From Arithmetic
To Differential Calculus
In Three Semesters

A Course of Study Based on Common Sense
With Inflammatory Notes
For the Mortification of "Educologists"
And the Vindication of "Just Plain Folks"[1]

A. Schremmer

July 18, 2006

[1]While, if need be, I take full responsability for coining the term "Educologist", the phrase "Just Plain Folks" first appeared in Calculus Syllabi, Report of the Content Workshop, J. Goldstein et al. in Toward a Lean and Lively Calculus, Ron. Douglas Ed., Tulane University 1986. MAA Notes.

Contents

objects
counter
represent
(black)board
semantic
syntactic

Chapter 1

Accounting For *Basic* Collections Of Money *On* A Counter

In this chapter we will be dealing with **objects** sitting on a **counter**/desk/table/etc and we will **represent** these *objects* on a **(black)board**/notebook/etc. We will then design *procedures* to be carried on the *board* to arrive at a representation of the result of what we did on the *counter*.

But, if the distinction between what sits on the *counter* and what we write on the *board* is quite clear in the classroom, it is not as easy to make in a *book* and here we will have to resort to various devices.

- Inasmuch as possible, we shall use *pictures* to stand for *objects* on the counter but, as this is not always possible, we shall also use their *usual name* but with a particular typeface so as to distinguish them from what we will write to represent them on the *board*.

  For instance, we will use ***dollar*** as an alternate for  to stand for a dollar-bill sitting on the counter while we will write **Dollar** to represent it on the *board*.
- Similarly, we shall use ONE, TWO, . . . , TEN, ELEVEN, etc, to stand for the *numbers* of objects sitting on the *counter*, with the firm understanding that, on the *board*, we can write only 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
- We shall use the symbol ♠ to signal that we are in the **semantic** mode, that is, *working on the counter* and the symbol ❖ to signal that we are in the **syntactic** mode, that is, *writing on the board*[1].

[1]If nothing else, this might help remind Educologists that what they write on the board

represent
denominator
collection
(counting) number-phrase
numerator
slash
/

1.1 Representing *Basic* Collections with (Counting) Number-Phrases

1. We begin with the issue of **representing** on the board money sitting on a *counter*. (Note by the way that banks used to be called "*counting* houses".)
For instance, given ***dollars*** and ***dimes*** on the counter, we use the words **Dollars** and **Dimes** as **denominators**[2], that is as names/symbols/denominations/etc to represent them on the *board*.

♠ On the *counter*.	❖ On the *board*.
We *have*  	We *write* **Dollar**, **Dollar**, **Dollar**, **Dollar**, **Dollar**, **Dime**, **Dime**

2. The first breakthrough in the development of arithmetic was the realization that objects of *different* kinds have to be accounted for *separately* because, when all the objects are of the *same* kind, we can *then* refer to them *collectively*, that is as a **collection**. For instance, we shall refer to ***dollar, dollar, dollar, dollar, dollar, dollar, dollar*** on the counter as a *collection* of ***dollars***. On the other hand, according to this agreement, ***dollar, dollar, dime, dime, dime, dime, dime*** will *not* be a *collection*. *What* would it be a collection of?
What this does is to allow us to represent a *collection* on the counter by *writing* on the board a **(counting) number-phrase**[3], that is a phrase consisting of:

- a **numerator** to indicate *how many* objects there are in the collection, which we do for the moment by writing on the board a **slash**, /, for each object in the collection on the counter, and,
- a *denominator* to indicate the *kind* of objects the collection is made of.

For instance,

ought to refer at least to *some* real world activity.

[2]An Educologist once warned me very, very sternly about "*the possible confusion in learners' minds between this use of the term and its association with fractions in the usual meaning of the word*"! More about this when we get to fractions.

[3]Words within parentheses can go without saying since they can always be recovered from the context. They are mainly meant to resolve ambiguities if and when these should occur.

♠ On the *counter*, we *have*:	❖ On the *board*, we *write*:
	/////// **Dollars**

number
quantity
quality
digit
1, 2, 3, ... , 9

We will say that the **number** of objects in a collection is *that* which[4] is represented by the *numerator* in the number-phrase that represents the collection on the board. Thus, while the *denominator* represents the *kind* of objects in a collection, the *numerator* represents the *number* of objects in a collection[5].

Observe that, even though a number-phrase is much more economical a way to represent on the board a collection of objects on the counter than writing one denominator for each object, there is *no* loss of information. Essentially, what we have done was merely to *separate* **quantity** *from* **quality** but, as it will turn out, this is a very powerful idea[6].

In particular, given, say, ***dollar***, ***dollar***, ***dollar*** on the counter, we can ask two very different questions:

- "*What* is on the counter?" whose answer on the board is the *number-phrase* /// **Dollars**
- "*How many* ***dollars*** are on the counter?" whose answer on the board is the *numerator* ///.

Note. We will need to make a distinction somewhat analogous to our use in English of "one dime" versus "a dime". We will distinguish between a *collection* consisting of one ***dime***, which we represent on the board by the *number-phrase* / **Dime**, and the *object* that a ***dime*** is, which we represent on the board by the *denominator* **Dime**. While this will surely appear as beyond nitpicking, not making the distinction would turn the development of board procedures into a nightmare.

3. The second breakthrough in the development of arithmetic occurred when Indian scribes introduced as *numerators* the **digits 1, 2, 3, ... , 9** to be used instead of /, //, ///, . . . , ///////// so that we now write, say,

[4]Educologists will surely note that this is a syntactic construction quite unfamiliar to most developmental students. However, it is a construction that we will find to be completely unavoidable.

[5]At least *one* reason for this roundabout, even underhanded, "definition" is that, with most people, *defining* numbers is counterproductive, if not insulting, as they know that, if they can do *one* thing, they definitely can count.

[6]As in TOPOLOGY versus GEOMETRY. In spite of which this is precisely the point where, in the name of "abstraction", Educologists cut their students away from *denominators* without noticing, of course, that this is exactly the point where they start losing them.

succession
procedure
count
basic collection

3 **Dollars** instead of /// **Dollars**.

a. Once we have *memorized* the **succession** 1, 2, 3, . . . , 9, what this does is to give us a **procedure** to find the *numerator* of the (counting) number-phrase that represents a given *collection* of objects on the board: we **count** the collection, that is we point in turn at each object in the collection, while reciting the succession of digits. The *numerator* we write on the board is the *last* digit recited in the count[7].

For instance,

♠ On the *counter*.	❖ On the *board*.
We *have* 	We *count* $\xrightarrow{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7}$ We *write* 7 **Dollars**

b. At this point, we can count only up to 9 **Dollars** because we cannot recite TEN as we have no symbol to represent TEN on the board. So, by a **basic collection**, we shall mean a collection with fewer than TEN objects which we can therefore count with just the above digits. At some fundamental level, *basic* collections are thus the only ones we can really *represent*[8]! Reaching "TEN" will be the signal for "bundling" as we shall see in Section 2.2.

Note. There is nothing sacred about TEN: it is simply how many fingers ("digit" is just a fancy word for finger) are on our two hands and we could have used just about any number of digits instead of TEN. For example, deep down, computers use a *machine language* based on TWO digits, 0 and 1, because any electronic device is either ***off*** or ***on***. At intermediate levels, computer software may use EIGHT (0, 1, 2, 3, 4, 5, 6, 7) or SIXTEEN digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f). The Babylonians used SIXTY digits, a historical remnant of which is the fact that there are SIXTY seconds to a minute and SIXTY minutes to an hour. The point here is that all that

[7]Educologists will surely note that, while we take a *cardinal* viewpoint on the *counter*, we take an *ordinal* viewpoint on the *board*. In other words, we use *ordinal* counting, a *syntactic* means, to arrive at a *semantic* end, the *cardinality* of a collection. This might indeed be an issue.

[8]Moreover, I shall use *basic* collections to illustrate issues that are independent of the number of objects in the collection(s).

we will do with TEN could easily be done with *any* number of digits[9].

default rule
compare
match one-to-one
relationship
leftover

c. A small complication is that the numerator 1 often "goes without saying" which has the unfortunate effect of obliterating the difference between *denominator* and *number-phrase*. This is often expressed as a **default rule**:

When no numerator is given, the numerator 1 is intended and goes without saying.

Note. Unfortunately, this default rule is often abbreviated as "when there is no numerator, the numerator is 1" which is dangerous because, when there is *no numerator*, it is tempting to think that there is *no object* either! To be on the safe side, *we* will avoid letting the numerator 1 go "without saying".

d. Finally, we note that we have not yet introduced the digit 0. This is only because, so far, we have had no *need* for it. In fact, historically, the digit 0 was a much later invention. It will be introduced in Section 1.5.

Note. Since we refer to, say, ***dollar, dollar, dollar, dollar, dollar, dollar, dollar*** as a collection of ***dollars***, it is tempting to "improve" a bit and write "a collection of 7 ***dollars***" but we should resist the temptation because ***dollars*** are *objects* that sit on the *counter* while 7 is something we write on the *board* and we don't want to mix what is written on the board with what sits on the counter. On the other hand, we can speak of a collection of SEVEN ***dollars***.

1.2 Comparing Collections: Equalities and Inequalities

We now want to **compare** collections—involving the *same* kind of objects. (We will compare collections involving *different* kinds of objects in Section 4.5.)

1. We begin with the *comparison* of two collections on the *counter* and with the board *procedure* for *getting* the result of the comparison. We will deal with the issue of how to *represent* this result on the board in sub-section **2.** below.

♠ On the *counter*, what we do is to **match one-to-one** the *objects* in the two collections; the particular **relationship** that stands between the two collections will depend on which of the two collections the **leftover** objects are in.

[9]Z. P. Dienes used to *start* with base-THREE ***arithmetic blocks*** and the digits 0, 1, 2.

count from ... to ...
is less numerous than
count forward
succeed
is more numerous than

❖ On the *board*, we count each one of the two collections and then we **count from** the numerator of the first number-phrase **to** the numerator of the second number-phrase, that is, starting *after* the numerator of the *first* number-phrase, we count to the numerator of the *second* number-phrase.

Either way, we then have three possibilities:

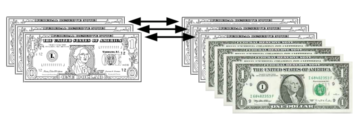
a. In the first case, that is

♠ When the leftover objects are in the *second* collection, we will say that the first collection **is less numerous than** the second collection[10].

❖ To count from the first numerator to the second one, *starting* with the digit *after* the first numerator, we must **count forward**, that is, we must call the digits that **succeed** it in the *succession* 1, 2, 3, 4, 5, 6, 7, 8, 9 and *end* with the second numerator.

For instance, $\xrightarrow{4,\,5,\,6,\,7}$ is a *forward* count that starts *after* 3 and ends with 7.

For instance,

♠ On the *counter*.	❖ On the *board*.
Jack has 	We *count* Jack's collection: $\xrightarrow{1,\,2,\,3}$
Jill has 	We *count* Jill's collection: $\xrightarrow{1,\,2,\,3,\,4,\,5,\,6,\,7}$
We match ***Jack***'s collection one-to-one with ***Jill***'s collection: 	We count from ***Jack***'s collection to ***Jill***'s collection: $\xrightarrow{4,\,5,\,6,\,7}$
Jack's collection is *less numerous than* ***Jill***'s collection	We must count *forward.*

b. In the second case, that is

♠ When the leftover objects are in the *first* collection, we will say that the first collection **is more numerous than** the second collection.

❖ To count from the first numerator to the second one, *starting* with the

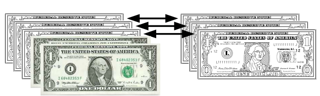
[10]Educologists may question this contrived term. Of course, the issue is to have different terms for use on the counter and to write on the board and only experience can tell if the difference is worth making.

digit *before* the first numerator, we must **count backward**, that is, we must call the digits that **precede** it in the *succession* 1, 2, 3, 4, 5, 6, 7, 8, 9 and *end* with the second numerator. For instance, $\xrightarrow{3,\ 4}$ is a *backward* count that starts *before* 5 and ends with 3.

count backward
precede
precession
is as numerous as

Note. Thus, the **precession** 9, 8, 7, 6, 5, 4, 3, 2, 1 should be memorized as well as the *succession* 1, 2, 3, 4, 5, 6, 7, 8, 9[11].

For instance,

♠ On the *counter*.	❖ On the *board*.
Jack has 	We *count* Jack's collection: $\xrightarrow{1,\ 2,\ 3,\ 4,\ 5}$
Jill has 	We *count* Jill's collection: $\xrightarrow{1,\ 2,\ 3}$
We match ***Jack***'s collection one-to-one with ***Jill***'s collection: 	We count from ***Jack***'s collection to ***Jill***'s collection: $\xleftarrow{3,\ 4}$
Jack's collection is *more numerous than* ***Jill***'s collection.	We must count *backward*.

c. In the third case, that is

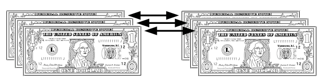
♠ When there are *no* leftover objects, we will say that the first collection **is as numerous as** the second collection.

❖ The two numerators are the same and we must count neither forward nor backward.

For instance,

[11]Should Educologists ask children to do so, they might discover that children actually *love* to count backward.

verb
sentence
$<$
is smaller than
$>$
is larger than
$=$
is equal to

♠ On the *counter*.	❖ On the *board*.
Jack has 	We *count* Jack's collection: $\xrightarrow{1,\ 2,\ 3}$
Jill has 	We *count* Jill's collection: $\xrightarrow{1,\ 2,\ 3}$
We match ***Jack***'s collection one-to-one with ***Jill***'s collection: 	We count from ***Jack***'s collection to ***Jill***'s collection:
Jack's collection is *equal* to ***Jill***'s collection.	We must count neither forward nor backward.

2. In order to *represent* on the board the *result* of comparing two collections, we first need to expand our *mathematical* language beyond *number-phrases*.

a. Given a *relationship* between two collections, we need a **verb** that represents the *relationship* between the two collections. Then we can write a **sentence** involving the two *number-phrases* that represent the collections and the *verb* that represents the *relationship* between the two collections:

- We will use the *verb* $<$ to represent the relationship *is less numerous than* and we will read it **is smaller than**. For instance, for the first of the above three examples, we will write the *sentence* 3 **Dollars** $<$ 7 **Dollars** which we will read "THREE ***dollars*** is smaller than FIVE ***dollars***".
- We will use the *verb* $>$ to represent the relationship *is more numerous than* and we will read it **is larger than**. For instance, for the second of the above three examples, we will write the *sentence* 5 **Dollars** $>$ 3 **Dollars** which we will read "FIVE ***dollars*** is larger than THREE ***dollars***".
- We will use the *verb* $=$ to represent the relationship *is as numerous as* and we will read it **is equal to**. For instance, for the third of the above three examples, we will write the *sentence* 3 **Dollars** $=$ 3 **Dollars** which we will read "THREE ***dollars*** is equal to THREE ***dollars***".

In other words,

strict inequality
equality
endpoint
bounded inequality
$\leqq$
less than or equal to
$\geqq$
more than or equal to
true
false
realize

When we must count *forward* $\xrightarrow{,\ \cdots\ ,}$	we write $<$	which is read as "is *smaller* than"
When we must count *backward* $\xleftarrow{,\ \cdots\ ,}$	we write $>$	which is read as "is *larger* than"
When we must *not* count either way	we write $=$	which is read "is *equal* to"

Note. Beware that the symbols $<$ and $>$ go in directions *opposite* to that of the arrowheads when we count from the first numerator to the second numerator. (If need be, one can think of $<$ as $\cdot :$ with $\cdot$ being "smaller" than $:$ and of $>$ as $: \cdot$ with $:$ being "larger" than $\cdot$.)

b. *Sentences* involving the *verbs* $>$ or $<$ are called **strict inequalities** while sentences involving the *verb* $=$ are called **equalities**. For example,

3 Dollars $<$ 7 Dollars and 8 Dollars $>$ 2 Dollars are *strict inequalities*
3 Dollars $=$ 3 Dollars is an *equality*

c. In English, when we say that we allow "up to" 5 Dollars, we mean that we allow 1 Dollar, 2 Dollars, 3 Dollars, 4 Dollars but that we do *not* allow the **endpoint** itself, 5 Dollars. If we do want also to allow the *endpoint*, 5 Dollars, we say "up to and including" 5 Dollars.
In mathematics we shall also need to make this distinction, that is, to allow or not to allow the *endpoint*, and, when we do allow it, we will say that the inequality is a **bounded inequality**:

- We will use the verb $\leqq$ to represent the relationship *is less numerous than or as numerous as* and we will read it **less than or equal to**.
- We will use the verb $\geqq$ to represent the relationship *is more numerous than or as numerous as* and we will read it **more than or equal to**.

d. Inasmuch as the sentences that we wrote above represented *actual* relationships between collections on the counter, they were **true** but there is of course nothing to prevent us from writing sentences that are **false** in the sense that there is no way that we could come up with *situations* that these sentences would represent. For example, the sentences

5 Dollars $=$ 3 Dollars and 5 Dollars $<$ 3 Dollars,

are *false* because there is no way that we could **realize** them on the counter, that is come up with actual collections with these *relationships*.
Observe, by the way, that the sentence

5 Dollars $\leqq$ 3 Dollars

negation
¬[ ]
slash
(linguistic) duality
(linguistic) symmetry
opposite

is *true*.

e. However, while occasionally useful, it is usually not very convenient to write sentences that are *false* because then we must not forget to write that they are false when we write them and we may miss that it says somewhere that they are false when we read them. So, inasmuch as possible, we shall write only sentences that are *true* and we will use the *default rule*:

When no indication of truth or falsehood is given, mathematical sentences will be understood to be true and this will go without saying.

When a sentence is *false*, rather than writing *it* and say that it is *false*, what we shall usually do is to write *its* **negation**—which is *true* and therefore which "goes without saying". We can do this either in either one of two manners:

- We can place the *false* sentence within the symbol ¬[ ],
- We can just **slash** the *verb* which is what we shall usually do.

For instance, instead of writing that

the sentence 5 **Dollars** $=$ 3 **Dollars** is *false*

we can either write the (true) sentence

¬[5 **Dollars** $=$ 3 **Dollars**]

or the (true) sentence

5 **Dollars** $\neq$ 3 **Dollars**

3. The **(linguistic) duality** that exists between $<$ and $>$ must not be confused with **(linguistic) symmetry**, a concept which we tend to be more familiar with[12].

a. Examples of linguistic *symmetry* include pairs of sentences—which may be *true* or *false*—such as the following:

- **Jack** is a child of **Jill** versus **Jill** is a child of **Jack**
- **Jill** beats **Jack** at poker versus **Jack** beats **Jill** at poker
- **Jack** loves **Jill** versus **Jill** loves **Jack**
- 9 **Dimes** $>$ 2 **Dimes** versus 2 **Dimes** $>$ 9 **Dimes**

In each example, the two sentences represent **opposite** relationships between the two people/collections because, even though the verbs are *the same*, the two people/collections are mentioned in *opposite* order.

