A Question About Content Architecture

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December 25, 2012

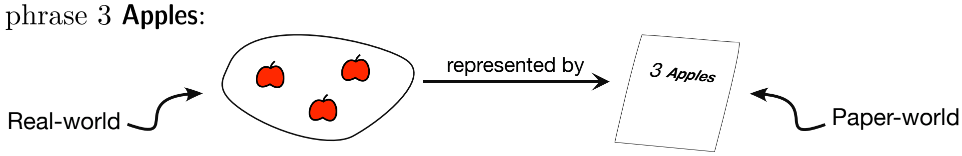
Abstract

The "content architecture" at the border of arithmetic and algebra is intrinsically a difficult one inasmuch as one is torn between looking back to (reviewing) arithmetic and looking forward to (introducing) algebra. It is even more delicate if one wants to facilitate the students' introduction to a mathematical world that makes enough sense to be predictable for them. A few specific questions are raised about which of (simple) formulas and (simple) functions should be preferred as starting point.

Background

The following is not directly relevant to the present issue and is only given to describe the environment in which the "content question" actually arose. The question remains in other environments.

1. The treatment is standard "model theoretic". For instance, a real-world collection of THREE ***apples*** is represented by the paper-world number-phrase 3 **Apples**:



Then, that the sentence 5 **Apples** > 2 **Apples** is TRUE is verified:

- In the real-world *cardinally*, that is by the fact that, when we try to match one-to-one a collection of FIVE ***apples*** with a collection of TWO ***apples***, there are ***apples*** remaining unmatched in the collection of FIVE ***apples***.
- In the paper-world *ordinally*, that is by the fact that, in order to count from 5 to 2 we must count *down*.

In *this* paper, the real-world will be taken for granted and the discussion will be entirely in the paper-world.

2. To fix the terminology used here[1], the standard, recursive definition of "sentence" is:

term
formula
sentence
data set
basic formula
connectors

a. Start with a vocabulary:

- Names: a, b, c
- Variables: x, y
- Verbs: unary: $\_P$, binary: $\_R\_$
- Operators: unary: $f(\_)$, binary: $g(\_,\_)$

b. Define **term** recursively:

i. a name is a term

ii. a variable is a term

iii. if u and v are terms. then $f(u)$ and $g(u, v)$ are terms

c. Define **formula** recursively:

i. if u and v are terms, then uP and uRv are atomic formulas

ii. If φ and ψ are formulas, $\neg\varphi$, $\varphi \,\&\, \psi$, $\varphi \vee \psi$ are formulas

iii. If φ is a formula, $\forall\varphi$ and $\exists\varphi$ are formulas

d. A **sentence** is a formula in which all variables are quantified

Formulas $\longrightarrow$ Operators

The following sketches what seemed to me to be a reasonable architecture: start with sentences and move to progressively more complicated formulas, eventually involving addition and multiplication. See `http://www.freemathtexts.org/Standalones/RBA/Downloads.php`.
A bit more precisely,

1. Given the vocabulary (for unsigned whole numbers):
Names: $1, 2, 3, \ldots$
Parametric Unary Verbs: $\_ = a$, $\_ \neq a$, $\_ < a$, $\_ > a$, $\_ \leqq a$, $\_ \geqq a$. Then,
a. Given a sentence, say $21 \leqq 17$, check its truth value.
b. Given a **data set**, for instance $\{3, 23, 16, 43, 2\}$, and a **basic formula**, say $x \leqq 16$, find the solution subset.

2. After having introduced addition, subtraction and multiplication using $\_ \xrightarrow{+b} \_$ (translation *i.e.* adding b) and $\_ \xrightarrow{a\,\cdot} \_$, (dilation *i.e.* multiply by a) and finally affine operators, extend the above, for instance with $x \oplus -6 > +5$, $+3x \leqq -12$, $-3x \oplus +7 > -8$.

3. Introducing the **connectors** BOTH, EITHER ONE OR BOTH, EITHER ONE BUT NOT BOTH, then:

[1] terms such as "open sentence", "statement", etc are often used instead.

a. Given two sentences, check the truth value of the two sentences connected by one of the connectors. Starting with, for instance:

$$\text{EITHER ONE OR BOTH} \begin{cases} -13.72 < +21.48 \\ -13.72 \geqq +3.05 \end{cases}$$

all the way up to, for instance

$$\text{BOTH} \begin{cases} -71.36 \odot -3.14 \oplus -13.62 \neq +21.78 \\ +21.28 \odot +4.2 \oplus -61.4 \leqq +3.05 \end{cases}$$

b. Given the signed decimal numbers as data set and two formulas connected by one of the connectors, graph the solution subset. For instance,

$$\text{EITHER ONE BUT NOT BOTH} \begin{cases} x \leqq +45.62 \\ x < -21.73 \end{cases}$$

all the way up to, for instance

$$\text{BOTH} \begin{cases} -71.36\, x \oplus -13.62 \neq +21.78 \\ +21.28\, x \oplus -61.4 \leqq +3.05 \end{cases}$$

Obviously, there are a number of issues with this approach. For instance:

- At what point, and how, should each of the four data sets

	Unsigned	Signed
Whole	$\mathbb{N}$	$\mathbb{Z}$
Decimal	$\mathbb{D}$	$\pm\mathbb{D}$

  be introduced?

- More generally, the interplay of the data sets with the above parametric unary operators, translation and dilation, is not even remotely touched upon and, even more of a drawback, this architecture leaves no room to do so. In other words, this is not just a sin of omission, it is a sin of commission.

- Another issue is that $x < a$ and $x + a$ look very much the same while the former is a formula that gives rise to a *sentence* and the latter is a ?[2] that gives rise to a *term*. It may appear to be rather inconsequential but, in fact, it sure does not help the students structure the contents.

[2]Not having a name for something to be discussed is a severe drawback with the intended student population.

Operators $\longrightarrow$ Formulas

reverse problem

Another seemingly reasonable architecture would then seem to be one in which the operators would be introduced first and where solving inequations would then appear as **reverse problems**. For instance, instead of asking, more or less out of nowhere, for the solution subset of $-3x \leqq +15$, it would seem more natural to ask "For which inputs does the function

$$x \xrightarrow{\;f\;} f(x) = -3x$$

return outputs less than or equal to +15".

This would allow the introduction of $\mathbb{Z}$, $\mathbb{D}$ and $\pm\mathbb{D}$ but there are difficulties with this architecture too. One is that functions might be less "natural" than formulas. After all, screening a collection according to some requirement is a time-honored activity, which certainly anti-dates functions. Another is that, even though it is only recently that addition has become *binary*, the teaching establishment does not seem even to recall that, for eons, adding has meant adding *to* and multiplying has meant multiplying *by*. The question then is how much of the binary view sticks to the students and what are the risks of conflict.

Comment

These questions arise only inasmuch as one wants to present the students with a treatment which flows "logically", both locally and globally. In my case, it arose because I am not satisfied with the discontinuities within each of Part I and Part II of Reasonable Basic Algebra as well as the lack of transition from one to the other.

Clearly, the goal is not to have anybody gape in admiration at how beautiful the flow is. Equally clearly, the flow cannot be entirely smooth: there will always be *gaps*. But what I want to avoid are *jumps*. I want the students to be able to ask "Why?" (instead of "How?") with a reasonable chance of their finding a reasonable answer. And, since my Reasonable Basic Algebra does not quite do that, I have been trying to modify it along the above lines. But one change calls for another and things tend to unravel.