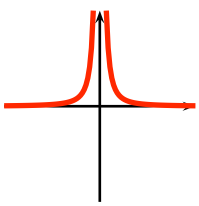


Figure 5.2: $x \mapsto \frac{1}{x^2}$

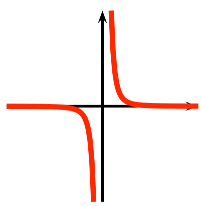


Figure 5.3: $x \mapsto \frac{1}{x^3}$

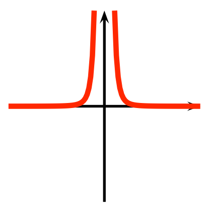


Figure 5.4: $x \mapsto \frac{1}{x^4}$

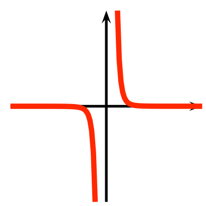


Figure 5.5: $x \mapsto \frac{1}{x^5}$

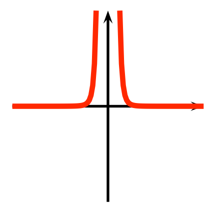


Figure 5.6: $x \mapsto \frac{1}{x^6}$

220 Example A few functions $x \mapsto \frac{1}{x^n}$ are shewn in figures 5.1 through 5.6.

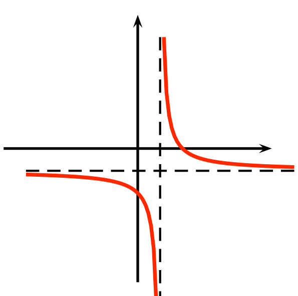


Figure 5.7: $x \mapsto \frac{1}{x-1} - 1$

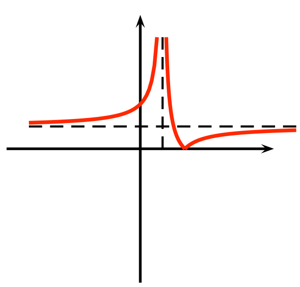


Figure 5.8: $x \mapsto \left|\frac{1}{x-1} - 1\right|$

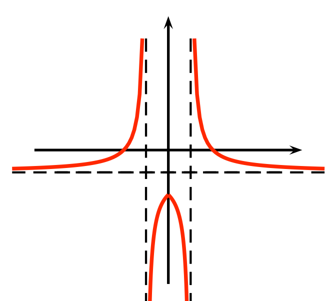


Figure 5.9: $x \mapsto \frac{1}{|x|-1} - 1$

221 Example Figures 5.7 through 5.9 shew a few transformations of $x \mapsto \frac{1}{x}$.

5.2 The Quotient Rule

222 Theorem If g is strongly differentiable at x and $g(x) \neq 0$ then

$$\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{(g(x))^2}.$$

Proof: *Let $a(x) = \frac{1}{x}$. Then $\frac{1}{g} = a \circ g$. By the Chain Rule (Theorem 159) and Theorem ??,*

$$\left(\frac{1}{g}\right)'(x) = a'(g(x))g'(x) = -\frac{g'(x)}{(g(x))^2},$$

as we needed to shew. ❑

223 Corollary (Quotient Rule) If f, g are strongly differentiable at x and if $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Proof: *Using the Product Rule (Theorem 156) and Theorem 222,*

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x) &= \left(f \cdot \frac{1}{g}\right)'(x) \\
&= f'(x)\frac{1}{g(x)} + f(x)\left(\frac{1}{g}\right)'(x) \\
&= f'(x)\frac{1}{g(x)} + f(x)\left(-\frac{g'(x)}{(g(x))^2}\right) \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2},
\end{aligned}$$

as desired. ❑

224 Example Find $b'(x)$ if $b(x) = \frac{1 + x + x^2}{(x - x^2)^2}$

Solution: Use the Quotient Rule and the Chain Rule:

$$\begin{aligned}
b'(x) &= \frac{(1 + 2x)(x - x^2)^2 - (1 + x + x^2)(2(1 - 2x)(x - x^2))}{(x - x^2)^4} \\
&= \frac{(1 + 2x)(x - x^2) - (1 + x + x^2)(2(1 - 2x))}{(x - x^2)^3} \\
&= -\frac{-2 + 3x + 3x^2 + 2x^3}{(x - x^2)^3}.
\end{aligned}$$

5.3 Rational Functions

225 Definition By a *rational function* $x \mapsto r(x)$ we mean a function r whose assignment rule is of the $r(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x) \neq 0$ are polynomials.

We now provide a few examples of graphing rational functions.

226 Example Draw the curve $x \mapsto \dfrac{x}{x^2+1}$.

Solution: Put $a(x) = \dfrac{x}{x^2+1}$. Observe that $a(-x) = -a(x)$, which means that a is an odd function and hence symmetric about the origin. Also

$$a'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}.$$

Since $(x^2+1)^2$ is always positive, a' changes sign when $1-x^2 = (1-x)(1+x)$ changes sign. Hence $a'(x) \geq 0$ if $x \in [-1;1]$ and $a'(x) < 0$ otherwise. This means that a is increasing for $x \in [-1;1]$ and decreasing otherwise. Moreover

$$a''(x) = \frac{d}{dx}\frac{1-x^2}{(x^2+1)^2} = \frac{-2x(x^2+1)^2 - 2(2x)(1-x^2)(x^2+1)}{(x^2+1)^4} = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

Again a'' will change sign when $2x(x^2-3) = x(x-\sqrt{3})(x+\sqrt{3})$ changes sign. By means of a sign diagram we see that $a''(x) \geq 0$ for $x \in [-\sqrt{3};0]\cup[\sqrt{3};+\infty[$, and so a is convex for $x \in [-\sqrt{3};0]\cup[\sqrt{3};+\infty[$ and concave otherwise. The graph is shewn in figure 5.10.