Observe that just because one of the two sentences is *true* (or *false*) does *not*, by itself, automatically force the other to be either *true* or *false* and that whether or not it does depends on the *nature* of the relationship.

b. Examples of linguistic *duality* include:

[12]This confusion is a most important *linguistic* stumbling block for students and one that Educologists utterly fail to take into consideration.

dual
specify
requirement
satisfy

- **Jack** is a *child* of **Jill** versus **Jill** is a *parent* of **Jack**
- **Jill** *beats* **Jack** at poker versus **Jack** *is beaten by* **Jill** at poker
- **Jack** *loves* **Jill** versus **Jill** *is loved by* **Jack**
- 9 **Dimes** > 2 **Dimes** versus 2 **Dimes** < 9 **Dimes**

In each example, the two sentences represent the *same* relationship between the two people/collections because, even though the people/collections are *mentioned in opposite order*, the two *verbs* are **dual** of each other which "undoes" the effect of the order so that only the *emphasis* is different.
Observe that here, as a result, if one of the two sentences is *true*(or *false*) this *automatically* forces the other to be *true* (or *false*) and this regardless of the *nature* of the relationship.

c. The following are examples of simultaneous (*linguistic*) *symmetry* and (*linguistic*) *duality* because the *verbs* are *the same* and the order does *not* matter.

- **Jack** is a sibling of **Jill** versus **Jill** is a sibling of **Jack**
- 2 **Nickels** = 1 **Dime** versus 1 **Dime** = 2 **Nickels**

Observe that, here again, as soon as one sentence is *true* (or *false*), by itself this *automatically* forces the other to be *true* (or *false*) and that it does not depend on the *nature* of the relationship.

1.3 Specifying Collections: Equations and "Inequations"

In real life, we often have to **specify** things we want by stating some **requirement(s)** that these things must **satisfy**.

Here, we will *specify* collection(s) by the *requirement* that they stand in a given *relationship* with a given *collection*, namely one or the other of the following,

- *is more numerous than* the *given* collection,
- *is less numerous than* the *given* collection,
- *is as numerous as* the *given* collection.

For instance, say that

Jack has THREE ***dollars***,
Jill has SEVEN ***dollars***,
Dick has THREE ***dollars***,
Jane has FOUR ***dollars***.

and that we specify the collection(s) that satisfy the *requirement* that they be *more numerous than* ***Jack***'s collection.

1. We could of course proceed as we did in Section 1.2:

efficient
data
solution

♠ On the counter, matching ***Jack***'s collection one-to-one with each one of the collections of ***Jill***, ***Dick*** and ***Jane*** shows that this specifies the collections of ***Jill*** and ***Jane***.

❖ On the board, counting *from* ***Jack***'s collection each one of the collections of ***Jill***, ***Dick*** and ***Jane*** specifies the same collections.

This approach, though, is somewhat short of ideal if only because it would become very time-consuming with large numbers of collections to compare. So, what we want is to develop a board procedure that is more **efficient** in that the time it requires will not go up appreciably as the number of collections and of objects in the collections goes up.

2. Before we do that, though, we need a way to phrase *requirements* that lends itself to *procedural manipulations.*

a. Essentially, what we will do is to introduce the *mathematical* version of something common in everyday life, namely *forms* such as

$\boxed{}$ was President of the United States.

which, when we *fill* it it with some **data**, say,

Kissinger

produces a *sentence*, namely

$\boxed{\text{Kissinger}}$ was President of the United States.

which happens to be *false* while, when we fill it with the *data*

Bill Clinton

it produces the *sentence*

$\boxed{\text{Bill Clinton}}$ was President of the United States.

which happens to be *true.*

b. In the case of the above example,

♠ On the counter, we want the collections of ***dollars*** that *satisfy* the *requirement* that they be more numerous than THREE ***dollars***.

❖ On the board, we want the **solutions** of the form

$$\boxed{}\ \textbf{Dollars} > 3\ \textbf{Dollars}$$

Thus, from what we did above, we have that

- the *data* 7 produces the *sentence* $\boxed{7}$ **Dollars** $>$ 3 **Dollars** which is *true*,
- the *data* 4 produces the *sentence* $\boxed{4}$ **Dollars** $>$ 3 **Dollars** which is *true*,
- the *data* 3 produces the *sentence* $\boxed{3}$ **Dollars** $>$ 3 **Dollars** which is *false.*

so that 7, 4 are *solutions* of the *form* $\square$ **Dollars** $>$ 3 **Dollars** while 3 is a **non-solution**.

c. Boxes, though, would soon turn out to be impossibly difficult to use and, instead, we will use **unspecified numerators**, such as for instance the letter x, as in

$$x \textsf{ Dollars}$$

and, instead of the form $\square$ **Dollars** $>$ 3 **Dollars** we shall write

$$x \textsf{ Dollars} > 3 \textsf{ Dollars}$$

We shall call:

- **equations** those forms whose *verb* is $=$,
- **strict inequations** those forms whose *verb* is either $<$ or $>$,
- **bounded inequations** those forms whose *verb* is either $\leqq$ or $\geqq$.

d. Instead of *filling the box* with the data, say, 3, we **replace** x by 3 and the **instruction** to do so will be

$$|_{\text{where } x:=3}$$

in which the symbol **:=**, borrowed from a computer language called Pascal, is read as "is to be replaced by". Thus

$$x \textsf{ Dollars}|_{\text{where } x:=3}$$

is a **specifying-phrase** in that it *specifies*

$$3 \textsf{ Dollars}$$

The following sentence

$$x \textsf{ Dollars}|_{\text{where } x:=3} = 3 \textsf{ Dollars}$$

is therefore "trivially" *true*; it is an example of a type of sentence called **identity** because it **identifies** the numerator specified by the *specifying-phrase.*

We also have

- x **Dollars**$|_{\text{where } x:=7} > 3$ **Dollars**,
- x **Dollars**$|_{\text{where } x:=4} > 3$ **Dollars**,
- x **Dollars**$|_{\text{where } x:=3} \not> 3$ **Dollars**.

3. We now turn to the simplest possible instance of a more **general** problem which is that we shall now want *all* the collections, if any, that stand in a given relationship with a given collection.

For example,

non-solution
unspecified numerator
equations
strict inequation
bounded inequation
replace
instruction
:=
specifying-phrase
identity
identify
general

solution set
break-even point
associated equation
pick
test
test-point

♠ Say ***Jack*** has FIVE ***dollars*** on the *counter*. We then want to find *all* collections of ***dollars*** that satisfy a given one of the following three *requirements*:
i. *is less numerous* than ***Jack***'s collection,
ii. *is more numerous* than ***Jack***'s collection,
iii. *is as numerous* as ***Jack***'s collection.
(In other words, we are looking here at *three* distinct problems at once.)

❖ On the *board*, we are looking for the **solution set** of the corresponding *inequation/equation*:
i. x **Dollars** < 5 **Dollars**
ii. x **Dollars** > 5 **Dollars**
iii. x **Dollars** $= 5$ **Dollars**

We now proceed to do just so.
Regardless of which one of the three requirements we are trying to satisfy, we begin by considering the *equation*

$$x \text{ Dollars} = 5 \text{ Dollars}$$

whose *solution set* contains of course one, and only one, numerator: 5.
Then,

a. If it was the *equation* we were trying to solve, we are of course done.

b. If we were trying to solve either one of the *inequations*

$$x \text{ Dollars} < 5 \text{ Dollars} \quad \text{or} \quad x \text{ Dollars} > 5 \text{ Dollars},$$

it remains to determine which side of the **break-even point** the *solution set* of the *inequation* is. (The *break-even point* is the solution of their **associated equation**, x **Dollars** $= 5$ **Dollars**, that is, of the *equation* obtained from the *inequation* by replacing the verb, $<$ or $>$, by the verb $=$.)

That the solution set must be a *whole side* of the break-even point is because if the solution set was only *part* of a whole side, then there would have to be both a *solution* and a *non-solution* on the *same* side of the break-even point and then there would have to be *another* break-even point in-between the two. But that cannot be since a break-even point is a solution of the *associated equation* x **Dollars** $= 5$ **Dollars** and we just saw that it has one and *only* one solution, namely 5.

So, on each side of the *break-even* point, all we need to do is to **pick** *one* numerator and **test** it against the wanted requirement, that is to ask whether this **test-point** is a *solution* or a *non-solution*: Then, every numerator on the same side of the break-even point as the test-point will be the same.

curly brackets
endpoint
operation
initial situation
action
terminal situation
direct problem

For instance, say we are looking at the *inequation*

$$x \text{ Dollars} > 5 \text{ Dollars}$$

The *associated equation* is

$$x \text{ Dollars} = 5 \text{ Dollars}$$

so that the *break-even point* of the *inequation* is 5. Then, on each side of 5, we pick a *test-point.* Say we *pick* 3 and 7. Since to count from 3 to 5 we have to count *forward*, 3 is *not* a solution and *all* numerators on the same side of 5 as 3 will *not* be solutions either. Since to count from 7 to 5 we have to count *backward*, 7 *is* a solution and *all* numerators on the same side of 5 as 7 *will* also be solutions so that the solution set of the *inequation*

$$x \text{ Dollars} > 5 \text{ Dollars}$$

is 6, 7, 8, ...

Note. It is customary, though, to write solutions sets in-between **curly brackets** as in $\{6, 7, 8, \dots\}$ and we shall follow the custom.

Observe that the time we spent with the above procedure does *not* depend anymore on the number of collections we are dealing with.

Observe that, here, the *break-even point* is also an **endpoint** in that *all* the numerators on the *one* side of the break-even point *are* solutions and *all* the numerators on the *other* side of the break-even point are *not* solutions. This, though, will *not* be always the case and we will encounter *break-even points* that will turn out *not* to be *endpoint*s.

1.4 Aggregating To A Collection. Addition.

Comparing collections is static in that nothing gets *created* and we now turn to **operations** on collections which are dynamic in that:

i. Starting from a given **initial situation**,
ii. We perform some **action** on the initial situation,
iii. Which results in some **terminal situation**.

Given an *operation*, we will be considering different types of *problems* that can be associated with the operation. In the simplest type, which we shall call **direct problems**, given an *initial* situation and an *action*, we want to find the *terminal* situation[13]. (We call this a *direct* problem because it "goes in the *same* direction" as the *operation.*)

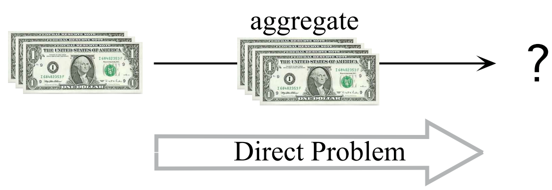
[13] Educologists will of course recognize this as the problem which consists, given an input, of finding the output under a given function.

aggregation
aggregate
aggregate collection
+
addition
specifying-phrase
specify

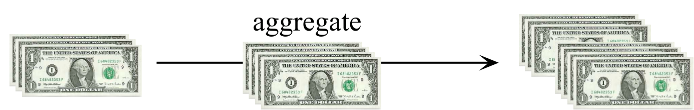
In this section, we consider *direct problems* associated with **aggregation**, an operation in which the *initial* situation involves a collection of objects, the *action* is to **aggregate** another collection of objects (of the same kind) and the *terminal* situation then involves the **aggregate collection**, namely the collection obtained by collecting all the objects in the two collections into one single collection[14].

For instance, a *direct problem* might be

♠ Aggregating FOUR ***dollars*** to THREE ***dollars***:



Performing the *action* of collecting all the objects in the two collections



gives the collection in the *terminal* situation:  .

❖ On the board,

i. In order to *state the problem* we use the symbol + to denote **addition**, the *procedure* that will give us the *numerator* of the number-phrase that represents the *aggregate* collection, and we write

$$3 \textsf{ Dollars} \xrightarrow{+\,4 \textsf{ Dollars}} 3 \textsf{ Dollars} + 4 \textsf{ Dollars}$$

where $3 \textsf{ Dollars} + 4 \textsf{ Dollars}$ is the **specifying-phrase** that represent on the board the *aggregate* collection *before we count it*. We shall call it a specifying-phrase because, while it is not a *number-phrase*, it does **specify** a *number-phrase* namely that which will be the *result* of the addition.

More succinctly, but less transparently, we shall usually write only the specifying phrase

$$3 \textsf{ Dollars} + 4 \textsf{ Dollars}$$

[14] In other words, we are not introducing addition as a *binary operation* but as a *unary operation*, the pro and con of which Educologists will readily see.

ii. In order to **identify** the collection specified by the specifying-phrase[15], we *count* the *initial* collection and then **forward count** the collection being aggregated, that is, starting *after* the count of the *initial* collection, we call the digits that *succeed* it in the succession 1, 2, 3, 4, 5, 6, 7, 8, 9 while pointing at the objects in the collection being *aggregated*. For instance, $\xrightarrow{4,\,5,\,6,\,7}$ is a *forward* count that starts *after* 3 and ends with 7. The numerator of the number-phrase that represents the *aggregate* collection is the *end* of the forward count.

identify
forward count
identifying sentence

iii. In order to *represent* the solution of the direct problem, we write a *sentence* which we will call an **identifying sentence** because it *identifies* the number-phrase that was specified by the specifying-phrase:

$$3\ \textsf{Dollars} \xrightarrow{+\ 4\ \textsf{Dollars}} 3\ \textsf{Dollars} + 4\ \textsf{Dollars} = 7\ \textsf{Dollars}$$

or, more succinctly,

$$3\ \textsf{Dollars} + 4\ \textsf{Dollars} = 7\ \textsf{Dollars}$$

Note. A *specifying-phrase* such as $3\ \textsf{Dollars} + 4\ \textsf{Dollars}$ is of course not to be confused with a *sentence* such as $3\ \textsf{Dollars} < 4\ \textsf{Dollars}$.

Altogether then,

[15] Educologists will surely wonder why not just say "compute". It is a matter of *connotation*: "identify" leaves the question open while "compute" implies that the question is already closed and displaces the student's attention to "how" to get the "answer".

general statement
reverse problem

♠ On the *counter*.	❖ On the *board*.
We have	We write the *specifying phrase*[16] 3 **Dollars** + 4 **Dollars**
	We count the 1st collection: $\xrightarrow{1,\,2,\,3}$
to which we aggregate 	We count the 2nd collection *forward* starting *after* 3: $\xrightarrow{4,\,5,\,6,\,7}$
The *aggregate collection* is:	The *numerator* of the result is 7.
	We write the *identifying sentence* 3 **Dollars** + 4 **Dollars** = 7 **Dollars**

So, we have the **general statement**:

When we aggregate on the counter, we add on the board by counting forward.

1.5 Removing From A Collection; Subtraction. (A *Reverse* Problem.)

We saw in Section 1.4 that, given an *operation*, a *direct* problem consists in performing a given action on a given initial situation and thus getting to some terminal situation—whatever that may turn out to be. In this section we consider a rather different kind of problem coming out of the fact that we are usually not ready to accept whatever terminal situation may happen to come up but, rather, that we usually have a *goal* in mind, namely a *specific* terminal situation that we *want*.

Generally speaking, we shall call **reverse problem** any problem involving a *wanted* terminal situation but there are two *types* of reverse problem depending on what else is *given* aside from the *wanted* terminal situation.

- If it is also the *initial situation* that is given, then what we must find is what *action* on this initial situation will get us the *wanted* terminal

[16] Educologists will surely appreciate this being "question oriented".

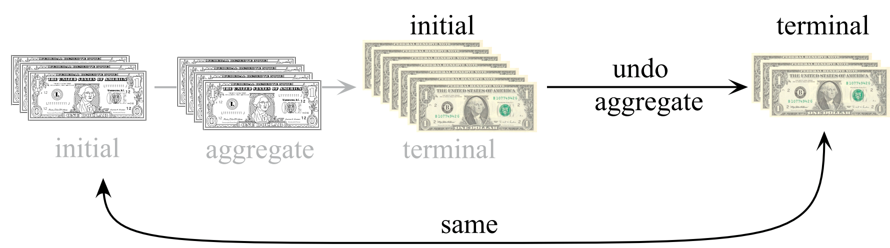
undo
remove

situation.

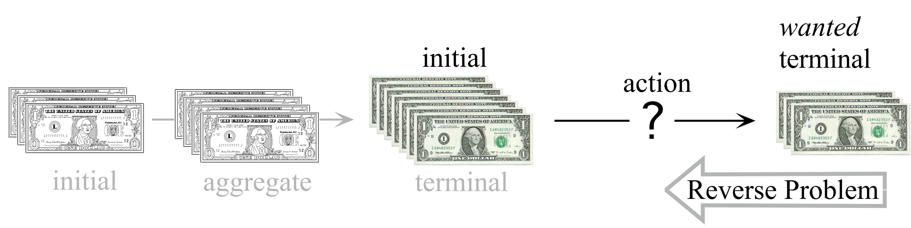
- If it is also the *action* that is given, then what we must find is for what *initial situation* will this action get us the *wanted* terminal situation.

In either case we call this a *reverse* problem because, since it starts from a wanted *terminal* situation, it "goes in the *opposite* direction" from the actual *operation*.

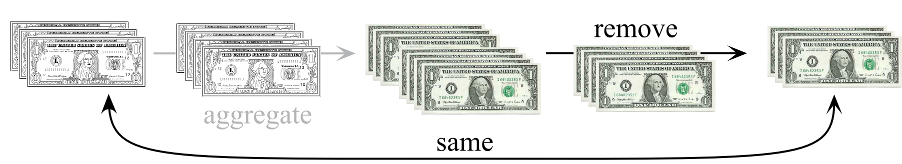
1. A special instance of a *reverse* problem arises when we want to **undo** the result of an *action*, that is, when we want to "return" from the *terminal* situation to the *initial* situation.
For instance, we might want to *undo* the *aggregation* of a collection to an *initial* collection.



The *reverse problem* associated with the "undo aggregate" operation then is to find what *action* on the *original* terminal situation will get us back to the *original* initial situation:



♠ On the *counter*, we must **remove** from the *original* terminal collection the collection that had been *aggregated* to the *original* initial collection.



❖ On the *board*, we count the terminal collection and then we count *backward* from the numerator of the *terminal* collection the collection that

subtraction
−

had been *aggregated*. (This makes sense since, in the original operation, we obtained the numerator of the *terminal* collection by counting *forward* from the numerator of the *initial* collection the collection being *aggregated*.)
We shall say that we **subtract** the numerator of the collection being *aggregated*—in the original operation—from the numerator of the *terminal* collection and we use the symbol − to write, for instance,

$$7\ \textsf{Dollars} \xrightarrow{-\ 4\ \textsf{Dollars}} 7\ \textsf{Dollars}\ -\ 4\ \textsf{Dollars}$$

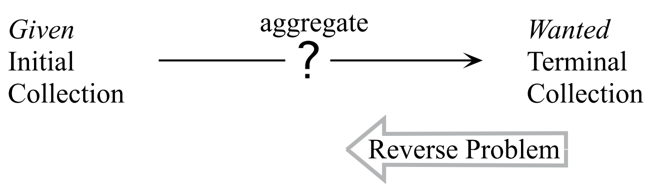
where 7 **Dollars** − 4 **Dollars** is the *specifying-phrase* that represent on the board the *leftover* collection *before we count it*, namely that which will be the *result* of the subtraction.

