Making Sense of Mathematics:

A Faster and Surer Pathway for Developmental Students?

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Community College of Philadelphia

Professional Development, Spring 2011

Outline

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2001 Report Of The Institutional Research Office.

Longitudinal Study of 1732 students entering Math 016

Percentages:

	016					
100.0%	40.9%					
	15.0%	017				
	44.1%	16.2%				
		6.6%	118			
		21.3%	5.9%			
			12.7%	161		
			2.7%	1.4%		
				0.5%	162	
				0.8%	0.3%	
					0.3%	171
					0.2%	
						0.2%

Failed
Passed but Quit
Passed & Went On

Outline

Looking at Physics.

In order really to address the problem we are facing in Developmental Mathematics, we may turn to David Hestenes, a physicist whose use of Clifford Algebras as “the” language for physics, *Geometric Algebra*, was recognized in 2002 with the Oersted Medal[1].

Following are four excerpts from his Oersted lecture.

[1] The American Association of Physics Teachers’ most prestigious award, it recognizes notable contributions to the teaching of physics. Recipients include many Nobel laureates.

Conceptual Learning

1. Conceptual learning is *a creative act*.
2. Conceptual learning is *systemic*.
3. Conceptual learning is *context dependent*.
4. The quality of learning is *critically dependent on the quality of conceptual tools* at the learner’s command.
5. Expert learning *requires deliberate practice* with critical feedback.

The Necessity Of Conceptual Tools.

For students and scientists alike, what they know and learn about physics is profoundly shaped by the conceptual tools at their command.

Reconstructing Course Content.

Course content is taken [by many] as given, so the research problem is how to teach it most effectively. This approach [...] has produced valuable insights and useful results. However, it ignores the possibility of improving pedagogy by reconstructing course content.

Where To Start

To reform the mathematical language of physics,
you need to start all over at the most elementary level.

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Mathematics As Physics.

What is important is the real world, that is physics, but it can be explained only in mathematical terms.

Denis Serre. Bulletin of the AMS, Vol 47 Number 1

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- Model Theory, a branch of mathematics, deals with the correspondence between real-world structures and their paper-world theories.
- The goal then is for the students to realize that by developing symbolic systems, they will be able to:
  - Represent different aspects of the real world,
  - Develop paper-world procedures to parallel real-world processes.

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The Governing Ideas.

The Content Architecture, presented below in 30 chapters suitable for a 15 weeks (5-0-0) course, “is not perfect” if only because the unfolding of the story line is not entirely smooth. However, it is based on the many connections between Arithmetic and Algebra by way of only a very few simple ideas:

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- Operations are seen as *unary* rather than as *binary*, that is as functions. (As, historically, they used to.)
- Equations and Inequations appear in the context of “undoing” what a function does.
- Laurent polynomials generalize decimal numbers.

Symbolic System To Represent Bunches Of Items.

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2. Combinations of real-world collections can be represented by combination-phrases: 2 **Apples** + 3 **Bananas** where $+$ does not stand for addition but for “and” so that we should really write 2 **Apples** & 3 **Bananas**. Of course, what we have is a “basket”, AKA vector.

   By the way, the “Arithmetic of Baskets” leads to Linear Algebra done in a completely painless way.

The Place Symbolic System.

3 With “large” collections, say four hundred and seven one-dollar bills, we must exchange: TEN one-dollar bills for a ten-dollar bill, TEN ten-dollar bills for a hundred-dollar bill, etc. We then get a combination of collections, each with less than TEN items and thus representable using only $1, 2, \dots 8, 9$, and we can write 4 **Franklins** & 7 **Washingtons**. For computational purposes, we rewrite it as the array

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Roosevelts**
	4		7	

which we can then encode with base-TEN number-phrases such as:

0.407 **Clevelands**,
40.7 **Hamiltons**,
4070. **Roosevelts**, etc

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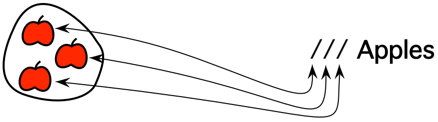
   - Exponential Suffixes to modify numerators. For instance,
     – $6.31 \times 10^{+3}$ codes[2] “6.31 with . $\xrightarrow{\text{3 places}}$” i.e. 6310.
     – 6.31×10^{-2} codes “6.31 with . $\xleftarrow{\text{2 places}}$” i.e. 0.0631

   Using arrays shows how the two work together.

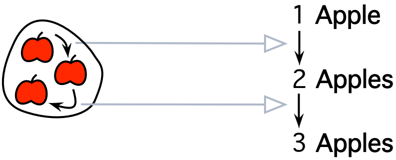
[2] At this point, $\times$ is only a separator and 10 is meaningless.

From The Real World To The Paper World.

By substituting for the cardinal viewpoint



the ordinal viewpoint



counting permits the development of procedures.

Procedures Based On Counting From . . . To . . . .

5. Comparison Of Collections: Equalities and Inequalities.

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5. Comparison Of Collections: Equalities and Inequalities.
6. States And Actions: Functions.
7. Attaching A Collection To A Collection.