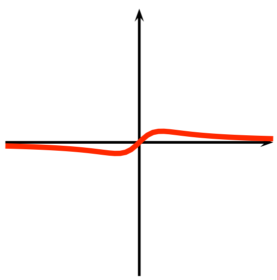


Figure 5.10: $x \mapsto \dfrac{x}{x^2+1}$

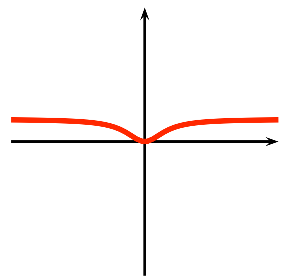


Figure 5.11: $x \mapsto \dfrac{x^2}{x^2+1}$

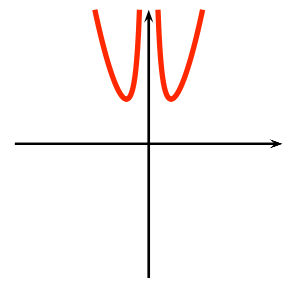


Figure 5.12: $x \mapsto x^2 + \dfrac{1}{x^2}$

227 Example Draw the curve $x \mapsto \dfrac{x^2}{x^2+1}$.

Solution: Put $b(x) = \dfrac{x^2}{x^2+1}$. Observe that $b(-x) = b(x)$, which means that b is an even function and hence symmetric about the y-axis. Also

$$b'(x) = \frac{(2x)(x^2+1) - 2x(x^2)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}.$$

Since $(x^2+1)^2$ is always positive, b' changes sign when x changes sign. Hence $b'(x) \geq 0$ if $x \geq 0$ and $b'(x) < 0$ otherwise. This means that b is increasing for $x \geq 0$ and decreasing otherwise. Moreover

$$b''(x) = \frac{d}{dx}\frac{2x}{(x^2+1)^2} = \frac{2(x^2+1)^2 - 2(2x)(2x)(x^2+1)}{(x^2+1)^4} = \frac{2-6x^2}{(x^2+1)^3}.$$

Again b'' will change sign when $2-6x^2 = 2(1-\sqrt{3}x)(1+\sqrt{3}x)$ changes sign. By means of a sign diagram we see that $b''(x) \geq 0$ for $x \in \left[-\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}}\right]$, and so b is convex for $x \in \left[-\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}}\right]$ and concave otherwise. The graph is shewn in figure 5.11.

228 Example Draw the curve $x \mapsto x^2 + \dfrac{1}{x^2}$.

Solution: Put $c(x) = x^2 + \frac{1}{x^2}$. Observe that $c(-x) = c(x)$, which means that c is an even function and hence symmetric about the y-axis. Also

$$c'(x) = 2x - \frac{2}{x^3} = \frac{2(x^4-1)}{x^3} = \frac{2(x-1)(x+1)(x^2+1)}{x^3}.$$

We make a sign diagram investigating the sign changes of c' near $x = -1$, $x = 0$, and $x = 1$. From this we gather that c is increasing for $x \in [-1; 0[\cup [1; +\infty[$. Moreover

$$c''(x) = \frac{d}{dx}\left(2x - \frac{2}{x^3}\right) = 2 + \frac{6}{x^4}.$$

We see that c'' is always positive and hence it is always convex. The graph is shewn in figure 5.12.

Analogous to theorem 213, we now consider rational functions $x \mapsto r(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials with no factors in common and splitting in $\mathbb{R}$.

229 Theorem Let $a \neq 0$ and the r_i are real numbers and the m_i be positive integers. Then the rational function with assignment rule

$$r(x) = K\frac{(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_k)^{m_k}}{(x-b_1)^{n_1}(x-b_2)^{n_2}\cdots(x-b_l)^{n_l}},$$

- has zeroes at $x = a_i$ and poles at $x = b_j$.
- crosses the x-axis at $x = a_i$ if m_i is odd.
- is tangent to the x-axis at $x = a_i$ if m_i is even.
- has a convexity change at $x = a_i$ if $m_i \geq 3$ and m_i is odd.
- both $r(b_j-)$ and $r(b_j+)$ blow to infinity. If n_i is even, then they have the same sign infinity: $r(b_i+) = r(b_i-) = +\infty$ or $r(b_i+) = r(b_i-) = -\infty$. If n_i is odd, then they have different sign infinity: $r(b_i+) = -r(b_i-) = +\infty$ or $r(b_i+) = -r(b_i-) = -\infty$.

Proof: *Since the local behaviour of $r(x)$ is that of $c(x-r_i)^{t_i}$ (where c is a real number constant) near r_i, the theorem follows at once from Theorem 148 and 219.* ❑

230 Example Draw a rough sketch of $x \mapsto \frac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$.

Solution: Put $r(x) = \frac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$. By Theorem 229, r has zeroes at $x = 1$, and $x = -2$, and poles at $x = -1$ and $x = 2$. As $x \to 1$, $r(x) \sim \frac{3}{2}(x-1)^2$, hence the graph of r is tangent to the axes, and positive, around $x = 2$. As $x \to -2$, $r(x) \sim -\frac{9}{16}(x+2)$, hence the graph of r crosses the x-axis at $x = -2$, coming from positive y-values on the left of $x = -2$ and going to negative y=values on the right of $x = -2$. As $x \to -1$, $r(x) \sim \frac{4}{9(x+1)}$, hence the graph of r blows to $-\infty$ to the left of $x = -1$ and to $+\infty$ to the right of $x = -1$. As $x \to 2$, $r(x) \sim \frac{4}{3(x-2)^2}$, hence the graph of r blows to $+\infty$ both from the left and the right of $x = 2$. Also we observe that

$$r(x) \sim \frac{(x)^2(x)}{(x)(x)^2} = \frac{x^3}{x^3} = 1,$$

and hence r has the horizontal asymptote $y = 1$. The graph of r can be found in figure 5.13.