In other words,

♠ On the *counter*.	❖ On the *board*.
From the *terminal* collection in the original operation, 	We write the *specifying phrase* 7 **Dollars** − 4 **Dollars**
	We count the *terminal* collection: $\xrightarrow{1,\,2,\,3,\,4,\,5,\,6,\,7}$
we *remove* the collection that had been *aggregated* 	We count the *aggregated* collection *backward* starting *after* 7: $\xleftarrow{3,\,4,\,5,\,6}$
The *leftover* collection	The *numerator* of the *initial* collection is 3.
 is the *initial* collection.	We write the *sentence* 7 **Dollars** − 4 **Dollars** = 3 **Dollars**

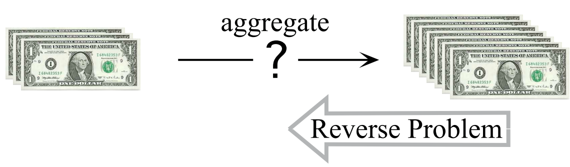
2. We now turn to a reverse problem that is more *general* in that, given *any* initial collection and *any* wanted terminal collection, we shall now want to find what collection (if possible) we have to *aggregate* to the *given* initial collection to get the *wanted* terminal collection.

empty



Here, though, because the *wanted* terminal collection need not have resulted from the aggregation of a collection, but can now be *any* collection, the reverse problem, as we shall see, may or may not have a *solution*. There are three cases.

a. The *wanted* terminal collection is *more numerous than* the *given* initial collection:



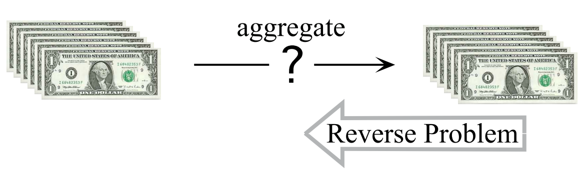
To find the collection to be aggragated, we proceed essentially as when we wanted to undo aggregation

♠ On the *counter*, we *remove* the *initial* collection from the wanted *terminal* collection.

❖ On the *board*, we *subtract* the numerator of the *initial* collection from the numerator of the wanted *terminal* collection, that is, we count the *initial* collection *backward* from the numerator of the wanted *terminal* collection.

partial undo

b. The wanted terminal collection is *as numerous as* the initial collection.



♠ There is *just* enough in the *wanted* terminal situation to remove the given first collection but, rather than to say that there *is* no *leftover* collection, we shall say that the leftover collection is **empty**.

❖ We subtract the *initial* numerator from the *terminal* numerator, that is, after we have counted the *wanted* terminal collection we count the

0 initial collection *backward* but here we must introduce a new digit, **0**, to end the backward count. The digit 0 is thus the numerator in the number-phrase that represents any *empty* collection[17].

Note. As a matter of historical fact, 0 was invented much later than the other digits and *not* for this purpose. We shall see the historical purpose in Section **??**

c. The wanted terminal collection is *less numerous than* the initial collection.

♠ There is *not* enough in the *wanted* terminal situation to remove the given first collection. The reverse problem has no solution.

❖ On the board, we cannot *subtract* the *initial* numerator from the *terminal* numerator because we cannot count *backward* more than we counted in the first place!
Thus, say, specifying-phrases such as 3 **Dollars** − 5 **Dollars** make no sense whatsoever.

So, we have the *general statement*:

When we remove on the counter, we subtract on the board by counting backward.

3. We can now look at more complicated problems in which we would be looking for the *solution set* of one of the following

i. 3 **Dollars** $+ x$ **Dollars** $<$ 7 **Dollars** or 3 **Dollars** $+ x$ **Dollars** $\leqq$ 7 **Dollars**
ii. 3 **Dollars** $+ x$ **Dollars** $>$ 7 **Dollars** or 3 **Dollars** $+ x$ **Dollars** $\geqq$ 7 **Dollars**
iii. 3 **Dollars** $+ x$ **Dollars** $=$ 7 **Dollars**

For instance, say the *initial* situation is that ***Jack*** has THREE ***dollars*** that he will donate to ***Jill*** but that the *wanted* terminal situation is that ***Jill*** should have a collection *more numerous than* SEVEN ***dollars***. The question thus is what collection should be aggregated to ***Jack***'s collection.

♠ On the counter, removing the THREE ***dollars*** in ***Jack***'s collection from SEVEN ***dollars***, we find that FOUR ***dollars*** are leftover. So, if we *aggregate* FOUR ***dollars*** to ***Jack***'s collection, then the aggregated collection will be *as numerous as* ***Jill***'s collection and aggregating a collection more numerous than FOUR ***dollars*** to ***Jack***'s collection will make ***Jill***'s collection *more numerous than* SEVEN.***dollars***

❖ On the board
   – ***Jack***'s collection is represented by 3 **Dollars** and ***Jill***'s collection is represented by 7 **Dollars** and thus we are trying to find the solution(s),

[17] Educologists will be interested to note that students have no trouble with 0 **Dollar** being different from 0 **Dime** while they certainly have trouble with "*the* empty set".

if any, of the *inequation*

$$3\ \textsf{Dollars} + x\ \textsf{Dollars} > 7\ \textsf{Dollars}$$

– To obtain the *break-even point*, that is the solution of the *associated equation*,

$$3\ \textsf{Dollars} + x\ \textsf{Dollars} = 7\ \textsf{Dollars}$$

we must *identify*

$$7\ \textsf{Dollars} - 3\ \textsf{Dollars}$$

that is we must count from 3 to 7:

$$3 \xrightarrow{4,\ 5,\ 6,\ 7} 7$$

which is a *forward* count of 4. Thus the *break-even point* is $7\ \textsf{Dollars} - 3\ \textsf{Dollars} = 4\ \textsf{Dollars}$.

– We pick a test-point on each side of the break-even point, say 2 **Dollars** and 5 **Dollars**.
By counting from 3, we get:

$$3\ \textsf{Dollars} + 2\ \textsf{Dollars} \not> 7\ \textsf{Dollars}$$

and

$$3\ \textsf{Dollars} + 5\ \textsf{Dollars} > 7\ \textsf{Dollars}$$

So the *solution set* of $3\ \textsf{Dollars} + x\ \textsf{Dollars} > 7\ \textsf{Dollars}$ is $\{5, 6, 7, \ldots\}$.

1.6 Combinations.

Situations in the real world are rarely that simple that they only involve one single kind of objects. As it turns out, though, only a small but far-reaching adjustment needs to be made to what we have done so far.

1. When the objects are *not all of the same kind*, that is when we do *not* have a collection and therefore we cannot represent them by a (counting) number-phrase.
For instance, say we have ***dime***, ***dime***, ***nickel***, ***nickel***, ***nickel***, ***nickel***, ***nickel***, on the counter. Of course, we could write 7 **Coins** but then we would be losing information, for instance, about how much *money* there is. Moreover, what could we write to represent, say, ***dollar***, ***dollar***, ***dime***, ***dime***, ***dime***, ***dime***, ***nickel***, ***nickel***, ***nickel***?

bunch
&
combination
attach

In the latter case, for instance, and in accordance with the "second breakthrough in the development of arithmetic" (Section 1.1), we begin by *separating* the objects into a **bunch**[18] of collections:

- the collection ***dollar***, ***dollar***, which we can represent by the (counting) number-phrase 2 **Dollars**
- the collection ***dime***, ***dime dime***, ***dime***, which we can represent by the (counting) number-phrase 4 **Dimes**
- the collection ***nickel***, ***nickel***, ***nickel***, which we can represent by the (counting) number-phrases 3 **Nickels**

Then, we represent the *bunch* on the board by writing 2 **Dollars** & 4 **Dimes** & 3 **Nickels** where **&** means "and". We will call this a **combination**[19] of **Dollars**, **Dimes** and **Nickels**. Thus *combinations* represent on the board *bunches* of collections on the counter.

Combinations are a very powerful concept that comes up again and again and that, in fact, is the subject of a whole part of mathematics called LINEAR ALGEBRA[20].

Note. Here again, it is usual to write, say, 2 **Dollars** & **Dime** but while we *see* of course that **Dollars** is the denominator in a number-phrase whose numerator is 2, we have to *remember* that **Dime** is not a *denominator* but really stands for a *number-phrase* whose numerator is 1 and whose denominator is **Dime** so that 2 **Dollars** & **Dime** stands for 2 **Dollars** & 1 **Dime**. *We* will always write, say, 2 **Dollars** & 1 **Dime** rather than 2 **Dollars** & **Dime**.

2. In the absence of any additional information, we cannot *compare* bunches of collections. We shall see in Section **??** what kind of information permits what kind of comparison.

3. When two collections consist of *different* kinds of objects, we cannot *aggregate* them since the result would not be a *collection* but a *bunch* of two collections.

However, we can **attach** bunches of collections and the result is still just

[18] I am avoiding the terms *set*, *group* and *bundle* since they are pre-empted.

[19] The term has better connotations than **vector**

[20] There, the **vector-phrase** $2\vec{i} + 3\vec{j}$ represents the answer to the question "*What* is on the counter?" while the **numerator-pair** $(2, 3)$ represents the answer to the question "*How many* are on the counter?".

By the way, in LINEAR ALGEBRA too, the *denominators* corresponding to the "dimensions" the numerators refer to are usually "understood". Which may explain why the *dual space* is never introduced up front. See NOTMU, Fall 2003.

another bunch represented by a combination[21]. For instance,

$$
\begin{aligned}
2\ \textsf{Dollars}\ \&\ 3\ \textsf{Nickels} + 4\ \textsf{Dollars}\ \&\ 5\ \textsf{Dimes} = {} & \textsf{Dollar},\ \textsf{Dollar}, \\
& \textsf{Nickel},\ \textsf{Nickel},\ \textsf{Nickel}, \\
& \textsf{Dollar},\ \textsf{Dollar},\ \textsf{Dollar},\ \textsf{Dollar}, \\
& \textsf{Dime},\ \textsf{Dime},\ \textsf{Dime},\ \textsf{Dime},\ \textsf{Dime}, \\
= {} & 6\ \textsf{Dollars}\ \&\ 5\ \textsf{Dimes}\ \&\ 3\ \textsf{Nickels}
\end{aligned}
$$

4. Neither, when two collections consist of *different* kinds of objects, can we cannot *remove* one from the other.
Occasionally, we can *detach* one bunch from another and the result being usually a bunch. For instance,

$$
\begin{aligned}
7\ \textsf{Dollars}\ \&\ 5\ \textsf{Nickels}\ \&\ 9\ \textsf{Dimes} & \ - 4\ \textsf{Dollars}\ \&\ 1\ \textsf{Dime} \\
& = 7\ \textsf{Dollars} - 4\ \textsf{Dollars}\ \&\ 5\ \textsf{Nickels}\ \&\ 9\ \textsf{Dimes} - 1\ \textsf{Dime} \\
& = 3\ \textsf{Dollars}\ \&\ 5\ \textsf{Nickels}\ \&\ 8\ \textsf{Dimes}
\end{aligned}
$$

However, most of the time we cannot as, for instance, in

$$7\ \textsf{Dollars}\ \&\ 5\ \textsf{Nickels} - 4\ \textsf{Dollars}\ \&\ 3\ \textsf{Dimes}$$

[21] Educologists will of course recognize this as addition in a *vector space*.

extended
bundle
parentheses

Chapter 2

Accounting For *Extended* Collections Of Money *On* A Counter

We now turn to the case of collections that are **extended** in that they have more *objects* than we have *digits* so that we cannot represent them on the board with *just* the digits we have[1].

2.1 Bundles and Exchanges

We begin by introducing two ideas relating to collections that we shall use *systematically*.

1. The first idea is that, given any *collection* of objects, we can **bundle** it into ONE object of a *new* kind.

a. For instance, banks *wrap* collections of FIFTY ***dimes*** into ONE ***roll-of-dimes***.

♠ Similarly, we may thus have on the counter, say, a ***bundle-of-THREE-dimes*** as a new kind of object.

❖ We shall represent on the board this *bundle*, a new kind of *object*, by (3 Dimes), a new kind of *denominator* in which the **parentheses** represent the *wrapping*.

[1]It is not clear, though, that this is the right place for something which, after all, is mostly a *syntactic* matter. For instance, there might be some advantage in delaying it until after the next chapter since the latter is also, if from a different viewpoint, a natural extension of the preceding chapter. So, which of the two progressions would make the development smoother for the students remains for Educologists to investigate.

unpack
multiplication tables
shorthand

b. Then, of course, we can count *collections* of bundles.

♠ For instance, we may have on the counter ***bundle-of-TWO-dimes, bundle-of-TWO-dimes, bundle-of-TWO-dimes***.

❖ We represent this on the board by writing (2 **Dimes**), (2 **Dimes**), (2 **Dimes**) and therefore 3(2 **Dimes**) in which 3 is the *numerator* and (2 **Dimes**) is the *denominator*.

c. Now we look at what happens when we **unpack** the bundles.

♠ For instance, unpacking ***bundle-of-TWO-dimes, bundle-of-TWO-dimes, bundle-of-TWO-dimes*** gives ***dime, dime, dime, dime, dime, dime***.

❖ On the board, we write, rather naturally

$$\begin{aligned} 3(2\ \textsf{\textbf{Dimes}}) &= (2\ \textsf{\textbf{Dimes}}),\ (2\ \textsf{\textbf{Dimes}}),\ (2\ \textsf{\textbf{Dimes}}) \\ &= \textsf{\textbf{Dimes}}, \textsf{\textbf{Dimes}}, \textsf{\textbf{Dimes}}, \textsf{\textbf{Dimes}}, \textsf{\textbf{Dimes}}, \textsf{\textbf{Dimes}} \\ &= 6\ \textsf{\textbf{Dimes}} \end{aligned}$$

where we obtained the numerator by *counting* as above. But we can also write

$$\begin{aligned} 3(2\ \textsf{\textbf{Dimes}}) &= 2\ \textsf{\textbf{Dimes}} + 2\ \textsf{\textbf{Dimes}} + 2\ \textsf{\textbf{Dimes}} \\ &= 6\ \textsf{\textbf{Dimes}} \end{aligned}$$

where we used *addition.* And, finally, we can just write

$$3(2\ \textsf{\textbf{Dimes}}) = 6\ \textsf{\textbf{Dimes}}$$

where we used the **multiplication tables** that we memorized as children[2].

d. There is of course nothing to prevent us to bundle collections that we cannot represent on the board.

♠ For instance, we can bundle ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime*** as a ***bundle-of-TEN-dimes***.

❖ And while we *cannot* represent
dime, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***
on the board, we *can* represent a ***bundle-of-TEN-dimes*** as
(**Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**).
Instead however, we shall write (TEN **Dimes**) as a **shorthand** for the bundle (**Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**).

[2]Educologists will note that we studiously avoid speaking of "multiplication". Indeed, the concept involved here is an entirely different one, namely **additive power**. The multiplication tables are used here merely as a *syntactic procedure* devoid of any *meaning*.

2. The second idea is that since a *bundle* is an object, we can **exchange** it for another object, of a different kind. This will be particularly useful as dealing with bundles can be quite cumbersome. exchange

♠ While we *cannot* represent TEN ***dimes*** on the board, we can *bundle* it as ***bundle-of-TEN-dimes***, and then *exchange* the bundle for a new object, a ***dollar***, which is an object of another kind that we *can* also represent on the board.

❖ On the board, we will write (TEN **Dimes**) = 1 **Dollar**.

2.2 Representing Extended Collections With Combinations of (Counting) Number-Phrases

The general idea will be to reduce the number of objects we have to deal with until we need only count *basic* collections.

We will achieve this by *bundling* the collections and then counting the *bundles* of which there are necessarily fewer than there were objects in the original collections. If the collection of bundles is *basic*, then we can count it. If not, then we bundle the collection of bundles. Etc.

Actually, in the case of *money*, rather than to deal with the *bundles* themselves, we usually *exchange* the bundles for other *objects*. Then we have collections of these other objects we can bundle. Etc. This avoids having to deal with bundles of bundles of bundles ..., which, however, is what we shall have to do when, in Section 4.1, we shall count extended collections of *discrete goods*.

In other words, when dealing with money, we will be using the two ideas that we discussed in Section 2.1.

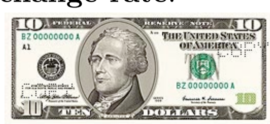
1. We begin with an example.

♠ Suppose we have, say, the following collection on the counter:
dime, ***dime***.
As we have no digit to represent on the board TWENTY-THREE ***dimes*** on the counter,
  – we bundle TEN ***dimes*** (out of the TWENTY-THREE ***dimes*** on the counter),
  we bundle another TEN ***dimes*** (out of the THIRTEEN ***dimes*** left on the counter),
  – we then exchange each bundle of TEN ***dimes*** for ONE ***dollar***.

And so, instead of the original collection, we now have on the counter a *bunch* of two *basic* collections:
dollar, ***dollar***
dime, ***dime***, ***dime***.
which we will be able to represent on the board as a *combination*.

❖ On the board, we write correspondingly:
Dime, **Dime**.
= (TEN **Dimes**), (TEN **Dimes**), **Dime**, **Dime**, **Dime**.
= **Dollar**, **Dollar**, **Dime**, **Dime**, **Dime**.
= 2 **Dollars** & 3 **Dimes**.
where we wrote (TEN **Dimes**) as a *shorthand* for the bundle (**Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**). (Section 2.1.)

2. Of course, the problem is that we will have to keep introducing *ever-larger* denominators. For the time being, we will stick to bills that change at a TEN to ONE **exchange rate**:

- ***Hamiltons***: 
- ***Franklins***: 
- ***Cleveland***: 

In other words, we shall disregard ***five-dollar-bills***, ***twenty-dollar-bills*** and ***fifty-dollar-bills***. (However, we will deal with these later on.)

To emphasize the distinction between the actual *bills* on the counter and the denominators on the board, **Hamiltons**, **Franklins**, **Clevelands** as denominators for ***ten-dollar-bills***, ***hundred-dollar-bills*** and ***thousand-dollar-bills*** and also, since consistency is a virtue we will want to practice, **Washingtons** instead of **Dollars**. But, for lack of a better word, we will keep writing **Dimes** as denominator for ***dimes***.

3. We are now in a position to deal with larger and larger collections, such as, for instance, occur when we aggregate *bunches* of collections.

♠ For instance, when we want to aggregate EIGHT ***dollars*** and FIVE ***dimes*** with SIX ***dollars*** and THREE ***dimes***, we bundle TEN of the resulting FOURTEEN ***dollars*** on the counter and exchange the bundle for a ***ten-dollar-bill***.

heading

❖ The (board) representation of this is:

[8 **Washingtons** & 5 **Dimes**] + [6 **Washingtons** & 3 **Dimes**]
= FOURTEEN **Washingtons** & 8 **Dimes**
= 1 **Hamilton** & 4 **Washingtons** & 8 **Dimes**

4. As a result of all this, we can now represent collections of all the way up to NINETY NINE THOUSAND NINE HUNDRED NINETY NINE ***dimes*** which we write on the board as 9 **Clevelands** & 9 **Franklins** & 9 **Hamiltons** & 9 **Washingtons** & 9 **Dimes**. In order to go beyond that, we would have to continue to create ever "larger" denominators. Instead, we shall eventually develop another, even more systematic approach.