   Adding To: $\text{5 Apples} \xrightarrow{\nearrow \text{2 Apples}} \text{7 Apples}$

   Detaching A Collection From A Collection.

   Subtracting From: $\text{5 Apples} \xrightarrow{\searrow \text{2 Apples}} \text{3 Apples}$

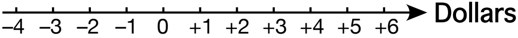
   $\text{5 Apples} \xrightarrow{\searrow \text{7 Apples}} ?$

Symbolic System To Represent Bunches of “Sided” Items.

The ancestor of Signed Counting is Double Entry Accounting which, to this day, remains a mathematical wonder.

Signed Arithmetic.

8 The balances of T-accounts can be encoded by signed number-phrases which can be imaged on rulers:



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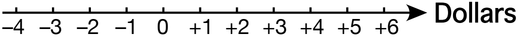

9. Absolute Comparisons versus Size Comparisons.

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   $-4 \quad -3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3 \quad +4 \quad +5 \quad +6 \longrightarrow$ Dollars

9. Absolute Comparisons versus Size Comparisons.
10. Translations: Moving back or forth from a given place:

$$\text{Milestone } +5 \xrightarrow{\leftarrow 7 \text{ Miles}} \text{Milestone } -2$$

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    - $\oplus$ defined in terms of $>$, $<$, $+$ and $-$ in **N**.
    - $\ominus$ shorthand for $\oplus$ Opposite.
      (Banks cannot *remove* an erroneous entry and can only cancel it by “adding the opposite” entry.)

Multiplication.

“*It Ain’t No Repeated Addition*”. (Devlin’s Angle, June 2008. Mathematical Association of America.)

> *Pretending there is just one basic operation on numbers (be they whole numbers, fractions, or whatever) will surely lead to [students] assuming that numbers are simply an additive system and nothing more. Why not do it right from the start?*

It’s Multiplications.

12 As:

- Times Table. (“Axiomatic” basis for *procedures*.)

  $3 \xrightarrow{2} 2 \times 3$ (In “times 2” table.)

- Dilation.

  $-3 \xrightarrow{-2} -3 \otimes -2$ (-3 flip-dilated by a factor of 2)

- Additive Power.

  4×7 shorthand for $\underbrace{4+4+4+4+4+4+4}_{\text{7 copies of 4}}$

- Measure.

  $(7,4) \longrightarrow 7 \times 4$ (Area of a 7 by 4 rectangle.)

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- Co-Multiplication: The real-world "Losing three bad apples that would have cost four cents each to dispose of is a twelve cents plus" is coded as

$$-3\,\cancel{\text{Apples}} \otimes -4\,\frac{\text{Cents}}{\cancel{\text{Apple}}} = +12\,\text{Cents}$$

where the co-denominator $\frac{\text{Cents}}{\text{Apple}}$ lives in the dual space.

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  - *The real real numbers are the decimal numbers.*
    (An Engineer.)
  - *No physical quantity has ever been measured with more than 15 or so digits of accuracy. Mathematicians, however, freely fantasize with infinite-precision real numbers. Nevertheless within pure mathematics the notion of a real number is extremely problematic.*
    (G. Chaitin. *How real are real numbers?* IBM Research.)

Decimal Arithmetic.

14 Smaller And Smaller:
– “That of which TEN can be exchanged for . . . ”,
– [...] read as “something too small to matter *here*”, that is as a precursor to Landau’s $o[h^n]$,
– Approximations.
– $7 \times 10^{\pm 3}$ is now shorthand[3] for 7 multiplied/divided by 3 copies of 10.

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Laurent Polynomials.

- Laurent polynomials are essentially decimal numbers with x instead of 10 as basis. Anticipating a bit:

$$7x^{+2} + 5x^{+1} + 8x^{0} + 6x^{-1} + 4x^{-2}\Big|_{x\leftarrow 10} = 758.64$$

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  (Where, here too, $+$ really ought to be read as “and”.)
- The algebra of Laurent polynomials is in fact “simpler” than the arithmetic of decimal numbers in that there is neither *carryover* nor *borrowing*.

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- Near x_0: $\frac{-12+23h-h^2-2h^3}{-3+2h} = +4 - 5h - 3h^2 + [...]$.

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  - explicit expression which note taking rarely produces.

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- Preempting many likely misunderstandings,
- Enabling the students to formulate their own questions,
- Leaving class time for discussions.

Explicative Texts, Though, Must Be Transparent.

- An idiosyncratic terminology is necessary because:
  – Every single thing requires a *specific* name.
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- There *can* be such a thing as *too much* explanation.

Explicative Texts Must Be Read (Pencil In Hand).

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- A difficulty remains in that learning from a text is something entirely new for Developmental Students.
- A simple, no-cost solution would be for the explicative text to be used as reading material in a linked Eng 098.

An English 098 Instructor’s View.