231 Example Draw a rough sketch of $x \mapsto \dfrac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$.

Solution: Put $r(x) = \dfrac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$. First observe that $r(x) = r(-x)$, and so r is even. By Theorem 229, r has zeroes at $x = \pm\dfrac{3}{4}$, and poles at $x = \pm 1$. As $x \to \dfrac{3}{4}$, $r(x) \sim -\dfrac{36}{7}(x-3/4)^2$, hence the graph of r is tangent to the axes, and negative, around $x = 3/4$, and similar behaviour occurs around $x = -\dfrac{3}{4}$. As $x \to 1$, $r(x) \sim \dfrac{49}{512(x-1)}$, hence the graph of r blows to $-\infty$ to the left of $x = 1$ and to $+\infty$ to the right of $x = 1$. As $x \to -1$, $r(x) \sim -\dfrac{49}{512(x-1)}$, hence the graph of r blows to $+\infty$ to the left of $x = -1$ and to $-\infty$ to the right of $x = -1$. Also, as $x \to +\infty$,

$$r(x) \sim \frac{(x)^2(x)^2}{(x)(x)} = x^2,$$

so $r(+\infty) = +\infty$ and $r(-\infty) = +\infty$. The graph of r can be found in figure 5.14.

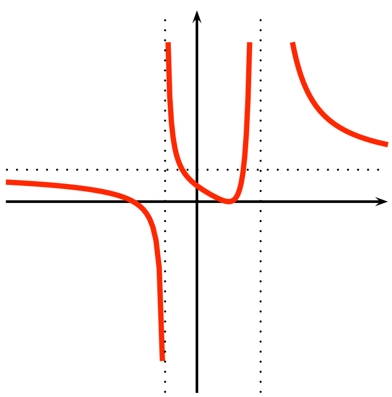


Figure 5.13: $x \mapsto \dfrac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$

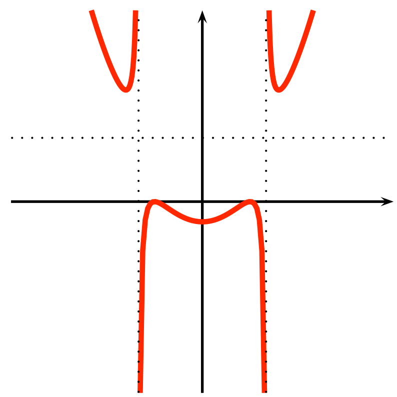


Figure 5.14: $x \mapsto \dfrac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$

5.4 Algebraic Functions

232 Definition We will call *algebraic function* a function whose assignment rule can be obtained from a rational function by a finite combination of additions, subtractions, multiplications, divisions, exponentiations to a rational power.

233 Theorem Let $q \neq 0$ be an integer. The strong derivative of the function $x \mapsto x^{1/q}$ is the function $x \mapsto \dfrac{1}{q} \cdot x^{1/q-1}$, whenever this last makes sense.

Proof: *Put $f(x) = x^{1/q}$, assuming this quantity is real. Then $(f(x))^q = x$. Differentiating both sides using the Chain Rule we have*

$$q(f(x))^{q-1}f'(x) = \implies qx^{(q-1)/q}f'(x) = 1.$$

Solving for f' gives

$$f'(x) = \frac{1}{q} \cdot x^{1/q-1},$$

if this quantity is a real number, proving the result. ❑

STOP *Theorems 146, 218, and 233, when combined with the Chain Rule, say that if α is a rational number, then $(x^{\alpha})' = \alpha x^{\alpha-1}$.*

With the derivatives of rational powers determined, we can now address how to graph them.

234 Theorem Let $|q| \geq 2$ be an integer. If

- if q is even then $x \mapsto x^{1/q}$ is increasing and concave for $q \geq 2$ and decreasing and convex for $q \leq -2$ for all $x > 0$ and it is undefined for $x < 0$.
- if q is odd then $x \mapsto x^{1/q}$ is everywhere increasing and convex for $x < 0$ but concave for $x > 0$ if $q \geq 3$. If $q \leq -3$ then $x \mapsto x^{1/q}$ is decreasing and concave for $x < 0$ and increasing and convex for $x > 0$.

Proof: *Let $h(x) = x^{1/q}$. By Theorem 233, $h'(x) = \dfrac{x^{(1-q)/q}}{q}$ and $h''(x) = \dfrac{(1-q)x^{(2q-1)/q}}{q^2}$.*

Assume first that q is even. Then $x^{1/q}$ is not real for $x < 0$ so we assume that $x > 0$. The quantity $h'(x) = \dfrac{x^{(1-q)/q}}{q}$ is > 0 for $q \geq 2$ and negative for $q \leq -2$. If $q \geq 2$ then $h''(x) = \dfrac{(1-q)x^{(2q-1)/q}}{q^2} < 0$ and if $q \leq -2$ then $h''(x) > 0$. Hence h is increasing and concave for $q \geq 2$ and decreasing and convex for $q \leq -2$

Assume now that q is odd. Then $1 - q$ is even and the sign of the quantity $h'(x) = \dfrac{1}{q} \cdot \left(x^{1/q}\right)^{1-q}$ is depends on the sign of $\dfrac{1}{q}$. Since $2q - 1$ is odd, the sign of $h''(x) = \dfrac{(1-q)x^{(2q-1)/q}}{q^2}$ is **signum** *$((1-q)(x))$. We have: if $q \geq 3$, $h'(x) > 0$, $h''(x) < 0$ for $x > 0$ and $h''(x) > 0$ for $x < 0$. Hence for $q \geq 3$, h is increasing and it is convex for $x < 0$ but concave for $x > 0$. If $q \leq -3$ then h is decreasing and it is concave for $x < 0$ and decreasing convex for $x > 0$.* ❑

A few of the functions $x \mapsto x^{1/q}$ are shewn in figures 5.15 through 5.26.