2.3 (Decimal) Headings

To simplify this representation, a somewhat natural idea would be to write the denominators only *once* and then just write the numerators, "as needed". The problem, though, is how to indicate which numerator goes with which denominator.

What we do is to write the *denominators* into a **heading** such as

Clevelands	Franklins	Hamiltons	Washingtons	Dimes

and then write the *numerators* directly under the corresponding denominators.
For instance, we write the combination 3 **Franklins** & 1 **Washington** & 7 **Dimes** as follows:

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
	3		1	7

Thus, each *column* corresponds to a collection of a *different kind* of objects. While this may look like going from cumbersome to *very* cumbersome, we will now see how easy it is to *work* with.

2.4 Adding Under A Heading

In Section 1.4, in the case of *basic* collections, we defined *addition* as the (board) procedure that represents *aggregation* of collections on the counter.

Here, we focus on how addition works out, under a heading, for *extended* collections.

❖ Suppose, for instance, that we wanted to add the three combinations,

8 **Hamiltons** & 7 **Dimes**
8 **Washingtons** & 2 **Dimes**
5 **Hamiltons** & 3 **Washingtons** & 4 **Dimes**

First, we rewrite the three combinations under the heading:

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		8		7
			8	2
		5	3	4

Adding up the **Dimes** by "counting on our fingers":

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		8		7
			8	2
		5	3	4
				THIRTEEN

gives us THIRTEEN **Dimes** but we cannot *write* that. So, we *bundle* TEN of the THIRTEEN ***dimes*** and, *if* we can *exchange* the bundle for ONE ***dollar*** then we can write[3]:

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
			1	
		8		7
			8	2
		5	3	4
				3

Adding up the **Washingtons** by "counting on our fingers":

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
			1	
		8		7
			8	2
		5	3	4
			TWELVE	3

[3]No Educologist has ever bothered to explain what "carry over" is supposed to *evoke* in this context.

gives us TWELVE **Washingtons** but we cannot *write* that. So, we *bundle* TEN of the TWELVE ***dollars*** and, *if* we can *exchange* the bundle for ONE ***ten-dollar-bill***, then we can write:

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		1	*1*	
		8		7
			8	2
		5	3	4
			2	3

Finally, adding the **Hamiltons** by "counting on our fingers":

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		1	*1*	
		8		7
			8	2
		5	3	4
		FOURTEEN	2	3

gives us FOURTEEN **Hamiltons** but we cannot *write* that. So, we *bundle* TEN of the FOURTEEN ***ten-dollar-bills*** and, *if* we can *exchange* the bundle for ONE ***hundred-dollar-bill***, then we can write:

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
	1	*1*	*1*	
		8		7
			8	2
		5	3	4
	1	4	2	3

Thus, *if* we have *changing facilities*, we can then carry out the addition and the result will be 1 **Franklin** & 4 **Hamiltons** & 2 **Washingtons** & 3 **Dimes**.

2.5 Subtracting Under A Heading

In Section 1.5, we defined *subtraction*, in the case of *basic* collections, as the (board) procedure that represents what is left on the counter of a collection after we have taken a number of objects. Here, we focus on how addition works out, under a heading, for *extended* collections.

1. Occasionally, we have enough of each kind of bills to take what we want.

♠ Suppose for instance that SEVENTY-EIGHT ***dollars***, in the form of SEVEN ***ten-dollar-bills*** and EIGHT ***one-dollar-bills***, are on the counter and that we want TWENTY-FIVE ***dollars***, in the form of TWO ***ten-dollar-bills*** and FIVE ***one-dollar-bills***.

❖ On the board, what we want to do is the subtraction

7 **Hamiltons** & 8 **Washingtons** − 2 **Hamiltons** & 5 **Washingtons**

which we rewrite under a heading as follows:

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		7	8	
		2	5	

Then, counting 5 steps backward from 8, $\xleftarrow{3,\,4,\,5,\,6,\,7,\,8}$, gives us 3 and counting 2 steps backward from 7, $\xleftarrow{5,\,6,\,7}$, gives us 5.

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		7	8	
		2	5	
		5	3	

that is 5 **Hamiltons** & 3 **Washingtons**.

2. Most of the time, though, we need to *exchange* and *unpack*.

♠ Suppose for instance that, instead of SEVENTY-EIGHT ***dollars***, there is only SEVENTY-THREE ***dollars*** on the counter, in the form of SEVEN ***ten-dollar-bills*** and THREE ***one-dollar-bills***. and that we want TWENTY-FIVE ***dollars***, in the form of TWO ***ten-dollar-bills*** and FIVE ***one-dollar-bills***. The problem we immediately run into is that THREE ***one-dollar-bills*** is not enough from which to take FIVE ***one-dollar-bills***. However, if we can *exchange*[4] ONE ***ten-dollar-bill*** for a bundle of TEN ***one-dollar-bills***, then we can unpack the bundle so that we now have THIRTEEN ***one-dollar-bills*** from which we can take the FIVE ***one-dollar-bills***. But now, of course, we have only SIX ***ten-dollar bills*** from which to take the TWO ***ten-dollar-bills***. Altogether, this leaves us with FOUR ***ten-dollar-bills*** and EIGHT ***one-dollar-bills***.

❖ On the board, we write:

[4]Educologists will readily explain why they use the term "borrowing" even though it is both completely unfounded and perfectly non-explanatory.

specify
reconstruct

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		7	3	
		2	5	

and then

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		~~7~~ 6	THIRTEEN	
		2	5	

from which we get

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		~~7~~ 6	THIRTEEN	
		2	5	
		4	8	

and so the result of the subtraction is 4 **Hamiltons** & 8 **Washingtons**.

2.6 Decimal Number-Phrases

1. While headings are convenient when we want to *work* with *several* combinations, they are much too cumbersome if all we want is just to *write* *one* combination. So we will now develop yet another way in which it will be convenient *both* to *write* a single combination and to *add* several combinations.

a. Suppose we wanted to **specify** by mail a whole lot of combinations. A natural idea would be to mail the *heading*

Clevelands	Franklins	Hamiltons	Washingtons	Dimes

once and for all[5] so that, in later mails, the heading could now go "without saying" and we would just have to mail the *numerators*.
For instance, rather than mailing the combination

7 **Franklins** & 2 **Hamiltons** & 4 **Washingtons**

we would just mail the numerators 7, 2, 4.
The problem with that, though, is that the recipient would not be able to **reconstruct** the specified combination as s/he wouldn't know under which

[5]This would be somewhat similar to what software engineers call a *declaration*.

(decimal) number-phrase
(select) denominator
pick
(decimal) pointer
decimal point
(decimal) numerator

denominators in the heading to write these numerators and thus wouldn't be able to recover the combination.

For instance, s/he wouldn't know if the numerators 7, 2, 4 should be placed under the heading this way

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
7	2	4		

or that way

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		7	2	4

or *any* other way.

So, we cannot just mail the *numerators* and, along with the numerators, we must also mail some information as to *how* the numerators are to be placed under the heading.

b. To that purpose, we introduce a new type of number-phrases, **(decimal) number-phrase**, which consists of:

- the *numerators* that we want to mail,
- a **(select) denominator** that *we* can " **pick**" from among the denominators in the heading, that is whose choice is entirely *ours*.
- a **(decimal) pointer** to indicate which of the numerators is to be placed under the (select) denominator, the agreement being that the other numerators are to be placed "accordingly".

Note. For a little while, rather than a *dot*, we shall use $\leftharpoondown$ for a *decimal pointer*. This is to remind us that, by tradition, a decimal pointer points to its *left* even though we read from left to right. However, after this section, we shall conform and use a **decimal point**. (The use of a dot as decimal pointer is not universal and, for instance, many languages use a *comma* instead of a *dot*.) The regrettable thing about decimal *points*, as opposed to decimal *pointers*, is that we will have to *remember* that the digit being pointed at is to the *left* of the decimal point.

All the individual numerators, together with the decimal pointer, will be collectively referred to as **(decimal) numerator**. Thus, just like a (counting) number-phrase consists of a *numerator* and a *denominator*, a (decimal) number-phrase consists of a (*decimal*) *numerator* and a (*select*) *denominator*.

For instance, after we have mailed the heading, we can specify the combination 7 **Franklins** & 2 **Hamiltons** & 4 **Washingtons** by mailing the (decimal) *number-phrase* 0

$72_{\leftarrow}4$ **Hamiltons**

where we “picked” **Hamiltons** as our (select) denominator and where $72_{\leftarrow}4$ is the corresponding (decimal) *numerator*. The recipient would then *reconstruct* the specified combination as follows:

- since 2 is being pointed at, s/he would place 2 under **Hamiltons** since it is the (select) denominator,
- since 7 is left of 2, s/he would place 7 under **Franklins**, the denominator left of **Hamiltons**, and,
- since 4 is right of 2, s/he would place 4 under **Washingtons**, the denominator right of **Hamiltons**.

that is, s/he would write

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
	7	2	4	

which is indeed 7 **Franklins** & 2 **Hamiltons** & 4 **Washingtons**.

c. There is however a problem with combinations such as 5 **Franklins** & 3 **Hamiltons** & 8 **Dimes** which, under a heading, looks like

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
	5	3		8

Say we “pick” **Franklins** as our (select) denominator. The problem is that what is immediately to the right of **Hamiltons** is **Washingtons** and not **Dimes**. So, we cannot write $5_{\leftarrow}38$ **Franklins** because the recipient would reconstruct *that* (decimal) number-phrase as

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
	5	3	8	

and thus as specifying 5 **Franklins** & 3 **Hamiltons** & 8 **Washingtons**.

In order to specify 5 **Franklins** & 3 **Hamiltons** & 8 **Dimes** we must indicate that there are no **Washingtons**. To that purpose, we use the digit **0** which we introduced in Section 1.5 and we write $5_{\leftarrow}308$ **Hamiltons** and the recipient will reconstruct *that* as

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
	5	3	0	8

which is 5 **Franklins** & 3 **Hamiltons** & 0 **Washingtons** & 8 **Dimes** and indeed the same as 5 **Franklins** & 3 **Hamiltons** & 8 **Dimes**.

Note. This is in fact what 0 was invented for.

2. There are two default rules. (We now use the decimal *point* . instead of the decimal *pointer* $\leftharpoondown$.)

- When the decimal point is to the right of the (decimal) numerator, as in 7204. **Dimes**, it is customary not to write the decimal point at all and just to write 7204 **Dimes**. The corresponding default rule is:

  When there is no decimal point, it goes without saying that the decimal point is to the right of the (decimal) numerator.

- When there is *no doubt* as to what the (select) denominator is, say that it is **Hamiltons**, it is customary not to write it and, for instance, just to write 72.04 instead of 72.04 **Hamiltons**. The corresponding default rule is:

  When there is no (select) denominator, it goes without saying that it is the (select) denominator that was picked earlier on.

  This, though, is *extremely dangerous* because it depends on us *remembering* what the (select) denominator is that was picked earlier on[6]. So, while it is being done all the time, *we* shall not.

Note. In the U.S., it is usual to write, for instance, .56 **Hamiltons** rather than 0.56 **Hamiltons**. *We* shall *not* do so in this text, if only because we don't like the idea of a pointer pointing, at least apparently, at nothing. In any case, the "energy saving" would not be worth it.

3. To add and subtract (decimal) number-phrases we can just place them back under a heading. When the (decimal) number-phrases have the same (select) denominators, we need not even write the heading but just make sure that the pointed digits are lined up and that, in fact, is exactly what we were taught to do in school.

When the (decimal) number-phrases have different (select) denominators, we may begin by changing to a *common* (select) denominator but, keeping the heading in mind and with a little bit of practice, it is almost as easy to line up the numerators "as if" they were under the heading.

4. The use of (decimal) number-phrases has several *advantages*:

a. As we already saw, *we* can pick as our (select) denominator any denominator *we* want. For instance, a rich person might pick **Clevelands** as her/his (select) denominator while the rest of us would probably pick **Washingtons** as our (select) denominator.

[6] This is perhaps one more occasion to remind Educologists that memory is the weakest part of the mind, that it is the first to go and that, so far, humans can be defined as thinking entities, that is entities amenable to logic, rather than mere memorizing ones.

b. Not only does using (decimal) number-phrases allow us to *pick* as our (select) denominator any denominator we want but, by placing the (decimal) number-phrase back under the heading, we can easily *change* the denominator we want to use as our (select) denominator. Of course we then need to adjust the decimal point accordingly.

For instance, placing 85.7 **Hamiltons** under a heading,

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
	8	5	7	

we see that the (decimal) number-phrase 85.7 **Hamiltons** could just as well be written as any of the following (decimal) number-phrases:

0.857 **Clevelands**
8.57 **Franklins**
857. **Washingtons**
8570. **Dimes**

Any of the above means the same as 8 **Franklins** & 5 **Hamiltons** & 7 **Dimes**.

c. Another advantage of (decimal) number-phrases is that if, for whatever reason, we needed to have a "smaller" or a "larger" (decimal) numerator, it would be easy to do. We would "move the decimal point" by placing the number-phrase back under a heading, change the *digit* being pointed and change the (select) *denominator* accordingly.

d. We can now *count* beyond 9: $\xrightarrow{1,\ 2,\ 3,\ \ldots,\ 9,\ 10,\ 11,\ \ldots,\ 99999.}$

e. Finally, there is another, intriguing, advantage to the use of (decimal) number-phrases. While we can now count all the way up to 9 **Clevelands** & 9 **Franklins** & 9 **Hamiltons** & 9 **Washingtons** & 9 **Dimes**, that is all the way up to 99999 **Dimes**, what if we wanted to add 1 **Dime**? Let us do it under a heading:

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
9	9	9	9	9 1

We have no trouble reaching the following stage (in the presence of exchange facilities):

in the hundreds
in the ones
in the tenths
in the hundredths
in the thousands
principal unit

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
1	*1*	*1*	*1*	
9	9	9	9	9
				1
TEN	0	0	0	0

At this point, although we can *bundle* the TEN ***thousand-dollar-bills***, we cannot *exchange* the bundle for any bill. Nevertheless, as long as we "pick" any of the *existing* denominators for our (select) denominator, then we *can* write a number-phrase. For instance, if we "pick" Franklins, we can write the result as 100. Franklins or, if we "pick" Dimes, as 100000. Dimes.
Note. To some extent this would be cheating because we *still* do no have a denominator corresponding to the leftmost 1 but, other than that, everything looks fine and we could even say that we wrote the (decimal) number phrases *pending the creation* of that new denominator! But, in case you should worry, we shall in fact develop *several* ways in Section **??** to get *automatically* as many denominators as we need.

2.7 Comparing Orders of Magnitude

While both 3 Franklins and 7 Washingtons are larger than 8 Dimes, they are not so in the same manner. We shall say that

- 3 Franklins is **in the hundreds**,
- 7 Washingtons is **in the ones**,
- 8 Dimes is **in the tenths**.

More generally, we shall say that
0.00875 Hamiltons is **in the hundredths** as we can see from
0.00875 Hamiltons = 8.75 Cents
and that 42187.53 Dimes is **in the thousands** as we can see from
42187.53 Dimes = 4.218753 Clevelands
All of the above is in terms of Washingtons being the **principal unit**.

state

Chapter 3

Accounting For Money Changing Hands *Over* The Counter

We now deal with collections that, for whatever reason, are *marked* either one of *two* ways[1].

The simplest example comes up when we want to undo aggregating ***steps*** on a path. When we wanted to undo aggregating ***apples***, what we did was to remove them. But, when we want to undo aggregating ***steps*** on a path, we cannot do that by *removing* these steps—which would require going back in time—but only by walking the same number of steps in the *opposite* direction, that is *backward*. So we have to be able to count both ***steps forward*** and ***steps backward***. But then, for instance, after we have walked forward three steps forward, we can aggregate five steps backward because, altogether, this amounts to walking two steps backward. However, in the case of steps on a path we will need to deal both with positions and walking steps.

3.1 States

We will call **state** a collection of objects that, *as a whole*, can be on *this-side* or *that-side* of some *benchmark*.

1. First, a few real-world examples.

[1]It is difficult to understand what causes Educologists to delay the introduction of integers until *after* fractions.

signed-number-phrase
(side-) sign
signed-numerator
standard side
opposite side
transaction

• Being in such and such *solar year*. Thus, with Christ as benchmark, we can have THREE HUNDRED FORTY FIVE ***years after*** (345 AD) as well as THREE HUNDRED FORTY FIVE ***years before*** (345 BC).
• Being at such and such *temperature*. Thus, we can have +15°C as well as −15°C with the temperature at which water starts *freezing* as benchmark.
• Being in such and such *financial state*. Thus, FIVE ***dollars*** "ahead of the game" and FIVE ***dollars*** "in the hole" are examples of states a gambler can be in while FIVE ***dollars*** "in the black" and FIVE ***dollars*** "in the red" are examples of states a business can be in.
• Being at such and such *point* on a path.

2. On the board, we will represent a *state* by a **signed-number-phrase** that consists of:

- a **(side-) sign** to represent the *side* of the benchmark the collection is,
- the *numerator* that represents the *number* of objects in the collection,
- the *denominator* that represents the *kind* of objects in the collection.

However, because this will make *procedures* on the *board* a lot simpler, we will lump the *side-sign* together with the *numerator* of the number-phrase that represents the number of objects in the state and speak of a **signed-numerator** which we will separate from the *denominator*.
First, we record on the board, once and for all, which *side* of the benchmark is to be the **standard side**. States on the *other* side of the benchmark will be said to be on the **opposite side**. Then we need only use, say, ↑ to represent the *standard* side and ↓ for the *opposite* side.
For instance, say that ***in-the-black*** is on the *standard* side so that ***in-the-red*** is on the *opposite* side. Then,

♠ On the *counter*, we *look* at:	❖ On the *board*, we *write*:
FIVE ***dollars*** ***in-the-black***	(5 ↑) **Washingtons**
THREE ***dollars*** ***in-the-red***	(3 ↓) **Washingtons**

where (5 ↑) and (3 ↓) are the *signed-numerators* and **Washingtons** is the *denominator*. Thus, *signed-number-phrases* will be to *states* what *number-phrases* are to *collections*.

3.2 Transactions

We will call **transaction** a collection of objects that, *as a whole*, can go *this-way* or *that-way* over the counter. Then, just as with *states*, together

direction
standard direction
opposite direction

with the *number* and *kind* of objects in the collection, we will need to represent the **direction** of the transaction, that is the *way* the collection is going over the counter which we do very much in the same manner as with *states*. First we record on the board, once and for all, which way is to be in the **standard direction**. Transactions going the *other* way will be in the **opposite direction**. Then we need only use, say, → to represent the *standard* direction and ← for the *opposite* direction.

For instance, say that going from **Jack** to **Jill** is in the *standard* direction so that going from **Jill** to **Jack** is in the *opposite* direction.

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:
	Jack **Jill**
FIVE ***dollars*** ***from-Jack-to-Jill***	(5 →) **Washingtons**
THREE ***dollars*** ***from-Jill-to-Jack***	(3 ←) **Washingtons**

where (5 →) and (3 ←) are the *signed-numerators* and **Washingtons** is the *denominator*. Thus, *signed-number-phrases* will be to *transactions* the same as what they are to *states* and what *number-phrases* are to *collections*.