“*I did appreciate Schremmer’s linguistic approach. The students who stayed to the end also appreciated his methods whether they passed or not. If we pursue another link, the English teacher should definitely read the math text with the students. Unfortunately, because I had my own reading to do, we did not read the math in English class as we should have done.*”

Outline

No Prerequisite.

Nothing in the above contents requires from the students anything more than “common sense”. All that is necessary for the students to take advantage of the exponential character of conceptual learning is for them to be able to persevere.

However, and whatever the reasons, some students cannot:

- Spare the necessary time,

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However, and whatever the reasons, some students cannot:

- Spare the necessary time,
- Conceive of learning as anything other than doing on the exam what they were shown how to do in class.

Attitude Assessment instead of Knowledge Test.

So, rather than on the basis of prior knowledge, placement into the “alternative” should be on the basis of the student’s ability:

- To commit to a sustained effort in return for the instructor’s availability and help,

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- To commit to a sustained effort in return for the instructor’s availability and help,
- To accept that learning mathematics can be learning how to make the case that things are true or false.

Outline

“Free as in Free Speech and Free as in Free Beer”.

A freely modifiable implementation of the above Content Architecture, including explicative text and ancillaries, i.e. daily homework, daily quiz, reviews and exams, is freely available from `http://www.freemathtexts.org/Standalones/RBA/Downloads.php`.

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- Within the “model theoretic” setting used here, other architectures are quite conceivable.
- As opposed to the “model theoretic” approach, a “formalist” approach would also be quite conceivable.

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- Seems unlikely to harm Developmental Students,
- Would give more Developmental Students immediate access to college level courses.

Pathway To Where?

In particular, since the contents of the proposed “alternative” were reverse-engineered from the contents of Math 165, the two semester sequence of (4-0-0) courses, Math 165 - 166, Differential Calculus I and II, AKA the Mathematics of Change, would be a natural continuation of the proposed “alternative”.

1992 Report Of The Institutional Research Office.

- "Of those attempting the first course in their respective calculus sequence, 654 did so between Fall 1988 and Fall 1989[4]. While 67 (12.5%) of the 538 traditional track students finished that sequence, 56 (48.3%) of the 116 differential calculus sequence did likewise, revealing an association between the latter sequence and completion $(\chi^2(1) = 82.14), p < .001)$."

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- "If the additional requirement of passing Math 172 or Engineering 172 is added, 27 (5.0%) of the tradtional sequence students completed all courses, while 6 (5.2 %) of the 116 differential calculus students did likewise."

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4. This would be a five-fold improvement over the current situation.

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- Other courses might be adjusted to take advantage of the proposed “alternative”,
- Other “alternatives” might be designed that would be better suited for other courses and/or sequences.

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About three quarters of the contents of Math 165 were re-implemented a few years ago for Math 161[5]:

- The emphasis is on the analysis of individual "algebraic functions".

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  - The global analysis uses the local analysis near a generic x_0 but beyond that is of course more or less ad hoc.
- Derivatives are not defined by way of limits but, following Lagrange, as the functions outputting $n! \times$ the coefficient of h^n in the polynomial approximation of f near x_0.

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  12. Global Analysis: The *concavity* is given by a constant function. The *slope* is given by an affine function, so must be 0 somewhere. The number of 0-height inputs depends on Sign $f(x)|_{x \text{ near } x_{0\text{-slope}}}$ compared to Sign $f(x)|_{x \text{ near } \infty}$.

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  13. Cubic Functions: Local Analysis near ∞, x_0.
  14. Global Analysis. The *concavity* is given by an affine function so must be 0 somewhere. The *slope* is given by a quadratic function so the number of 0-slope inputs depends on Discriminant f'.

Part III Of Math 161

- Rational Functions.

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  15 Algebra Review:

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  15. Algebra Review:
      - Binomial Expansions ("Alternative" 29)
      - Divisions in Descending and Ascending Exponents ("Alternative" 30)
  16. Local Analysis near ∞ (Requires division in *descending* exponents).
  17. Local Analysis near x_0 (Requires division in *ascending* exponents).
  18. Global Analysis. Contrary to (plain) polynomial functions, rational functions have a wide variety of bounded graphs.

Differential Calculus.

The fourth hour in Math 165 then provides ample time to:

- Develop the recursive aspect of the derivatives,

Differential Calculus.

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- Focus more formally on continuity and smoothness,

Differential Calculus.

The fourth hour in Math 165 then provides ample time to:

- Develop the recursive aspect of the derivatives,
- Focus more formally on continuity and smoothness,
- Prove all the usual theorems.

Four Implementations of 165-166. Sort of.

- The materials developed under the 1989 NSF grant in the form of “task cards” about which I now have mixed feelings.

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- The materials that Mason developed subsequently.
- The materials mentioned above for 161 which make up about 3/4 of what would be needed in 165. As for the materials for 166, there are on my list of todos.
- Santos just put online a text which I have not yet read, `http://www.openmathtext.org/lecture_notes/Very_Basic_Calculus.pdf`. He mentions only 165 in the Preface but the contents include Exponential, Logarithmic and “Goniometric” Functions.