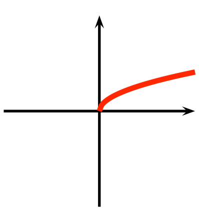


Figure 5.15: $x \mapsto x^{1/2}$

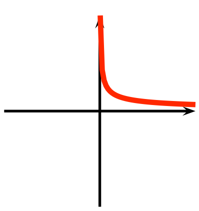


Figure 5.16: $x \mapsto x^{-1/2}$

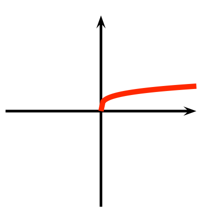


Figure 5.17: $x \mapsto x^{1/4}$

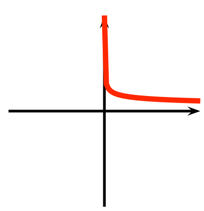


Figure 5.18: $x \mapsto x^{-1/4}$

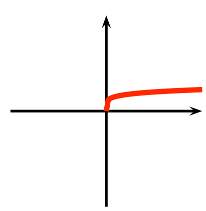


Figure 5.19: $x \mapsto x^{1/6}$

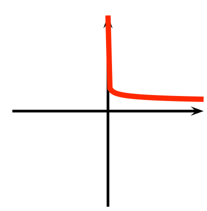


Figure 5.20: $x \mapsto x^{-1/6}$

We finish this section with an example.

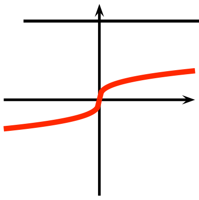


Figure 5.21: $x \mapsto x^{1/3}$

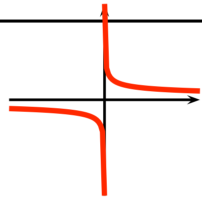


Figure 5.22: $x \mapsto x^{-1/3}$

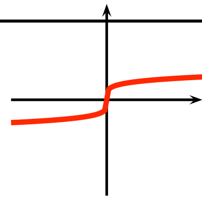


Figure 5.23: $x \mapsto x^{1/5}$

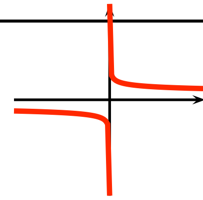


Figure 5.24: $x \mapsto x^{-1/5}$

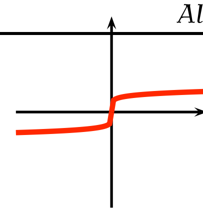


Figure 5.25: $x \mapsto x^{1/7}$

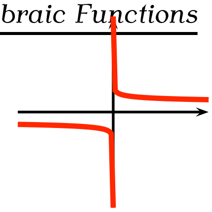


Figure 5.26: $x \mapsto x^{-1/7}$

235 Example Consider the function $y = f(x) = \sqrt{x-1} + \sqrt{2-x}$.

1. For which x will the output of f be a real number?
2. Find $f'(x)$.
3. By examining f', prove that f increases for $x < \dfrac{3}{2}$ and decreasing for $x > \dfrac{3}{2}$.
4. Prove that for all x in the domain of f one has $f(x) \leq \sqrt{2}$.
5. Find $f''(x)$.
6. Determine in which intervals f is convex or concave.
7. Graph f.

Solution:

1. We need, simultaneously, $x \geq 1$ and $2 \geq x$. This means that $x \in [1;2]$.
2. Via the Chain Rule:
$$\frac{\mathrm{d}}{\mathrm{d}x}\,(x-1)^{1/2} + (2-x)^{1/2} = \frac{1}{2}(x-1)^{-1/2} - \frac{1}{2}(2-x)^{-1/2} = \frac{1}{2(x-1)^{1/2}} - \frac{1}{2(2-x)^{1/2}}$$
3. f has a stationary point when $f'(x) = 0$, that is, if
$$\frac{1}{2(x-1)^{1/2}} = \frac{1}{2(2-x)^{1/2}} \implies x - 1 = 2 - x \implies x = \frac{3}{2},$$
so f' has only one zero in $[1;2]$. Since f' is continuous in $]1;2[$ and has only one zero there, it must be negative in a portion of the interval and positive in the other. Examining values in $]1;\frac{3}{2}[$ we see that f' is positive for there and negative in $]\frac{3}{2};2[$.
4. By the above, $x = \dfrac{3}{2}$ is a global maximum in $[1;2]$, and hence
$$f(x) \leq f\left(\frac{3}{2}\right) = \sqrt{\frac{3}{2} - 1}\sqrt{2 - \frac{3}{2}} = 2\sqrt{\frac{1}{2}} = \sqrt{2}.$$
5. Via the Chain Rule:
$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\,\frac{1}{2}(x-1)^{-1/2} - \frac{1}{2}(2-x)^{-1/2} = -\frac{1}{4}(x-1)^{-3/2} - \frac{1}{4}(2-x)^{-3/2} = -\frac{1}{4(x-1)^{3/2}} - \frac{1}{4(2-x)^{3/2}}.$$
6. Observe that
$$f''(x) = -\frac{1}{4(x-1)^{3/2}} - \frac{1}{4(2-x)^{3/2}} = -\frac{1}{4}\left(\frac{1}{(x-1)^{3/2}} + \frac{1}{(2-x)^{3/2}}\right).$$
Since the quantity in parenthesis is always positive, f'' is always negative, and hence it is everywhere concave.