3.3 Standard Representations: Signed-Number-Phrases versus T-Accounts

Of course, in practice, we do not use *arrows* but the ways *mathematicians* and *accountants* represent *states* and *transactions* are quite different.

1. Instead of arrows, *mathematicians* "re-use" the + sign for the *standard* side and the *standard* direction and the − sign for the *opposite* side and the *opposite* direction and write the sign *ahead* of the numerator. Moreover, the parentheses are usually omitted. Thus,

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:
FIVE ***dollars*** ***in-the-black***	+5 **Washingtons**
THREE ***dollars*** ***in-the-red***	−3 **Washingtons**

and,

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:
	Jack **Jill**
FIVE ***dollars*** ***from-Jack-to-Jill***	+5 **Washingtons**
THREE ***dollars*** ***from-Jill-to-Jack***	−3 **Washingtons**

T-account
double-entry bookkeeping
balance

Note. Actually, the "usual way" is to let the + sign "go without saying" and to mark only, with the − sign, the states on the *opposite* side and the transactions in the *opposite* direction. The problem with this practice, though, is that it tends to blur on the board the distinction between *states* on the *standard side* or transactions in the *standard direction* and *collections* just sitting on the counter. So, *we* shall always write +5.

2. *Accountants* use **T-accounts**. While the rules for operating with T-accounts can, at least initially, appear a bit intricate, **double-entry bookkeeping** is tremendously powerful and well worth the effort of understanding its basic principles[2]. Here, we will just give a few indications. (For lack of space, we will use here $ instead of **Washington** as *denominator*.)

a. A *state*, called **balance** in ACCOUNTING, is represented by a line in the corresponding T-account.

In the following examples, money in the *black* is represented on the *left* side of the T-account and money in the *red* is represented on the *right* side of the T-account.

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:	
	Jill	
	Black	*Red*
Jill is THREE ***dollars*** in the ***red***		$3

and

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:	
	Jack	
	Black	*Red*
Jack is FIVE ***dollars*** in the ***black***	$5	

b. A *transaction* is represented by a line in the T-accounts of the *two* individuals involved in the *transaction*.

In the following example, money *coming into* the account is represented on the *black* side of the T-accounts while money *going out of* the account is represented on the *red* side of the T-accounts.

[2]Of course, Educologists have never shown any interest in such crass matters which is quite regrettable in view of the Grothendieck construction of $\mathbb{Z}$ as $\mathbb{N}^2/\sim$ where $\sim$ is the equivalence relation of debit-credit pairs modulo the balance, that is $(a,b) \sim (c,d)$ iff $a + d = b + c$. Moreover, the "law of money conservation", $\int_{start}^{end} Net\ Income(t) = Position(t)|_{start}^{end}$, is a rather nice instance of the Fundamental Theorem of the Calculus.

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:			
	Jack		**Jill**	
	Black (*In*)	*Red* (*Out*)	*Black* (*In*)	*Red* (*Out*)
FIVE ***dollars from-Jack-to-Jill***		\$5	\$5	
THREE ***dollars from-Jill-to-Jack***	\$3			\$3

merge
string
addition of signed-number-phrases
⊕

Note. Instead of "*write the signed-number-phrase for* a transaction", accountants say "*enter* a transaction" just as, instead of saying "*write the number-phrase for* a collection", we say "*count* a collection".

However, with the advent of computerized accounting, T-accounts are increasingly giving way to signed-number-phrases.

3.4 Adding Signed-Number-Phrases.

Suppose that, just like we *aggregated* collections *on* the counter, we now

- **merge** *states*, each on this or that side of the same benchmark

or

- **string** *transactions*, each going one way or the other (but between the same two people).

followed by

Then, just like *addition of number-phrases* was the board procedure that gave us the number-phrase that represents the result of *aggregating collections*, **addition of signed-number-phrases** will be the procedure that will give us the signed-number-phrase that represents the *result* of *merging states* or *stringing transactions*.

We will need a new symbol to distinguish addition of *signed*-number-phrases from addition of *counting*-number-phrases. But, as usual, mathematicians dislike introducing new symbols! So, we will try to have it both ways by re-using, yet another time, the symbol + but, at least for the time being, put it within a circle: $\oplus$. Later, we will learn to rely on the *context*.

1. In order to help us picture things while dealing with signed-numerators, we revert temporarily to the "arrow notation" that we used just above.
In what follows, we deal with *transactions* but everything applies to *states* (just use $\uparrow$ and $\downarrow$ instead of $\rightarrow$ and $\leftarrow$).
Then, for instance, we look at

$3 \rightarrow$ as standing for $\rightarrow \rightarrow \rightarrow$

$5 \leftarrow$ as standing for $\leftarrow \leftarrow \leftarrow \leftarrow \leftarrow$

In other words, we look at $\rightarrow$ and $\leftarrow$ as if they were *denominators* that, furthermore, "cancel each other":

$$\cancel{\rightarrow}\ \cancel{\leftarrow} \qquad \text{and} \qquad \cancel{\leftarrow}\ \cancel{\rightarrow}$$

so that

$$1\rightarrow\ \oplus\ 1\leftarrow\ =\ 0\leftarrow\ =\ 0\rightarrow \qquad \text{and} \qquad 1\leftarrow\ \oplus\ 1\rightarrow\ =\ 0\leftarrow\ =\ 0\rightarrow$$

2. When we string transactions, we must distinguish two cases.

a. The two transactions go in the *same* direction.

♠ Say we have two transactions:	❖ On the *board*, we *write*:
FIVE ***dollars from-Jack-to-Jill***	$5\rightarrow$ **Washingtons**
THREE ***dollars from-Jack-to-Jill***	$3\rightarrow$ **Washingtons**
Stringing the transactions	Adding the signed-numerators
	$5\rightarrow\ \oplus\ 3\rightarrow$
gives	$\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow$
	$8\rightarrow$
EIGHT ***dollars from-Jack-to-Jill***	$+8$ **Washingtons**

Accountants would represent this as follows:

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*: **Jack**		**Jill**	
	Black (*In*)	*Red* (*Out*)	*Black* (*In*)	*Red* (*Out*)
FIVE ***dollars from-Jack-to-Jill***		\$5	\$5	
THREE ***dollars from-Jack-to-Jill***		\$3	\$3	
EIGHT ***dollars from-Jack-to-Jill***		\$8	\$8	

In other words, when we add signed-numerators that have the *same* sign, we *add* the numerators and the sign of the resulting signed-numerator is of course the sign common to the signed-numerators being added.

b. The two transactions go in *opposite* directions[3].

[3]The lack of syntactic parallel between"in the same direction" and "in opposite directions" can be troublesome. Moreover, "the two transactions are in opposite directions" does not mean the same as "the two transactions are in *the* opposite direction".

subtraction of signed-number-phrases
incorrect

♠ Say we have the two transactions:	❖ On the *board*, we *write*:
THREE ***dollars from-Jack-to-Jill***	$3 \rightarrow$ **Washingtons**
FIVE ***dollars from-Jill-to-Jack***	$5 \leftarrow$ **Washingtons**
Stringing the transactions	Adding the signed-numerators
	$3 \rightarrow \oplus\ 5 \leftarrow$
	$\rightarrow \rightarrow \rightarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow$
gives	$\rightarrow \rightarrow$ /// /// $\leftarrow \leftarrow \leftarrow \leftarrow$
	$\rightarrow$ /// /// $\leftarrow \leftarrow \leftarrow$
	/// /// $\leftarrow \leftarrow$
	$2 \leftarrow$
TWO ***dollars Jill-to-Jack***	-2 **Washingtons**

Accountants would represent the above as follows:

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:			
	Jack		**Jill**	
	Black (*In*)	*Red* (*Out*)	*Black* (*In*)	*Red* (*Out*)
THREE ***dollars from-Jack-to-Jill***		\$3	\$3	
FIVE ***dollars from-Jill-to-Jack***	\$5			\$5
TWO ***dollars from-Jill-to-Jack***	\$2			\$2

In other words, when we add signed-number-phrases that have *opposite* signs, we *subtract* one numerator from the other and since this can only be done *one* way, this gives us the sign of the resulting signed-numerator.

3.5 Subtracting Signed-Number-Phrases.

The next issue is the **subtraction of signed-number-phrases**. However, (i) what a subtraction *represents*, and, (ii) what the *procedure* should be, are not immediately obvious. So, first, here is an example of how subtraction could come up. Suppose we had just added a long string of signed-number-phrases, say

-2 **Dollars** $\oplus$ -7 **Dollars** $\oplus$ $+5$ **Dollars** $\oplus$ $\dots$ $\oplus$ $+3$ **Dollars**

and say, for the sake of the argument, that we had found that the total was, say, -132 **Dollars**.

Now suppose we then realized that, somewhere along the line, one of the signed-number-phrases, say the second one, -7 **Dollars**, was **incorrect** in that it should not have appeared in the addition, so that the total too is

subtract
$\ominus$
cancel
adjustment

incorrect. A priori, to obtain the new, *corrected total*, we have the following three choices.

1. We could *strike out* the incorrect signed-number-phrase and *redo* the entire addition:

-2 **Dollars** $\oplus$ ~~-7 **Dollars**~~ $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars**

Of course, if the addition is really long, this is going to involve a lot of unnecessary work, redoing a lot that had been done correctly.

2. We could **subtract** the incorrect signed-number-phrase from the incorrect *total*:

-132 **Dollars** $\quad\ominus\quad$ -7 **Dollars**

The trouble, though, is that we have no idea what *procedure* to use for $\ominus$!

3. We can **cancel** the *effect* of the incorrect signed-number-phrase on the incorrect total by *adding the opposite* of the incorrect signed-number-phrase to the incorrect total. Accountants call this entering an **adjustment**.

That this *must* give us the same correct result as would choice **1.** is easy to see by comparing:

• The addition in which -7 **Dollars**, the incorrect signed-number-phrase, was *struck out*:

-2 **Dollars** $\oplus$ ~~-7 **Dollars**~~ $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars**

• The addition in which -7 **Dollars**, the incorrect signed-number-phrase has been *left in* but has been *cancelled* by the adjustment $+7$ **Dollars** that was *added* at the end:

-2 **Dollars** $\oplus$ ~~-7 **Dollars**~~ $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars** $\oplus$ ~~$+7$ **Dollars**~~

Either way, the signed-number-phrases that are *actually* involved are:

-2 **Dollars** $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars**

which makes the case.

Accountants would represent the above as follows:

subtract
effect
initial state
final state

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:			
	Striking out		Cancelling	
	Black (*In*)	*Red* (*Out*)	*Black* (*In*)	*Red* (*Out*)
TWO ***dollars*** out		$2		$2
SEVEN ***dollars*** out		~~$7~~		$7
FIVE ***dollars*** in	$5		$5	
...	...		...	
THREE ***dollars*** in	$3		$3	
SEVEN ***dollars*** in (Adjustment)			$7	

In other words,

- *Subtracting* the incorrect signed-number-phrase (choice **2.**):

$$-132 \text{ Dollars} \quad \ominus \quad -7 \text{ Dollars}$$

amounts to exactly the same as

- *Adding the opposite* of the incorrect signed-number-phrase (choice **3.**):

$$-132 \text{ Dollars} \quad \oplus \quad +7 \text{ Dollars}$$

Since, as already pointed out, we have no ready-made *procedure* for *subtraction*, we will say that "adding the opposite" *is* the procedure and that, *in the case of signed-number-phrases*, to **subtract** something *is just short for* "to add its opposite"[4].

3.6 Effect of Transactions on States

We now look at the **effect** of a *transaction* on *state*s. Given an **initial state** and a transaction involving that state, we will call **final state** the state *after* the transaction. (This will fit exactly what we did in Section 1.5.)

For instance,

♠ Looking at ***Jill***, suppose that:

- In the *initial* state, ***Jill*** is THREE ***dollars in-the-red***.

[4]This is indeed the *definition* of subtraction in a group. Yet, Educologists usually express this as an *operating prescription*: "To subtract a signed number, change the sign of the number being subtracted and add". It does of course work in the *short* run but *not* stressing that the term "subtraction" in the case of counting number-phases means something different than in the case of signed number-phrases does nothing for lucidity in the *long* term.

gain
loss
change of state
algebra-smaller
algebra-larger
algebra-equal

- Then, a *transaction* occurs, say FIVE ***dollars from-Jack-to-Jill***.
- Now, in the *final* state, ***Jill*** is TWO ***dollars in-the-black***.

Thus, the *effect* of a FIVE ***dollars from-Jack-to-Jill*** *transaction* is a FIVE ***dollars*** **gain** on ***Jill***'s *state*—as well as a FIVE ***dollars*** **loss** on ***Jack***'s *state*. A transaction in the *opposite* direction would have the *opposite* effects.

❖ On the board, to find the **change of state**, we *subtract* the *initial* state from the *final* state to *remove* from the final state the effect of all *previous* transactions.

$$\begin{aligned}\textsf{Change of State} &= \textsf{Final State} \ominus \textsf{Initial State}\\ &= +2 \textsf{ Washingtons} \ominus -3 \textsf{ Washingtons}\\ &= +2 \textsf{ Washingtons} \oplus +3 \textsf{ Washingtons}\\ &= +5 \textsf{ Washingtons}\\ &= \xrightarrow{\to 5 \textsf{ Washingtons}}\end{aligned}$$

We thus have that

$$\textit{Effect of a Transaction} = \textit{Change of State}$$

This seemingly trivial statement will have in fact far-reaching generalizations.

3.7 Comparing States

Essentially, we compare *states* just the way we compared *collections* in Section 1.2: Where we used *forward* and *backward* counts, here we use transactions in the *standard* and *opposite* directions:

- When it takes a transaction in the *standard* direction to go from an *initial* state to a *final* state, we say that the *initial* state is **algebra-smaller** than the *final* state.
- When it takes a transaction in the *opposite* direction to go from an *initial* state to a *final* state, we say that the *initial* state is **algebra-larger** than the *final* state.
- When it takes *no* transaction to go from an *initial* state to a *final* state, we say that the initial state is **algebra-equal** to the final state.

3.8 Equations and Inequations

We can now return to the problems we encountered in Section 1.5.

discrete
continuous

Chapter 4

Accounting For *Discrete* Goods

We now turn to the accounting for *goods*. While *essentially* the same as the accounting for *money*, there are a few technical issues and further developments. Here, though, we will deal with the accounting for

- **discrete** *goods* such as, say, ***apples*** or ***bananas***

as opposed to the accounting for

- **continuous** *goods* such as, say, ***lengths*** or ***liquids***

because the latter involve the *English* System of denominators which present problems of their own and which, therefore, we shall leave to a later chapter.

4.1 Counting And Adding Collections Of Discrete Goods

1. The main difference between counting *money* and *discrete goods* is that:

- When counting *money*, when we have more than TEN of a kind, we
  - *bundle* TEN of a kind, and then we
  - *exchange* the bundle of TEN of a kind for 1 of the next kind up—for which we already have a *denominator*.

The question is how to read 23 **Apples**

discrete
worth
unit-worth
value
unit-value

- To count **discrete** *goods* such as, say, ***apples***, and while we will still *bundle* collections of TEN ***objects***, we will usually *not* be able to *exchange* these bundles the way we did with *money*, so that the *denominators* will usually have to represent *bundles* rather than *objects* as was the case with *money*.

Note. Bundling by TEN is rather recent. Historical remnants of a time when *discrete* goods were bundled by TWELVE include ***eggs*** and ***pencils***.

2. As far as *aggregating* collections of *different* goods goes, the situation is the same as in Section 1.6, where we already saw that combinations can *always* be added.

♠ If, say Jack has THREE ***apples***, FIVE ***bananas*** and THIRTEEN ***cents*** and Jill has TWO ***apples***, EIGHT ***bananas*** and ELEVEN ***cents***, when they put everything together, they have FIVE ***apples***, THIRTEEN ***bananas*** and TWENTY-FOUR ***cents***.

❖ On the board we proceed as follows:

$$[3 \textsf{ Apples} \,\&\, 5 \textsf{ Bananas} \,\&\, 13 \textsf{ Cents}] \,\&\, [2 \textsf{ Apples} \,\&\, 8 \textsf{ Bananas} \,\&\, 11 \textsf{ Cents}]$$
$$= 5 \textsf{ Apples} \,\&\, 13 \textsf{ Bananas} \,\&\, 24 \textsf{ Cents}$$

4.2 Evaluating Collections of Discrete Goods: Co-multiplication

We seldom deal with *goods* alone and, often, we will want to know how much *money* a given collection of *goods* is **worth**. For that purpose, we shall need a **unit-worth** for the objects. On the board, we shall represent a *worth* by a **value** and a *unit-worth* by a **unit-value**.

Note. Just as we say "To *count* a collection" as a short for "To find the number-phrase that represents a collection", we shall say "To *evaluate* a collection" as a short for "To find the number-phrase that represents the *worth* of a collection".

1. For instance, given a collection of ***apples***, we may be interested in how many ***dimes*** it is *worth* and for that we need to know how many ***cents*** each ***apple*** is *worth*.

♠ Suppose Jack has THREE ***apples*** with a *unit-worth* of, say, SEVEN ***cents-per-apple***. Thus, at a unit-worth of SEVEN ***cents-per-apple***, ***Jack's worth*** would be TWENTY-ONE ***cents***.

co-multiply
multiplication tables
vector
co-vector
co-number-phrase
percentage
extend

❖ On the board, we write $7\ \frac{\textsf{Cents}}{\textsf{Apple}}$ to represent the unit-value SEVEN ***cents-per-apple*** and we **co-multiply**:

$$\begin{aligned}\textsf{Jack's Value} &= [3\ \cancel{\textsf{Apples}}] \times \left[7\ \frac{\textsf{Cents}}{\cancel{\textsf{Apple}}}\right] \\ &= [3 \times 7]\ \textsf{Cents} \\ &= 21\ \textsf{Cents}\end{aligned}$$

In other words, the (board) procedure for co-multiplication is as follows:
• Get the *denominator* by "canceling" the **Apples**, and
• Get the *numerator* from the **multiplication tables** we had to memorize as children.

2. *Co-multiplication* is in fact an extremely important concept.

a. The "cancelling" of denominators, as that of **Apples** above, is in fact something that *scientists* and *engineers* do all the time and it is in fact at the heart of a part of *mathematics* called DIMENSIONAL ANALYSIS.

b. Moreover, *co-multiplication* is also at the heart of LINEAR ALGEBRA[1], a part of mathematics already mentioned at the very beginning of this text, in which 3 **Apples** would be a **vector** while $7\ \frac{\textsf{Cents}}{\textsf{Apple}}$ would be a **co-vector**. Accordingly, since we call 3 **Apples** a *number-phrase*, we will call $7\ \frac{\textsf{Cents}}{\textsf{Apple}}$ a **co-number-phrase**.

c. More modestly, co-multiplication arises as at least one aspect of **percentage**:

$$[3\ \cancel{\textsf{Dollars}}] \times \left[7\ \frac{\textsf{Cents}}{\cancel{\textsf{Dollar}}}\right] = [3 \times 7]\ \textsf{Cents} = 21\ \textsf{Cents}$$

4.3 Effect of Transactions on States: Signed Co-Multiplication

We now **extend** the concept of *co-multiplication* to *signed*-number-phrases. The idea is of course to obtain the *gain* or *loss* caused by a transaction.

For illustration purposes, we will look at transactions occurring in an ***apple*** store. Suppose that, not only can ***apples***, for whatever reason to be left to the reader's imagination, *appear in* or *disappear from* the store, but the ***apples*** can *also* be ***good*** or ***bad***.

1. We begin with the issue of representing the *transactions* of ***apples***. This is no different from representing the *transactions* of ***dollars*** which we saw earlier on.

[1]Being the *action* on an element of a vector space of an element of its dual.

unit-profit-worth
unit-loss-worth
co-signed-number-phrase
unit-profit
unit-loss-value

♠ We have ***apples*** that can *appear* or *disappear* from the store. (Just like, earlier on, we had ***dollars*** going from Jack to Jill and/or from Jill to Jack.)

❖ We choose the *standard direction*, for which we use the + sign, to be that of ***apples*** that *appear in* the store and so the *opposite direction*, for which we use the − sign, is that of ***apples*** that *disappear out* of the store.

Thus, for instance,

♠ In the real world, we may have:	❖ We then write:
THREE ***apples*** *appear in* the store,	[+3 **Apples**]
or	
FIVE ***apples*** *disappear from* the store.	[−5 **Apples**]

2. Because the resulting *gain* or *loss* of such transactions will of course depend on what unit-worth the ***apples*** have, we now look at the kinds of unit-worth ***apples*** can have.

♠ The ***apples*** may be either ***good*** or ***bad*** and, if those that are ***good*** can be *sold* at a **unit-*profit*-worth**, those that are ***bad*** must be *disposed of* at a **unit-*loss*-worth**.

❖ To represent *unit-profit-worth* and *unit-loss-worth* on the board, we use **co-signed-number-phrase** as we did with *co-number-phrases* and we use a + sign for a **unit-*profit***-value and a − sign for a **unit-*loss*-value**.

Thus, for instance,

♠ In the store, we may have:	❖ We then write:
apples that are ***good*** and could be *sold* at a unit-*profit*-worth of, say, SEVEN ***cents-per-apple***	$+7\ \frac{\textsf{Cents}}{\textsf{Apple}}$
or	
apples that are ***bad*** and must be *disposed of* at a unit-*loss*-worth of, say, SEVEN ***cents-per-apple***	$-7\ \frac{\textsf{Cents}}{\textsf{Apple}}$

Note. The choice of symbols, + to represent ***good*** and − to represent ***bad***, is not really an arbitrary choice because of the way they will interact with the symbols for *appearing* and *disappearing*. We leave it as an exercise for the reader to investigate what happens when other choices are made.

3. We are now finally, in a position to look at the four kinds of *effect* that *transactions* of goods can have on money *states*.

make pictures of good in = good, good out = bad, bad in = bad, bad out = good

represent by $++ = +$, $+- = -$, $-+ = -$, $-- = +$

♠ In the *real world*, we have	❖ On the *board*, we write
Case 1.	
THREE ***apples*** *appear* in the store.	$[+3 \text{ Apples}]$
They are ***good***, with a unit-*profit*-worth of SEVEN ***cents-per-apple***.	$\left[+7 \frac{\text{Cents}}{\text{Apple}}\right]$
The result is	$[+3 \cancel{\text{Apples}}] \times \left[+7 \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$
	$= [+3] \times [+7] \text{ Cents}$
a *profit*-worth of TWENTY-ONE ***cents***.	$= +21 \text{ Cents}$
Case 2.	
THREE ***apples*** *appear* in the store.	$[+3 \text{ Apples}]$
They are ***bad***, with a unit-*loss*-worth of SEVEN ***cents-per-apple***.	$\left[-7 \frac{\text{Cents}}{\text{Apple}}\right]$
The result is	$[+3 \cancel{\text{Apples}}] \times \left[-7 \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$
	$= [+3] \times [-7] \text{ Cents}$
a *loss*-worth of TWENTY-ONE ***cents***.	$= -21 \text{ Cents}$
Case 3.	
THREE ***apples*** *disappear* from the store.	$[-3 \text{ Apples}]$
They are ***good***, with a unit-*profit*-worth of SEVEN ***cents-per-apple***.	$\left[+7 \frac{\text{Cents}}{\text{Apple}}\right]$
The result is	$[-3 \cancel{\text{Apples}}] \times \left[+7 \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$
	$= [-3] \times [+7] \text{ Cents}$
a *loss*-worth of TWENTY-ONE ***cents***.	$= -21 \text{ Cents}$
Case 4.	
THREE ***apples*** *disappear* from the store.	$[-3 \text{ Apples}]$
They are ***bad***, with a unit-*loss*-worth of SEVEN ***cents-per-apple***.	$\left[-7 \frac{\text{Cents}}{\text{Apple}}\right]$
The result is	$[-3 \cancel{\text{Apples}}] \times \left[-7 \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$
	$= [-3] \times [-7] \text{ Cents}$
a *profit*-worth of TWENTY-ONE ***cents***.	$= +21 \text{ Cents}$

4.4 Common Denominator

Usually, in order to deal with collections of *goods* of *different* kinds, these must still have "something in common".

For instance, say ***Jack*** has SIX ***apples*** and FIVE ***bananas*** while ***Jill*** has TWO ***apples*** and EIGHT ***bananas***. The question, say, of who has more and who has less is meaningless as we cannot compare ***apples*** and ***bananas***.

Collections can have "something in common" in two ways.

1. When both kinds of goods can be *exchanged* for a *same* kind of goods.
♠ For instance, say Jack has SIX ***apples*** and FIVE ***bananas*** while Jill has TWO ***apples*** and EIGHT ***bananas*** but that ONE ***apple*** can be exchanged for FOUR ***nuts*** and ONE ***banana*** can be exchanged for SEVEN ***nuts***.
Jack can then exchange his SIX ***apples*** for TWENTY-FOUR ***nuts*** and his FIVE ***bananas*** for THIRTY-FIVE ***nuts*** for a total of FIFTY-NINE ***nuts*** while ***Jill*** can exchange her TWO ***apples*** for EIGHT ***nuts*** and her EIGHT ***bananas*** for FIFTY-SIX ***nuts*** for a total of SIXTY-FOUR ***nuts***.
❖ On the board we use the *multiplication tables* to write:

$$\begin{aligned}\textsf{Jack's Holding} &= 6\ \textsf{Apples}\ \&\ 5\ \textsf{Bananas}\\ &= 6(4\ \textsf{Nuts}) + 5(7\ \textsf{Nuts})\\ &= 24\ \textsf{Nuts} + 35\ \textsf{Nuts}\\ &= 59\ \textsf{Nuts}\end{aligned}$$

and

$$\begin{aligned}\textsf{Jill's Holding} &= 2\ \textsf{Apples}\ \&\ 8\ \textsf{Bananas}\\ &= 2(4\ \textsf{Nuts}) + 8(7\ \textsf{Nuts})\\ &= 8\ \textsf{Nuts} + 56\ \textsf{Nuts}\\ &= 64\ \textsf{Nuts}\end{aligned}$$

2. Usually, though, we don't deal with just collections of *goods* but also with *money*. So, even being able to exchange the given kinds of *goods* for a "common" kind of *goods* would still leave us with both these goods *and* money.
♠ Suppose, for instance, that ***Jack*** not only has SIX ***apples*** and FIVE ***bananas*** as above but also THIRTEEN ***cents*** and that ***Jill*** not only has TWO ***apples*** and EIGHT ***bananas*** as above but also THIRTY-ONE ***cents***. Even if both ***apples*** and ***bananas*** could be exchanged for ***nuts*** as above, this would still leave ***Jack*** and ***Jill*** holding both ***nuts*** and ***cents***:

- ***Jack*** would be holding FIFTY-NINE ***nuts*** and THIRTEEN ***cents***
- ***Jill*** would be holding SIXTY-FOUR ***nuts*** and THIRTY-ONE ***cents***

In this kind of situations, it will be convenient to be able to *evaluate* both kinds of *goods* in terms of *money* so that, eventually, we will be left only with money. This will require that, corresponding to each *good*, we have a *unit-worth* for that *good*.

For instance, if, say, ***apples*** have a ***unit-worth*** of SEVEN ***cents-per-apple*** and ***bananas*** have a ***unit-worth*** of FOUR ***cents-per-banana***, *then*:

common denominator

- ***Jack***'s SIX ***apples*** will be worth FORTY-TWO ***cents*** and his FIVE ***bananas*** will be worth TWENTY ***cents*** so that, with the THIRTEEN ***cents*** he already had, ***Jack's worth*** will be SEVENTY-FIVE ***cents***.
- ***Jill***'s TWO ***apples*** will be worth FOURTEEN ***cents*** and her EIGHT ***bananas*** will be worth THIRTY-TWO ***cents*** so that, with the THIRTY-ONE ***cents*** she already had, ***Jill's worth*** will be SEVENTY-SEVEN ***cents***.

❖ On the board we proceed as follows

$$\begin{aligned}\textsf{Jack's Value} &= [3\ \cancel{\textsf{Apples}}] \times \left[7\ \frac{\textsf{Cents}}{\cancel{\textsf{Apple}}}\right] + [5\ \cancel{\textsf{Bananas}}] \times \left[4\ \frac{\textsf{Cents}}{\cancel{\textsf{Banana}}}\right] + 13\ \textsf{Cents}^{2}\\ &= [3 \times 7]\ \textsf{Cents} + [5 \times 4]\ \textsf{Cents} + 13\ \textsf{Cents}\\ &= 21\ \textsf{Cents} + 20\ \textsf{Cents} + 13\ \textsf{Cents}\\ &= 75\ \textsf{Cents}\end{aligned}$$

and

$$\begin{aligned}\textsf{Jill's Value} &= [2\ \cancel{\textsf{Apples}}] \times \left[7\ \frac{\textsf{Cents}}{\cancel{\textsf{Apple}}}\right] + [8\ \cancel{\textsf{Bananas}}] \times \left[4\ \frac{\textsf{Cents}}{\cancel{\textsf{Banana}}}\right] + 11\ \textsf{Cents}\\ &= [2 \times 7]\ \textsf{Cents} + [8 \times 4]\ \textsf{Cents} + 11\ \textsf{Cents}\\ &= 14\ \textsf{Cents} + 32\ \textsf{Cents} + 31\ \textsf{Cents}\\ &= 77\ \textsf{Cents}\end{aligned}$$

We shall refer to **Cents** as a **common denominator**.

4.5 Comparing Collections of Different Kinds of Goods

To compare collections of goods, they *must* consist of the *same* kind of objects. On the board, this means that we must have a *common* denominator. But, once we do, the comparison proceeds quite smoothly.

1. Indeed, if the collections consist of the *same* kinds of objects, we need of course concern ourselves only with the *numbers* of objects in the collections as we did in Section 1.2 .

- Say Jack has THREE ***apples*** and Jill has FIVE ***apples***. Who's ***worth*** is *smaller*?

---

[2] In other words, $\textsf{Jack's Value} = \begin{bmatrix} 3\ \textsf{Apples} & 5\ \textsf{Bananas} \end{bmatrix} \circ \begin{bmatrix} 7\ \frac{\textsf{Cents}}{\textsf{Apple}} \\ 4\ \frac{\textsf{Cents}}{\textsf{Banana}} \end{bmatrix} + 13\ \textsf{Cents}$ as Educologists will be sure to note.

one-to-one
one-to-one

♠ After matching their ***apples*** **one-to-one**, we find that ***Jill*** is left with un-matched ***apples*** which means that ***Jack*** has fewer ***apples*** than ***Jill***. So, ***Jack's worth*** is *smaller* than ***Jill's worth*** regardless of the ***unit-worth*** of ***apples***.
❖ On the board, since we go from 3 **Apples** to 5 **Apples** by counting *forward*, $\xrightarrow{3,\,4,\,5}$, we write that

$$3 \textsf{ Apples} < 5 \textsf{ Apples}$$

and therefore

$$\textsf{Jack's Value} < \textsf{Jill's Value}$$

• Say Jack has EIGHT ***apples*** and Jill has TWO ***apples***. Who's worth is *larger*?
♠ After matching their ***apples*** **one-to-one**, ***Jack*** has ***apples*** left over and so *more* ***apples*** than Jill. Thus, ***Jack's worth*** is *larger* than ***Jill's worth*** regardless of the worth of ***apples***.
❖ On the board, since we go from 8 **Apples** to 2 **Apples** by counting *backward*, $\xleftarrow{2,\,3,\,4,\,5,\,6,\,7,\,8}$, we write that

$$8 \textsf{ Apples} > 2 \textsf{ Apples}$$

so that

$$\textsf{Jack's Value} > \textsf{Jill's Value}$$

2. Usually, though, we need to compare collections of *different* goods and we first need to get a *common denominator*.

♠ For instance, in the last example of Section 4.4 we found that ***Jack's worth*** would be SEVENTY-FIVE ***cents*** and that ***Jill's worth*** would be SEVENTY-SEVEN ***cents***. So, after matching ***cents*** *one-to-one*, we find that ***Jack's worth*** is smaller than ***Jill's worth***

❖ On the board, after we get to the point where we have:

$$\textsf{Jack's Value} = 75 \textsf{ Cents}$$
$$\textsf{Jill's Value} = 77 \textsf{ Cents}$$

we proceed as in Section 1.2 and since we go from 75 **Cents** to 77 **Cents** by counting forward, $\xrightarrow{76,\,77}$, we write that

$$\textsf{Jack's Value} < \textsf{Jill's Value}$$

4.6 Computing Unit-Values: Co-division

Division
co-division
quotient
algebra
solve
equation

Division comes up in at least two different types of situation but, as we shall see, from the *procedural* viewpoint, these situations will turn out to be really the same. Here we shall deal with the type of situations in which we want to arrive at *unit-values*. We shall refer to is as **co-division**.

We begin by looking at *basic* collections, that is collections involving *less* than TEN ***objects*** (See Section 1.1) because our purpose here is only to show how *division* is connected with *co-multiplication*. We leave the more technical aspects of the procedure to a later chapter.

1. We begin with the somewhat rare case in which "the division comes out even".

a. Suppose we wanted to buy TWO ***apples*** at an auction and that we had EIGHT ***dimes*** available. We want to know what is the ***highest possible bid*** that we could make.

♠ Essentially, we proceed by *trial and error*.

• Bidding ONE ***dime-per-apple*** would require TWO ***dimes*** to buy TWO ***apples*** with SIX ***dimes*** left over.

• Bidding TWO ***dime-per-apple*** would require FOUR ***dimes*** to buy TWO ***apples*** with FOUR ***dimes*** left over.

• Bidding THREE ***dimes-per-apple*** would require SIX ***dimes*** to buy TWO ***apples*** with TWO ***dimes*** left over.

• Bidding FOUR ***dimes-per-apple*** would require EIGHT ***dimes*** to buy TWO ***apples*** with NO ***dime*** left over.

So, our ***highest possible bid*** would be FOUR ***dime-per-apple***.

❖ On the board, we use the "2 times" *multiplication table* to accelerate the trial and error process. We start with 1

• $2 \times 1 = 2$ which is *less* than 8 so we try the next one, 2,
• $2 \times 2 = 4$ which is *less* than 8 so we try the next one, 3,
• $2 \times 3 = 6$ which is *less* than 8 so we try the next one, 4,
• $2 \times 4 = 8$ which is *equal* to 8 so we stop with 4.

The **quotient**, namely what we write on the *board* to represent our ***highest possible bid***, is thus $4\ \frac{\textsf{Dimes}}{\textsf{Apples}}$.

b. From the **algebra** viewpoint, we are trying to **solve** the **equation**

$$2\ \textsf{Apples} \times ?\ \frac{???}{¿¿¿} = 8\ \textsf{Dimes}.$$

that is, to figure out what *co-number-phrase* would fit.

solution
÷
divided by

- ¿¿¿ must be **Apple** to allow for the cancellation and ??? must be **Dimes** because of the right hand side:

$$2\ \cancel{\textsf{Apples}} \times ?\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}} = 8\ \textsf{Dimes}.$$

$$(2 \times ?)\ \textsf{Dimes} = 8\ \textsf{Dimes}$$

- Since the *denominators* are now the same, we saw in Section **??** that we must solve:

$$2 \times ?\ = 8$$

  Then, using the "2 times" multiplication table, we find that 4 fits in the place of ?.

The *procedure* to solve the equation 2 **Apples** $\times ?\ \frac{???}{¿¿¿} = 8$ **Dimes** is thus to "divide both sides by 2 **Apples**":

$$\frac{2\ \textsf{Apples}}{2\ \textsf{Apples}} \times ?\ \frac{???}{¿¿¿} = \frac{8\ \textsf{Dimes}}{2\ \textsf{Apples}}$$

and, since, $\frac{2\ \textsf{Apples}}{2\ \textsf{Apples}} = 1$, we get

$$?\ \frac{???}{¿¿¿} = \frac{8\ \textsf{Dimes}}{2\ \textsf{Apples}}$$
$$= 4\ \frac{\textsf{Dimes}}{\textsf{Apples}}$$

So, $4\ \frac{\textsf{Dimes}}{\textsf{Apple}}$ is the **solution** of the equation 2 **Apples** $\times ?\ \frac{???}{¿¿¿} = 8$ **Dimes** and we will say that it is "given" by

$$8\ \textsf{Dimes} \div 2\ \textsf{Apples} = 4\ \frac{\textsf{Dimes}}{\textsf{Apple}}$$

which we read as "8 **Dimes divided by** 2 **Apples** is equal to $4\ \frac{\textsf{Dimes}}{\textsf{Apple}}$" and where the procedure for ÷ is the one above.
In other words,

$$2\ \textsf{Apples}\ \times 4\ \frac{\textsf{Dimes}}{\textsf{Apple}} = 8\ \textsf{Dimes}.$$

and

$$8\ \textsf{Dimes} \div 2\ \textsf{Apples} = 4\ \frac{\textsf{Dimes}}{\textsf{Apple}}$$

are *dual* sentences.

c. If, instead of writing 8 **Dimes** ÷ 2 **Apples** we move 8 **Dimes** in place of the upper dot and 2 **Apples** in place of the lower dot, we get

$$\frac{8\ \textsf{Dimes}}{2\ \textsf{Apples}}$$

divide into
quotient
remainder

as another way to write 8 **Dimes** ÷ 2 **Apple** and thus

$$\frac{8\ \textsf{Dimes}}{2\ \textsf{Apples}} = 4\ \frac{\textsf{Dimes}}{\textsf{Apple}}$$

d. In anticipation of the more complicated cases to be investigated later on, we set up the division as follows:

$$\begin{array}{r|l} & 4\ \frac{\textsf{Dimes}}{\not{\textsf{Apple}}} \\ \hline 2\ \not{\textsf{Apples}}\ ) & 8\ \textsf{Dimes} \\ & 8\ \textsf{Dimes} \\ \hline & 0\ \textsf{Dime} \end{array}$$

which we read as **divide** 2 **Apples into** 8 **Dimes**.
Note. Observe that the order of the two number-phrases in "*divide* 2 **Apples** *into* 8 **Dimes**" is the opposite of their order in "8 **Dimes** *divided by* 2 **Apples**" but that both result in the same co-number-phrase, namely 4 $\frac{\textsf{Dimes}}{\textsf{Apple}}$. This is another instance of *linguistic duality.*

2. We now look at a case where "the division does *not* come out even"

a. Suppose we wanted to buy THREE ***apples*** but that we had only SEVEN ***dimes***. We want to know what would be our ***highest possible bid***.
♠ Again, we proceed by *trial and error.*
• Bidding ONE ***dime-per-apple*** would require THREE ***dimes*** to buy THREE ***apples*** with FOUR ***dimes*** leftover.
• Bidding TWO ***dime-per-apple*** would require SIX ***dimes*** to buy THREE ***apples*** with ONE ***dime*** leftover.
• Bidding THREE ***dimes-per-apple*** would require NINE ***dimes*** to buy THREE ***apples*** which is more than the SEVEN ***dimes*** we have.
So, our ***highest possible bid*** is TWO ***dime-per-apple*** at which unit-value we would be able to buy TWO ***apples*** with ONE ***dime*** left over.
❖ On the board, we use the "3 times" *multiplication table* to accelerate the trial and error process. We start with 1.