7. The graph appears in figure 5.27.

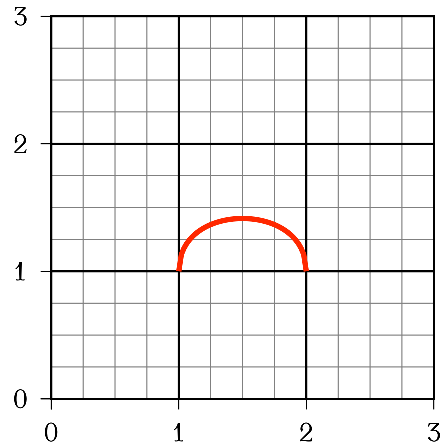


Figure 5.27: $y = \sqrt{x-1} + \sqrt{2-x}$

6 Exponential and Logarithmic Functions

6.1 Exponential Functions

6.2 Differentiating Exponential Functions

6.3 Graphs of Exponential Functions

6.4 Integrating Exponential Functions

6.5 Logarithmic Functions

6.6 Algebraic Properties of Logarithms

6.7 Differentiating Logarithmic Functions Functions

6.8 Graphs of Exponential Functions

6.9 Integrating Exponential Functions

6.10 Exponential and Logarithmic Equations

7 Goniometric Functions

7.1 The Sine and Cosine Functions

7.2 Differentiation of the Sine and Cosine

7.3 Integration of the Sine and Cosine

7.4 Graphs of the Sine and Cosine

7.5 Addition Formul

7.6 Inverse Sine and Cosine

7.7 Equations in Sine and Cosine

7.8 Other Goniometric Functions

A Answers to Selected Problems

1.1.1 There are infinitely many answers. A rational number between then is 0.101. An irrational number between then can be obtained by letting $c = 0.12345678910\ldots$ be the Champernowne-Mahler constant, and then taking

$$0.1 + 0.1 \cdot c = 0.101234567891011\ldots.$$

1.3.1

1.

$]-\infty;-1[$	$]-1;0[$	$]0;1[$	$]1;+\infty[$
$-$	$+$	$-$	$+$

2. $]-\infty\,;-1[\ \cup\ ]0\,;1[$
3. $]-1\,;0]\ \cup\ [1\,;+\infty[$
4. We have

$$\frac{x(x-1)}{x+1} \geq -1 \iff \frac{x(x-1)}{x+1} + 1 \geq 0 \iff \frac{x(x-1)}{x+1} + \frac{x+1}{x+1} \geq 0 \iff \frac{x^2+1}{x+1} \geq 0.$$

The numerator is always positive, since it is a square plus 1. The denominator is strictly positive for $x > -1$. Hence the solution set is $]-1\,;+\infty[$.

1.4.1 $\{0\}$

1.4.2 $\varnothing$, the empty set.

1.4.3 $]0\,;1[\cup]1\,;+\infty[$

1.4.4

1. $(f+g)(2) = f(2) + g(2) = \sqrt{5} + \sqrt{3}$.
2. Determine $(fg)(2) = f(2)\cdot g(2) = \sqrt{5}\cdot\sqrt{3} = \sqrt{15}$.
3. Determine $(f\circ g)(2) = f(g(2)) = f(\sqrt{3}) = 2$.
4. Determine $(g\circ f)(2) = g(f(2)) = g(\sqrt{5}) = 2$.

1.5.1 $a(x)b(x) = 2x + o(x)$ as $x \to 0$.

1.6.1 $a(x)b(x) = x^4 + 2x^3 + o\left(x^3\right)$ as $x \to +\infty$.

1.7.1 This is $\sqrt{(1-(-3))^2 + (-2-4)^2} = \sqrt{16+36} = 2\sqrt{13}$.

1.7.2 The slope of the line is $\dfrac{-2-4}{1-(-3)} = \dfrac{-6}{4} = -\dfrac{3}{2}$. The equation of the line is of the form $y = -\dfrac{3}{2}x + k$. Since $(1,-2)$ is on the line

$$-2 = -\frac{3}{2}(1) + k \implies k = -2 + \frac{3}{2} = -\frac{1}{2}.$$

The equation sought is thus $y = -\dfrac{3}{2}x - \dfrac{1}{2}$.

1.7.3 $-\dfrac{b}{a}$

1.7.4 $y = -x + b + a$

1.7.5 Let required point be (x, y). The distance of this point to its projection on the x-axis is $|y|$ and similarly, the distance of this point to its projection on the x-axis is $|x|$. We need

$$|y| = |x| \implies |6 - 2x| = |x| \implies 6 - 2x = x \qquad \text{or} \qquad 6 - 2x = -x.$$

The first case gives $x = 2$ and the point is $(2, 2)$, and the second case gives $x = 6$ and the point is $(6, -6)$.

1.7.6 Let $(x, 0)$ be the coordinates of S. Since the slope of the line segment SM is $\dfrac{1}{2}$, we have

$$\frac{2-x}{2-0} = \frac{1}{2} \implies x = 1,$$

whence S is the point $(1, 0)$. Let $(a, 0)$ be the coordinates of A. Since $SM = MA$, we have

$$\sqrt{(a-2)^2 + (0-2)^2} = \sqrt{(1-2)^2 + (0-2)^2} \implies (a-2)^2 + 4 = 5 \implies a \in \{1, 3\}.$$

This means that A is the point $(3, 0)$. Let B be point $(0, y)$. Since A, B, M are collinear, we may compute the slope in two different ways to obtain,

$$\frac{y-2}{0-2} = \frac{2-0}{2-3} \implies y - 2 = 4 \implies y = 6.$$

Thus B is the point $(0, 6)$.