- $3 \times 1 = 3$ which is *less* than 7 so we try the next one, 2,
- $3 \times 2 = 6$ which is *less* than 7 so we try the next one, 3,
- $3 \times 3 = 9$ which is *more* than 7 so we stop with 2.

Since 2 $\frac{\textsf{Dimes}}{\textsf{Apple}}$ represents the ***highest possible bid***, we will again call it the **quotient** and, since 1 **Dimes** represents the ***lowest possible leftover***, we will call it the **remainder** of the division.

b. From the *algebra* viewpoint, we are trying to solve the equation

$$3\ \textsf{Apples} \times ?\ \frac{???}{¿¿¿} = 7\ \textsf{Dimes}$$

≈
approximately equal
(...)

However, even though the quotient is $2\ \frac{\textsf{Dimes}}{\textsf{Apple}}$, we cannot just replace $?\ \frac{???}{¿¿¿}$ by $2\ \frac{\textsf{Dimes}}{\textsf{Apple}}$ because the way *division* is related to *co-multplication* is now complicated by the *remainder* and we can only write

$$7\ \textsf{Dimes} \approx 3\ \cancel{\textsf{Apples}} \times 2\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}}$$

where ≈ means that $7\ \textsf{Dimes}$ is **approximately equal** to $3\ \cancel{\textsf{Apples}} \times 2\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}}$ by which we mean that the difference is "too small to matter". However, we shall not use the symbol ≈ because we shall prefer to keep track of this difference, even though it is "too small to matter" and we shall write

$$7\ \textsf{Dimes} = 3\ \cancel{\textsf{Apples}} \times 2\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}} + (...)\ \textsf{Dimes}$$

where **(...) Dimes** represents a *number* of ***dimes*** "too small to matter". In fact, there will be situations where we will have to be more precise about what is being ignored and, in particular, we may even replace (...) by the *remainder*:

$$7\ \textsf{Dimes} = 3\ \cancel{\textsf{Apples}} \times 2\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}} + 1\ \textsf{Dimes}$$

Since the *denominators* are the same, we can focus on the *numerators* and write:

$$7 = 3 \times 2 + (...)$$

as well as

$$7 = 3 \times 2 + 1$$

Note. More generally, keeping track of things even though they are too small to be taken into account, one way or the other, will turn out to be an extremely powerful approach called ASYMPTOTIC ANALYSIS[3], one that we shall make extensive and systematic use of in Part 2 and 3 of this proto-text.

c. It is imperative to realize how this extends to the use of ÷.
First recall that, in the previous case, we wrote

$$8\ \textsf{Dimes}\ \div\ 2\ \textsf{Apples} =\ (4\ \frac{\textsf{Dimes}}{\textsf{Apple}})$$

as just the *dual* of

$$8\ \textsf{Dimes}\ = 2\ \textsf{Apples}\ \times\ (4\ \frac{\textsf{Dimes}}{\textsf{Apple}})$$

[3]Educologists will surely have recognized in "(...)" a *precursor* of "little ohs", the only difference being that the latter give the *order of magnitude* of what is *not* taken into account.

mixed-numbers notation

But, *here*, we have

$$7\ \textsf{Dimes} \ = \ 3\ \cancel{\textsf{Apples}} \ \times \ (2\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}}) + (\ldots)\ \textsf{Dimes}$$

so that

$$7\ \textsf{Dimes} \ \neq \ 3\ \cancel{\textsf{Apple}} \ \times \ (2\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}})$$

and therefore

$$7\ \textsf{Dimes} \div 3\ \textsf{Apple} \ \neq \ 2\ \frac{\textsf{Dimes}}{\textsf{Apple}}$$

However, we *can* write

$$7\ \textsf{Dimes} \div 3\ \textsf{Apples} = 2\ \frac{\textsf{Dimes}}{\textsf{Apple}} + (\ldots)\frac{\textsf{Dime}}{\textsf{Apple}}$$

and, if we want to be more specific,

$$7\ \textsf{Dimes} \div 3\ \textsf{Apples} = 2\ \frac{\textsf{Dimes}}{\textsf{Apple}} + \frac{1\ \textsf{Dime}}{3\ \textsf{Apple}}$$

where the 3 **Apples** under the remainder 1 **Dime** are to remind us that the remainder is yet to be divided among 3 **Apples**.

d. If we rewrite 7 **Dimes** ÷ 3 **Apples** as $\frac{7\ \textsf{Dimes}}{3\ \textsf{Apples}}$ then we can rewrite the above as

$$\frac{7\ \textsf{Dimes}}{3\ \textsf{Apple}} = 2\ \frac{\textsf{Dimes}}{\textsf{Apple}} + \frac{1\ \textsf{Dime}}{3\ \textsf{Apple}}$$

and, with a slight abuse of language,

$$\frac{7}{3}\ \frac{\textsf{Dimes}}{\textsf{Apple}} = 2\ \frac{\textsf{Dimes}}{\textsf{Apple}} + \frac{1}{3}\ \frac{\textsf{Dimes}}{\textsf{Apple}}$$

and therefore:

$$\frac{7}{3}\ \frac{\textsf{Dimes}}{\textsf{Apple}} = \left(2\ + \frac{1}{3}\right)\frac{\textsf{Dimes}}{\textsf{Apple}}$$

whence the so-called **mixed-numbers notation**:

$$\frac{7}{3} = 2\ \tfrac{1}{3}$$

which we shall encounter in a later chapter, in a somewhat different context.

e. In anticipation of the more complicated cases to be investigated later on, we set up the division as follows:

$$\begin{array}{r|l} & 2\ \frac{\cancel{\textsf{Dimes}}}{\textsf{Apple}} \\ \hline 3\ \cancel{\textsf{Apples}}\ ) & 7\ \textsf{Dimes} \\ & \underline{6\ \textsf{Dimes}} \\ & 1\ \textsf{Dime} \end{array}$$

f. In an attempt to save time and energy, though, it is usual *not* to write the denominators and to write only the numerators[4]

$$\begin{array}{r|l} & 2 \\ \hline 3\,) & 7 \\ & 6 \\ \cline{2-2} & 1 \end{array}$$

4.7 Signed Co-division

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

TO DO

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

4.8 Equations and Inequations For Collections of Goods

We saw in Section 2.5 that, given *collections* of *money*, a type of problem one often encounters in the real-world is not only how they *compare* but how they *differ*. Here we shall investigate how collections of *goods* differ in *worth*.

1. We begin with the case where ***Jack*** has a collection of ***apples*** and ***Jill*** has ***money***.

♠ Say that ***Jack*** has TWO ***apples*** and that ***Jill*** has EIGHT ***dimes***. We want to know for which ***unit-worth*** will ***Jack's worth*** be the same as ***Jill's worth***.
This is exactly the problem we dealt with in Section 4.6: If ONE ***apple*** is worth FOUR ***cents***, then ***Jack's worth*** will be the same as ***Jill's worth***.

❖ On the board, we write, again as in Section 4.6,

$$\begin{array}{cc} \textsf{Jack's Value} & \textsf{Jill's Value} \\ 2\ \textsf{Apples} \times ?\ \frac{???}{¿¿¿} & = 8\ \textsf{Cents} \end{array}$$

[4]While, of course, students must eventually *come* to the "short" manner, Educologists may well consider the benefit of spending at least *some* time *getting* there, if only because the "long" manner, by the very fact that it uses *denominators*, reflects more immediately "real life" situations and thus helps providing meaningful checks.

equitable manner
separate

Dividing both sides by 2 **Apples** gives

$$\frac{\not{2}\ \not{\text{Apples}}}{\not{2}\ \not{\text{Apples}}} \times ?\ \frac{???}{¿¿¿} = \frac{8\ \textsf{Cents}}{2\ \textsf{Apples}}$$
$$1 \times ?\ \frac{???}{¿¿¿} = \frac{8\ \textsf{Cents}}{2\ \textsf{Apple}}$$

and therefore

$$?\ \frac{???}{¿¿¿} = 4\ \frac{\textsf{Cents}}{\textsf{Apple}}$$

2. We now look at the case where both ***Jack*** and ***Jill*** hold both a collection of ***apples*** and ***money***. What we will do is to change ***Jack's holding*** and ***Jill's holding*** so as to obtain a situation such as the previous one, that is where ***Jack*** has *only* a collection of ***apples*** and ***Jill*** has *only* ***money***. Of course, we will have to proceed in an **equitable manner**, that is, whatever we do onto either one, we must do onto the other.

Since this is in fact a very general approach called *Separation of Variables*, we will say that we **separate** the ***apples*** from the ***money***.

♠ Say that ***Jack*** has SEVEN ***apples*** and THREE ***cents*** and that ***Jill*** has FIVE ***apples*** and ELEVEN ***cents***. We want to know for which ***unit-worth*** will ***Jack's worth*** be the same as ***Jill's worth***.
  - Since we do not want ***Jack*** to have any ***money*** and since he has THREE ***cents***, we take them away from him and, to be equitable, we also take THREE ***cents*** away from ***Jill*** which leaves her with only EIGTH ***cents***.
  - Since we do not want ***Jill*** to have any ***apple*** and since she has FIVE ***apples***, we take them away from her and, to be equitable, we also take FIVE ***apples*** away from ***Jack*** which leaves him with only TWO ***apples***.

❖ On the board, we can proceed in either one of two ways:
  - We can start by separating the **Combinations**

Jack′s Combination	Jill′s Combination
7 **Apples** & 3 **Cents**	5 **Apples** & 11 **Cents**

Subtracting 3 **Cents** from each **Combination** leaves

7 **Apples**	5 **Apples** & 8 **Cents**

Subtracting 5 **Apples** from each **Combination** leaves

2 **Apples** 8 **Cents**

Moving to **Values** now gives the *equation*

$$2\ \textsf{\textbf{Apples}} \times ?\ \frac{???}{¿¿¿} = 8\ \textsf{\textbf{Cents}}$$

– Alternatively, we can start with **Values** up front:

$$\overset{\textsf{\textbf{Jack's Value}}}{7\ \textsf{\textbf{Apples}} \times ?\ \frac{???}{¿¿¿} + 3\ \textsf{\textbf{Cents}}} = \overset{\textsf{\textbf{Jill's Value}}}{5\ \textsf{\textbf{Apples}} \times ?\ \frac{???}{¿¿¿} + 11\ \textsf{\textbf{Cents}}}$$

Subtracting 3 **Cents** from each side of the *equation* gives

$$7\ \textsf{\textbf{Apples}} \times ?\ \frac{???}{¿¿¿} = 5\ \textsf{\textbf{Apples}} \times ?\ \frac{???}{¿¿¿} + 8\ \textsf{\textbf{Cents}}$$

Subtracting 5 **Apples** $\times\ ?\ \frac{???}{¿¿¿}$ from each side of the *equation* gives

$$2\ \textsf{\textbf{Apples}} \times ?\ \frac{???}{¿¿¿} = 8\ \textsf{\textbf{Cents}}$$

So, either way, by *separating* the ***apples*** from the ***money***, we have reduced the problem to the preceding one.

3. Inequation

Test

Test at infinity

4.9 Equations and Inequations For States

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

TO DO

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

The exact same approach works for *states* as long as one keeps in mind that:

- "*to subtract*" means "*to add the opposite*" Section 3.5
- "law of signs for signed co-multiplication" Section 4.3
- "law of signs for signed division" Section 4.7

1. -2 **Apples** $\times$? $\frac{???}{¿¿¿}$ $= -14$ **Cents**

xxxxxxxxxxxxxx
xxxxxxxxxxxxxx
xxxxxxxxxxxxxx
xxxxxxxxxxxxxx

2. xxxxxxxxxxxxxx

♠ Say that ***Jack*** is SEVEN ***apples in-the-red*** and THREE ***cents in-the-black*** and that ***Jill*** is FIVE ***apples in-the-red*** and ELEVEN ***cents in-the-red***. We want to know for which ***profit/loss-unit-worth*** will ***Jack's worth*** be the same as ***Jill's worth***.

- Since we do not want ***Jack*** to have any ***money***, and since he is THREE ***cents in-the-black***, we take them away from him and, to be equitable, we also take THREE ***cents*** away from ***Jill*** which, since she already was ELEVEN ***cents in-the-red***, puts her now FOURTEEN ***cents in-the-red***.
- Since we do not want ***Jill*** to have any ***apple***, and since she is FIVE ***apples in-the-red***, we give her FIVE ***apples*** and, to be equitable, we also give FIVE ***apples*** to ***Jack*** which, since he was SEVEN ***apples in-the-red***, puts him now only TWO ***apples in-the-red***.

❖ Again, on the board, we can proceed in either one of two ways:

- We can separate the **Combinations** and then move to **Values**

Jack's Combination	**Jill's Combination**
-7 **Apples** & $+3$ **Cents**	-5 **Apples** & -11 **Cents**

To *subtract* $+3$ **Cents** from each side, we *add the opposite*, -3 **Cents**, to each side which results in

-7 **Apples**	-5 **Apples** & -14 **Cents**

To subtract -5 **Apples** from each side, we *add the opposite*, $+5$ **Apples**, to each side which results in

-2 **Apples**	-14 **Cents**

Moving to **Values** now gives the *equation*

$$-2 \textbf{ Apples} \times ?\ \frac{???}{¿¿¿} = -14 \textbf{ Cents}$$

- Alternatively, we can move to **Values** up front:

3. Inequation

Test

Test at infinity

discrete
continuous

Chapter 5

Accounting For *Continuous* Goods *on* the Counter (I)

- When we count *money*, what we do each time we have more than TEN of a kind is two things (See Chapter I):
  - We *bundle* TEN of a kind
  - We then *exchange* the bundle of TEN of a kind for 1 of the next kind up—for which we usually already have a denominator.

- When we count *goods*, what we will do will very much depend on the *kind* of goods we are counting.
  - When we count **discrete** *goods* such as, say, ***apples***, and while we will still *bundle* collections of TEN ***objects***, we will usually *not* be able to *exchange* these bundles for other objects, as we do when dealing with *money*, and the *denominators* will usually have to represent *bundles* rather than *objects* as was the case with *money*.
  - When we count **continuous** *goods* such as ***lengths*** or ***liquids***, what we will do will not involve any *bundling* but will involve *changes of denominator*. In most of the world, because of the *metric system*, accounting for this kind of *goods* is thus essentially the same as accounting for *money* and just as easy. However, under the English system, the process, while it remains essentially the same, involves much memorization.

5.1 Counting Goods *on* the Counter

In the U. S., accounting for *goods* is usually much more difficult than accounting for *money* because, contrary to what is the case with money, when dealing with goods, we still use English denominators and the English denominators do not change at the rate of TEN to 1.

1. Consider for instance the problem presented by, say, the number-phrase 27. **Inches** which corresponds to 2 TEN-**Inches** & 7 **Inches** which however *changes* to 2 **Feet** & 3 **Inches**. There are two ways to look at it.
One way would be to deplore that the English did not match our TEN digits with a denominator for a collection of TEN ***inches***. The other way would be to deplore that we do not have six fingers on each hand because then we would probably be using TWELVE digits which would match the fact that **Foot** is a denominator corresponding to a collection of TWELVE ***inches***.
However, and to make the problem even worse, English denominators do not even all change at the same rate with the result that there is no way that numerators and denominators could ever be matched. For instance, while 1 **Foot** changes for TWELVE **Inches**, 1 **Yard** changes for 3 **Feet**, 1 **Furlong** changes for TWO-HUNDRED AND TWENTY **Yards**, 1 **Mile** changes for 8 **Furlongs**, etc.

2. Compare counting money
1 **Dime**, 2 **Dime**, . . . , 9 **Dime**,
1 **Dollar**, 1 **Dollar** & 1 **Dime**, 1 **Dollar** & 2 **Dime**, . . . , 1 **Dollar** & 9 **Dime**,
2 **Dollar**, 2 **Dollar** & 1 **Dime**, 2 **Dollar** & 2 **Dime**, . . . ,
⋮
. . . , 9 **Dollar** & 9 **Dime**,
1 **DekaDollar**, 1 **DekaDollar** & 1 **Dime**, . . . ,
⋮

with counting lengths

1 **Inch**, 2 **Inch**, . . . , 9 **Inch**, TEN **Inch**, ELEVEN **Inch**,
1 **Foot**, 1 **Foot** & 1 **Inch**, 1 **Foot** & 2 **Inch**, . . . , 1 **Foot** & ELEVEN **Inch**,
2 **Foot**, 2 **Foot** & 1 **Inch**, . . . , 2 **Foot** & ELEVEN **Inch**,
1 **Yard**, 1 **Yard** & 1 **Inch**, . . . , 1 **Yard** & ELEVEN **Inch**, 1 **Yard** & 1 **Foot**, 1 **Yard** & 1 **Foot** & 1 **Inch**,
⋮
. . . , TWO-HUNDRED-NINETEEN **Yard** & 2 **Foot** & ELEVEN **Inch**,
1 **Furlong**, 1 **Furlong** & 1 **Inch**, . . .
⋮

5.2 Adding Goods *on* the Counter

1. Since, regardless of the denominators, we work with number-phrases that are based on TEN digits, this makes addition *very* awkward, even quite tricky.

♠ Say we want to weld the two pipes in Figure 5.1.

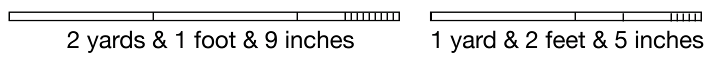


Figure 5.1: A 2 yrd, 1 ft, 9 in pipe and a 1 yrd, 2 ft, 5 in pipe.