2.1.1 There are infinitely many graphs satisfying the specified conditions. Figure A.1 shews one.

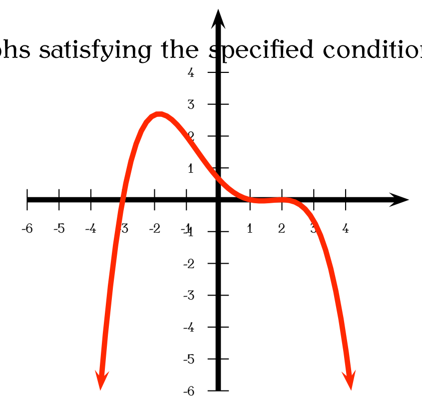


Figure A.1: Problem 2.1.1.

2.1.2 There are infinitely many graphs satisfying the specified conditions. Figure A.2 shews one.

2.1.3 There are infinitely many graphs satisfying the specified conditions. Figure A.3 shews one.

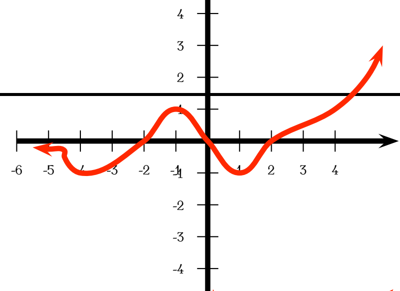


Figure A.2: Problem 2.1.2.

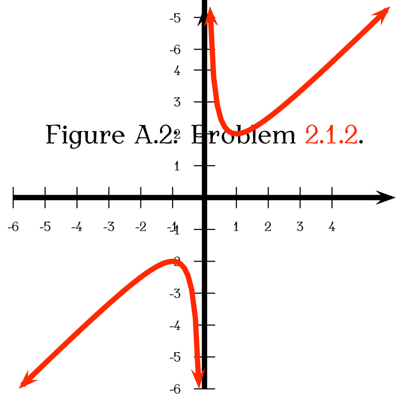


Figure A.3: Problem 2.1.3.

2.2.1 Here is the graph of $x \mapsto f(x+1)$.

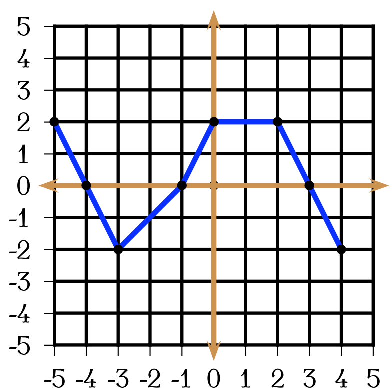


Figure A.4: $y = f(x+1)$.

Here is the graph of $x \mapsto f(-x+1)$.

Here is the graph of $x \mapsto -f(-x+1)$.

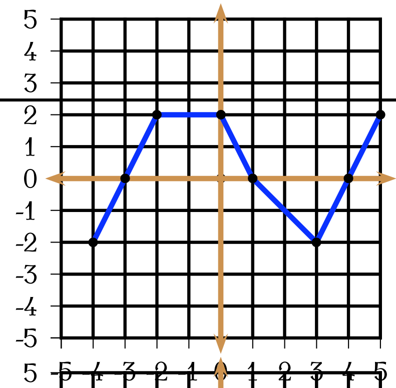


Figure A.5: $y = f(1 - x)$.

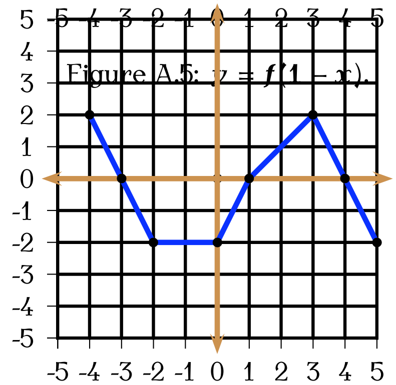


Figure A.6: $y = -f(1 - x)$.

3.1.1 Let $\alpha < \beta < \gamma$ be the roots. Then

$$p(-1) = -4, \quad p(0) = 1, \quad p(1) = 0, \qquad p\left(\frac{3}{2}\right) = -\frac{7}{128}, \quad p(2) = 41.$$

Hence $\alpha \in \left]-1\,;0\right[$, $\beta = 1$, $\gamma \in \left]\frac{3}{2}\,;2\right[$.

3.1.2 If p had odd degree, then, by the Intermediate Value Theorem it would have a real root. Let α be its largest real root. Then

$$0 = p(\alpha)q(\alpha) = p(\alpha^2 + \alpha + 1)$$

meaning that $\alpha^2 + \alpha + 1 > \alpha$ is a real root larger than the supposedly largest real root α, a contradiction.

3.1.3 Observe that $f(1000)f(f(1000)) = 1 \implies f(999) = \frac{1}{999}$. So the range of f include all numbers from $\frac{1}{999}$ to 999. By the intermediate value theorem, there is a real number a such that $f(a) = 500$. Thus

$$f(a)f(f(a)) = 1 \implies f(500) = \frac{1}{500}.$$

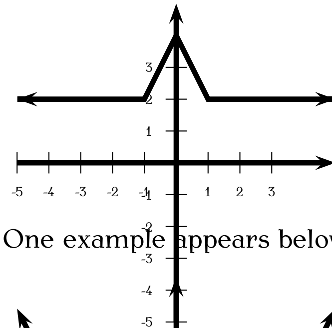
3.1.4 If either $f(0) = 1$ or $f(1) = 0$, we are done. So assume that $0 \geq f(0) < 1$ and $0 < f(1) \leq 1$. Put $g(x) = f(x) + x - 1$. Then $g(0) = f(0) - 1 < 0$ and $g(1) = f(1) > 0$. By Bolzano's Theorem there is a $c \in \left]0\,;1\right[$ such that $g(c) = 0$, that is, $f(c) + c - 1 = 0$, as required.

3.1.5 Consider $g(x) = f(x) - f(x + 1/n)$, which is clearly continuous. If g is never 0 in $[0\,;1]$ then by the Law of Inertia g must be either strictly positive or strictly negative. But then

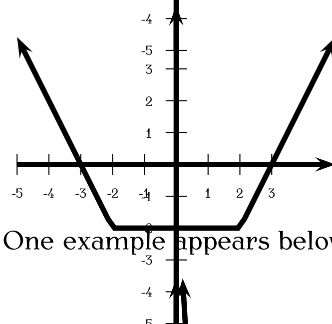
$$0 = f(0) - f(1) = \left(f(0) - f\left(\frac{1}{n}\right)\right) + \left(f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right)\right) + \left(f\left(\frac{2}{n}\right) - f\left(\frac{3}{n}\right)\right) + \cdots + \left(f\left(\frac{n-1}{n}\right) - f\left(\frac{n}{n}\right)\right).$$

The sum of each parenthesis on the right is strictly positive or strictly negative and hence never 0, a contradiction.

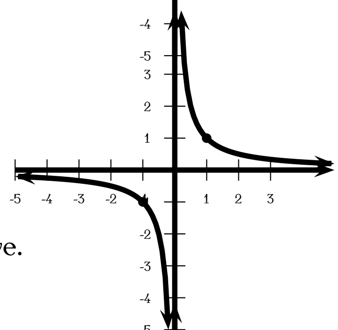
3.2.1 There are multiple answers here. One example appears below.



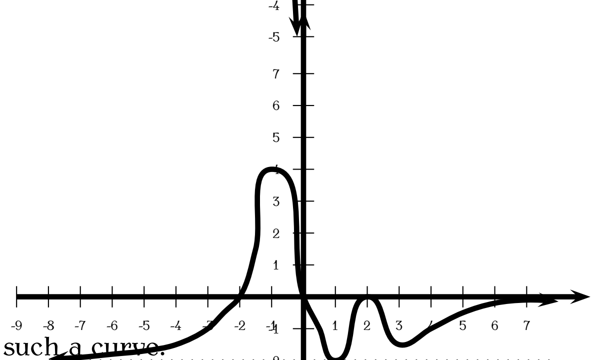
3.2.2 There are multiple answers here. One example appears below.



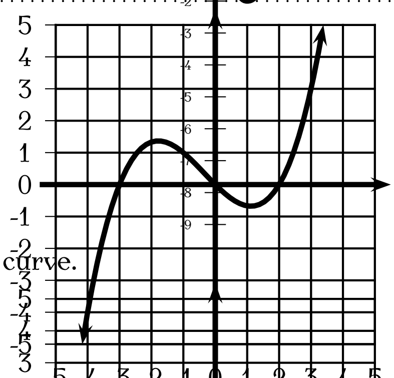
3.2.3 There are multiple answers here. One example appears below.



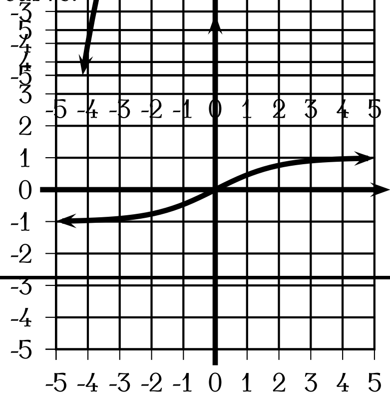
3.2.4 Here is an example of such a curve.



3.4.1 Below is an example of such a curve.



3.4.2 Below is an example of such a curve.



3.5.2 The integral gives the area of the triangle with vertices at $A(0,0)$, $B(t,0)$, $C(t,t)$. This is an isosceles right triangle of area $\frac{t^2}{2}$.

3.5.3 Observe that $x \mapsto 2x+1$ is a straight line that crosses the x-axis at $x = -\frac{1}{2}$. Since $\int_{-2}^{3}(2x+1)\mathrm{d}x = \int_{-2}^{-1/2}(2x+1)\mathrm{d}x + \int_{-1/2}^{3}(2x+1)\mathrm{d}x$, we decompose the region into two triangles: one with negative area and the other with positive area. The triangle with negative area has vertices $A(-2,0)$, $B(-2,-3)$, $C(-\frac{1}{2},0)$ and area $-\frac{1}{2}\cdot\frac{3}{2}\cdot 3 = -\frac{9}{4}$. The triangle with positive area has vertices $C(-\frac{1}{2},0)$, $D(3,0)$, $B(3,7)$, and area $\frac{1}{2}\cdot\frac{7}{2}\cdot 7 = \frac{49}{4}$. Thus the total signed area is $-\frac{9}{4}+\frac{49}{4} = 10$.

3.5.4 By example 65, the graph of $f:[-1\,;1]\to\mathbb{R}$, $x\mapsto\sqrt{1-x^2}$ is an upper semicircle of radius 1. Hence, its area is half of that of a circle of radius 1, that is $\frac{\pi}{2}$.

4.4.1

1. By the Product Rule:

$$\begin{aligned} f'(x) &= (x-1)(x+1)+x(x+1)+x(x-1) \\ &= x^2-1+x^2+x+x^2-x \\ &= 3x^2-1 \end{aligned}$$

2. Using the Chain Rule:

$$g'(x) = 4(ax+(bx+(cx+1)^2)^3)^3(a+3(bx+(cx+1)^2)^2)(b+2c(cx+1))$$

3. Using the Product Rule and the Chain Rule:

$$\begin{aligned} h'(x) &= 2(3x^2)(x^3+2)(x^2+2x+2)^3+(x^3+2)^2 3(2x+2)(x^2+2x+2)^2 \\ &= 6x^2(x^3+2)(x^2+2x+2)^3+6(x^3+2)(x+1)(x^2+2x+2)^2 \end{aligned}$$

4. Using the Product Rule and the Chain Rule:

$$u'(x) = (x-1)^2(x+2)^3+2x(x-1)(x+2)^3+3x(x-1)^2(x+2)^2$$

5. Put $f(x)=1-x^2$. Then $v(x)=f(f(f(f(f(x)))))$. Hence

$$\begin{aligned} v'(x) &= f'(x)\cdot f'(f(x))\cdot f'(f(f(x)))\cdot f'(f(f(f(x))))\cdot f'(f(f(f(f(x))))) \\ &= -2x\cdot(-2(1-x^2))\cdot(-2(1-(1-x^2)^2))\cdot(-2(1-(1-(1-x^2)^2)^2))\cdot(-2(1-(1-(1-(1-x^2)^2)^2)^2)) \\ &= -32x\cdot(1-x^2)\cdot(1-(1-x^2)^2)\cdot(1-(1-(1-x^2)^2)^2)\cdot(1-(1-(1-(1-x^2)^2)^2)^2) \end{aligned}$$

4.4.2 Use the Product Rule:

$$\begin{aligned} a'(x) &= (1+x+x^2)(1+x+x^2+x^4)+(1+2x)(1+x)(1+x+x^2+x^4)+(1+x)(1+x+x^2)(1+2x+4x^3) \\ &= 3+10x+15x^2+15x^4+16x^3+12x^5+7x^6. \end{aligned}$$

4.4.3 We have

$$100a_0x^{99} + 99a_1x^{98} + 98a_2x^{97} + \cdots + 2a_{98}x + a_{99} = p'(x) = 50(x-2)^{49}(2x-1)^{50} + 100(x-2)^{50}(2x-1)^{49}$$

and the desired quantity is

$$100a_0 + 99a_1 + 98a_2 + \cdots + 2a_{98} + a_{99} = p'(1) = -50 + 100 = 50.$$

4.4.4 We have

$$g(x) = f(f(x)) \implies g'(x) = f'(x)f'(f(x)) \implies g'(1) = f'(1)f'(f(1)) = 1 \cdot f'(2) = 1.$$

Also,

$$g(x) = f(f(x)) \implies g'(x) = f'(x)f'(f(x)) \implies g''(x) = f''(x)f'(f(x)) + (f'(x))^2f''(f(x)),$$

which gives

$$g''(1) = f''(1)f'(f(1)) + (f'(1))^2f''(f(1)) = -1f'(2) + (1)^2f''(2) = -1(1) + 1(-1) = -2.$$

4.4.5 Since f is even, f' is odd and f'' is even. Thus we have

$$f(1) = 2; \quad f(-1) = 2; \quad f(-2) = 1; \quad f(2) = 1$$

$$f'(1) = -1 \quad f'(-1) = 1; \quad f'(2) = 2; \quad f'(-2) = -2;$$

$$f''(1) = 0; \quad f''(-1) = 0; \quad f''(2) = 3; \quad f''(-2) = 3.$$

We have $a'(x) = f(x^2) + 2x^2f'(x^2)$ and

$$a''(x) = 2xf'(x^2) + 4xf'(x^2) + 4x^3f''(x^2),$$

whence

$$a''(-1) = -2f'(1) - 4f'(1) - 4f''(1) = -2(-1) - 4(-1) = 6.$$

4.4.6 Put $F(x) = \int_a^x \sqrt{1+t^2}\,dt$, where a is a constant. Then

$$\int_x^{x^2} \sqrt{1+t^2}\,dt = \int_a^{x^2} \sqrt{1+t^2}\,dt - \int_a^x \sqrt{1+t^2}\,dt = F(x^2) - F(x).$$

By the Fundamental Theorem of Calculus, $F'(x) = \sqrt{1+x^2}$, and by the Chain Rule,

$$\frac{d}{dx} F(x^2) - F(x) = 2xF'(x^2) - F'(x) = 2x\sqrt{1+x^4} - \sqrt{1+x^2}$$

4.5.1 We have

$$\int_{-1}^{a} (x+1)^{100}dx = \int_{-1}^{a} d\left(\frac{(x+1)^{101}}{101}\right) = \frac{(x+1)^{101}}{101}\bigg|_{-1}^{a} = \frac{(a+1)^{101}}{101}.$$

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4.5.3 We have, integrating by parts,

$$\begin{aligned}
\int_0^1 xf'(x)dx &= \int_0^1 x\,df(x) \\
&= xf(x)\Big|_0^1 - \int_0^1 f(x)dx \\
&= 1f(1) - 0f(0) - c \\
&= b - c,
\end{aligned}$$

and integrating by parts twice

$$\begin{aligned}
\int_0^1 xf''(x)\mathrm{d}x &= \int_0^1 x\,\mathrm{d}f'(x) \\
&= xf'(x)\Big|_0^1 - \int_0^1 f'(x)\mathrm{d}x \\
&= 1f'(1) - 0f'(0) - (f(1) - f(0)) \\
&= 2a - (b - a). \\
&= 3a - b.
\end{aligned}$$

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