When we measure the resulting pipe we find that its length is FOUR ***yards***, ONE ***foot***, TWO ***inches***. (Although we are of course much more likely to say that it is THIRTEEN ***feet***, TWO ***inches*** long.) ❖ On the board, we want to add 2 **Yard** & 1 **Foot** & 9 **Inch** and 1 **Yard** & 2 **Foot** & 5 **Inch** under the heading

Yards	Feet	Inches

The danger is to proceed with these *goods* just as if we were dealing with *money*:

Hamiltons	Washingtons	Dimes
	1	
2	1	9
1	2	5
3	4	4

Yards	Feet	Inches
2	1	9
1	2	5

and to conclude that the result of the addition is 3 **Yard** & 4 **Feet** & 4 **Inches** which of course does *not* represent what we found in the real world. The reason again is that it takes TWELVE **Inches** instead of TEN to get ONE **Foot** and it takes THREE **Feet** instead of TEN to get ONE **Yard**. So, of course, the addition should really proceed with the English rates of exchange, as follows:

Yards	Feet	Inches
1	*1*	
2	1	9
1	2	5
4	1	2

which indeed gives 4 **Yard** & 1 **Foot** & 2 **Inches**. This can be confusing

particularly if one does not write the denominators[1].
It is no wonder then that even the English gave up on English denominators! Note that, in the U. S., convenience prevailed over tradition in only a very few cases: money is exchanged TEN to ONE and surveying tapes are marked in ***tenths of a foot*** rather than ***inches***. And of course, systematic denominators prevail in all scientific matters.

2. In contrast, here is an example of how *addition* would go in the metric system.

5.3 Subtracting Goods *on* the Counter

text

5.4 Multiplication Goods *on* the Counter

We now come to *multiplication* which will turn out to be quite a bit more difficult than *addition* to introduce and to discuss.

The problems come from the fact that multiplication occurs in the representation of at least three very different environments which therefore need to be clearly differentiated.

- Multiplication as *additive power* of number-phrases. We saw in Chapter I that when counting the collection of objects we get from unpacking a collection of bundles, the numerator

- Multiplication as *co-multiplication* of number-phrases. In this case, we are multiplying *goods* by they *unit-price* to get their *money* equivalent.

- Multiplication as *multiplication* of number-phrases. While there are a lot of real life situations in which *addition* of number-phrases occurs naturally, there are a lot fewer real life situations in which *multiplication* of number-phrases does[2]. In the case of *addition* of number-phrases, we were able to start from its *meaning*, the aggregation of collections, and there was thus no doubt as to what the *result* was to be. This then allowed us to focus on developing the (board) procedure. By contrast, in

[1] This one is dedicated to the very special attention of those Educologists who claim that dispensing with denominators makes it all so much simpler and hence so much easier.

[2] Educologists will surely have noted that, while there are infinitely many vector spaces, there are only *four* (division) algebras—even if this was proven not that long ago! Bott and Milnor (1958)

the case of the *multiplication* of number-phrases, we must start by *finding* situations in which *multiplying* number-phrases will *mean* something. *If and when* it does mean something, then this will tell us what the resulting number-phrase might be and only then will it make sense to look for a (board) procedure that will give this resulting number-phrase. ×

5.4.1 Can Money Be Multiplied By Money?

First, and independently of whether or not *multiplying* counts might or might not mean anything, we introduce the symbol for *multiplication* that we will be using for it when writing on the board. We recall that, when we were dealing with *addition* and *subtraction*, we would write expressions involving two number-phrases with an addition symbol or a subtraction symbol in-between. For instance, we might have written 3 **Dimes** + 2 **Dimes** or 3 **Dimes** − 2 **Dimes**.

Similarly, multiplying counts would have to involve writing expressions involving two number-phrases with the multiplication symbol × in-between. For instance, we might write 3 **Dimes** × 2 **Dimes**.

At this point, though, we must clear up a frequent confusion: an expression like 3 **Dimes** × 2 **Dimes** is absolutely *not* the same as the expression 3 (2 **Dimes**)[3].

Now, we saw in Section **??** on page ?? that the expression 3 (2 **Dimes**) is nothing more than a number-phrase whose *numerator* is 3 and whose *denominator* (2 **Dimes**) represents a ***collection of two dimes*** so that, when we unpack, we get:

$$\begin{aligned} 3(2\ \textsf{Dimes}) &= (2\ \textsf{Dimes}), (2\ \textsf{Dimes}), (2\ \textsf{Dimes}) \\ &= \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime} \\ &= 6\ \textsf{Dimes} \end{aligned}$$

However, the fact that an expression on the board such as 3 (2 **Dimes**) makes perfect sense, that is, represents something on the counter, does *not* imply that an expression such as 3 **Dimes** × 2 **Dimes** also makes sense since they are expressions of a *different* kind.

For an expression such as 3 **Dimes** × 2 **Dimes** to make sense it would have to represent the result of doing something with THREE ***dimes*** and TWO

[3] Educologists, enamored of abstraction as they are, obviously have no trouble distinguishing 3 × 2 from the additive power 3(2) but it certainly does not seem to carry over very well to their students and, of course, even less to fractions.

length
construct
rectangle

dimes and coming up with a number of ***dimes*** the same way as 3 **Dimes** + 2 **Dimes** represented the result of *aggregating* THREE ***dimes*** and TWO ***dimes*** and the way 3 **Dimes** − 2 **Dimes** represented the result of *removing* TWO ***dimes*** from THREE ***dimes***.

The question then is: what could an expression on the board such as 3 **Dimes** × 2 **Dimes** possibly represent on the counter? The answer is: Absolutely nothing and expressions of the form 3 **Denominator** × 2 **Denominator** are *usually* completely meaningless.

5.4.2 Multiplying *Certain* Goods *on* the Counter

In the case of *certain goods*, though, expressions of the form 2 **Denominator** × 3 **Denominator** *can* represent the result of doing something with the collections represented by 2 **Denominator** and 3 **Denominator**. The objects with which this can be done are quite particular. The example we shall use is that of **length**, as in "a length of material" and we will thus draw our inspiration from "building materials" in which people speak, for instance, of a "four-by-eight" sheet of plywood.

We will begin with a very simple case and work our way up. At each stage, we will start with the more familiar *English* denominators and then look at the "same" example with *metric* denominators. The first stage will *not* involve any carryover because, as we already saw in the case of *addition*, English denominators do not lend themselves easily to computation since the English exchange rates are not always the same as is the case in the metric system. Only after we will have figured out what multiplying number-phrases might mean and what the resulting number-phrase then is, will we deal with the technical issue of "carryovers".

1. The point of this example is to observe that, contrary to what was the case with *addition* and *subtraction*, where the denominator in the result of the operation was the *same* as the denominator in the number phrases being operated on, in the case of *multiplication*, the denominator in the result is *different* from the denominator in the number phrases being operated on.

a. Here it is with English denominators.

♠ Given a TWO ***inch length*** and a THREE ***inch length***,

- We can **construct** on the counter (Figure 5.2 on the facing page) a ***two-by-three rectangle***, that is a **rectangle** that is TWO ***inch long*** one way and THREE ***inch long*** the other way:

tile
multiplication tables

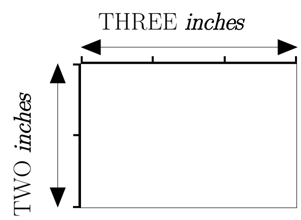


Figure 5.2: A “TWO by THREE” rectangle.

- We may then want to **tile** this rectangle (Figure 5.3) with ***one-inch-by-one-inch mosaics***:

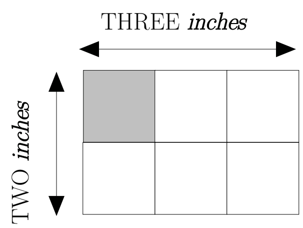


Figure 5.3: The TWO by THREE rectangle tiled with ***mosaics***

Counting the ***mosaics*** shows that we will need SIX ***one-inch-by-one-inch mosaics***.

❖ The expression 2 **Inch** × 3 **Inch** then represents on the board the ***mosaics*** that will be needed to tile the rectangle[4].
Since, as children, we are usually enjoined to memorize the **multiplication tables**, the (board) procedure for multiplication in this case consists in looking up the relevant multiplication table. We find that

$$2\ \textsf{Inch} \times 3\ \textsf{Inch} = 6\ [\textsf{Inch} \times \textsf{Inch}].$$

where 6 is the numerator and where [**Inch** × **Inch**] is the denominator that represents ***one-inch-by-one-inch mosaics*** on the board.

b. We now look at the “same” example but with *metric* denominators.
♠ For instance, given a TWO ***meters*** length and a THREE ***meters*** length, we can construct on the counter (Figure 5.4 on the following page) a ***two-meter-by-three-meter rectangle***, that is a rectangle that is TWO ***meters*** long one way and THREE ***meters*** long the other way and then we can tile it with ***one-meter-by-one-meter tiles*** See Figure 5.4 on the next page:

[4]Educologists will surely deplore that this definition is **robust** in the sense that it will withstand generalization.

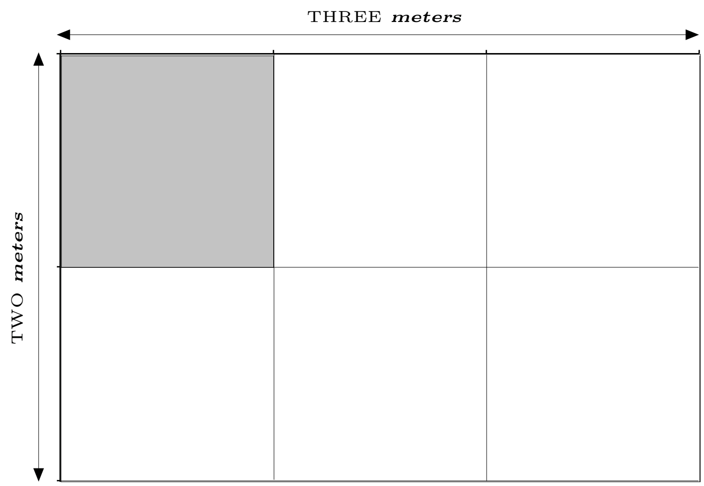


Figure 5.4: A TWO ***meter*** by THREE ***meter*** rectangle tiled with ***one-meter-by-one-meter tiles.***

Counting the ***tiles*** shows that we will need SIX ***one-meter-by-one-meter tiles***.

❖ The expression 2 Meter × 3 Meter then represents on the board the ***tiles*** that will be needed to tile the rectangle. Looking up the relevant multiplication table gives

$$2 \text{ Meter} \times 3 \text{ Meter} = 6 \ [\text{Meter} \times \text{Meter}].$$

where 6 is the numerator and where [Meter × Meter] is the denominator that represents ***one-meter-by-one-meter tiles*** on the board.

2. The point of this example is to show that, also contrary to what was the case with *addition*, where the two denominators in the number-phrases being added *had* to be the *same*, in the case of *multiplication*, the denominators of the number-phrases being multiplied *can* be *different*. Indeed, the two sides of a rectangle are often measured with different denominators.

a. We begin with an example involving the more familiar English denominators.

♠ For instance, given a THREE ***inch*** length and a TWO ***foot*** length, we can cut on the counter a ***three-inch-by-two-foot plank***, that is a rectangle that is THREE ***inches*** long one way and TWO ***feet*** long the other way. We may then want to tile this plank with ***one-inch-by-one-foot strips***

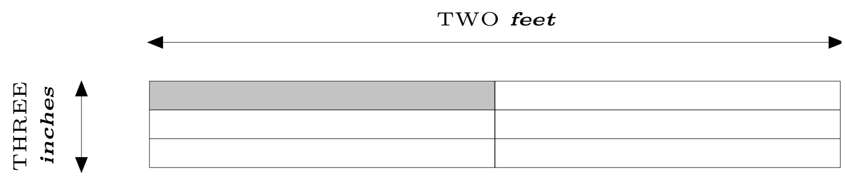


Figure 5.5: A THREE ***inch*** by TWO ***foot*** rectangle tiled with ***one-inch-by-one-foot strips***

Counting the ***strips*** shows that we will need SIX ***one-inch-by-one-foot strips***.

❖ The expression 3 **Inch** × 2 **Foot** then represents on the board the ***strips*** that will be needed to tile the rectangle on the counter. We find that

$$3 \text{ Inch} \times 2 \text{ Foot} = 6 \text{ [Inch} \times \text{Foot]}.$$

where 6 is the numerator and where [**Inch** × **Foot**] is the denominator that represents ***one-inch-by-one-foot strips*** on the board.

b. We now look at the "same" example but with metric denominators

♠ For instance, given a THREE ***meters*** length and a TWO ***dekameters*** length, we can cut on the counter a ***three-meters-by-two-dekameters rectangle***, that is a rectangle that is THREE ***meters*** long one way and TWO ***dekameters*** long the other way. We may then want to tile this rectangle with ***one-meter-by-one-dekameters strips*** (Figure 5.6)

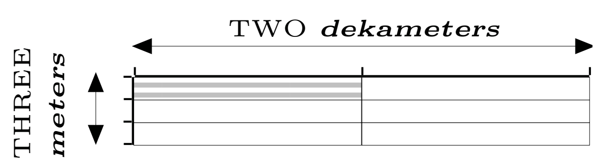


Figure 5.6: A THREE ***meters*** by TWO ***dekameters*** rectangle tiled with ONE ***meters*** by ONE ***dekameters*** strips

❖ The expression 3 **Meter** × 2 **DekaMeter** then represents on the board the ***strips*** that will be needed to tile the rectangle on the counter. We find that

$$3 \text{ Meter} \times 2 \text{ DekaMeter} = 6 \text{ [Meter} \times \text{DekaMeter]}.$$

where 6 is the numerator and where [**Inch** × **Foot**] is the denominator that representsONE ***dekameters*** strips on the board.

3. The point of this example is to show that, essentially in the same manner, we can multiply *combinations* of lengths.

a. We begin with English denominators.

square
rectangular

♠ For instance, given a TWO ***foot***, TWO ***inch*** length and a THREE ***foot***, ONE ***inch*** length, we can construct and tile the rectangle as in Figure 5.7)

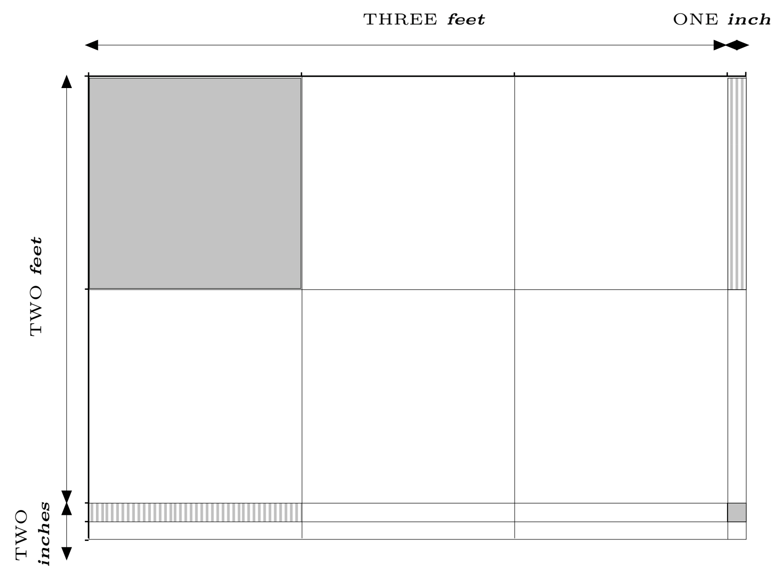


Figure 5.7: The ***three-foot, one-inch by two-foot, three-inch rectangle*** tiled with four different kinds of tiles

Counting the ***tiles*** shows that we will need:
the following **square** tiles

- SIX ***one-foot-by-one-foot tiles***,
- TWO ***one-inch-by-one-inch tiles***,

and the following **rectangular** tiles

- TWO ***one-foot-by-one-inch tiles***,
- SIX ***one-inch-by-one-foot tiles***,

The ***one-inch-by-one-foot tiles*** and the ***one-foot-by-one-inch tiles*** are counted *separately* if only because of the different ways they are striped.
❖ The expression [3 **Foot** & 1 **Inch**] × [2 **Foot** & 3 **Inch**] then represents on the board the ***tiles*** that we will need to tile it.
The (board) procedure for multiplication in this case is a bit more complicated. First we set up:

$$\begin{array}{rrcr} & 3\ \textsf{Foot} & \& & 1\ \textsf{Inch} \\ \times & 2\ \textsf{Foot} & \& & 2\ \textsf{Inch} \\ \hline \end{array}$$

The next step is to get the different kinds of tiles using the appropriate *multiplication tables*. Observe that we are handling **Inch**×**Foot** and **Foot**×**Inch** separately:

				3 **Foot**	&	1 **Inch**
			×	2 **Foot**	&	2 **Inch**
				6 **Inch**×**Foot**	&	2 **Inch**×**Inch**
6 **Foot**×**Foot**	&	2 **Foot**×**Inch**				

Altogether, we thus find:

3[3 **Foot** & 1 **Inch**] × [2 **Foot** & 2 **Inch**] = 6 **Foot**×**Foot**,
& 2 **Foot**×**Inch**
& 6 **Inch**×**Foot**
& 2 **Inch**×**Inch**

which is the (board) representation of the above.

b. We now look at the "same" example but with metric denominators ♠ Given a TWO ***dekameter***, TWO ***meter*** length and a THREE ***dekameter***, ONE ***meter*** length , we can construct and tile a rectangle as in Figure 5.8)

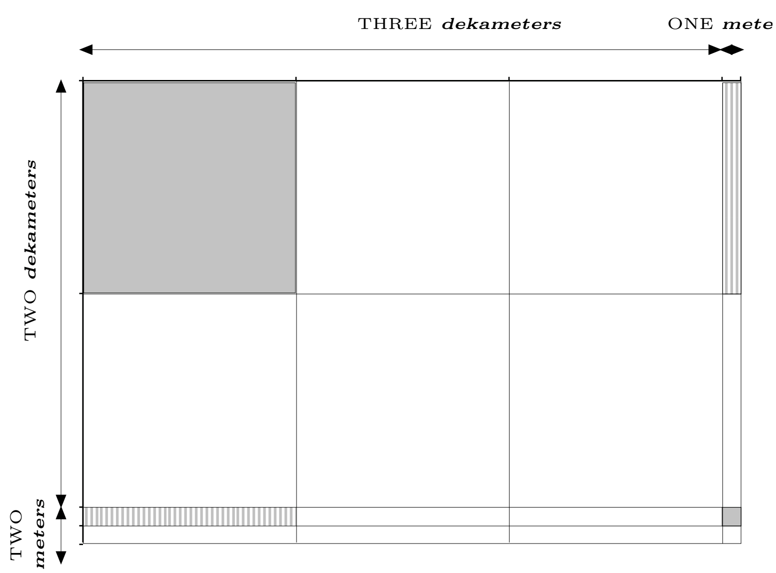


Figure 5.8: The ***two-dekameter, two-meter by three-dekameter, one-meter rectangle*** tiled with four different kinds of tiles

Counting the ***tiles*** shows that we will need:
The following *square* tiles

- SIX ***one-dekameter-by-one-dekameter tiles***,
- TWO ***one-meter-by-one-meter tiles***,

The following *rectangular* tiles

- TWO ***one-dekameter-by-one-meter tiles***,
- SIX ***one-meter-by-one-dekameter tiles***,

The rectangular tiles, ***one-meter-by-one-dekameter tiles*** and ***one-dekameter-by-one-meter tiles***, are counted *separately* if only because of the different ways they are striped.

❖ Exceptionally, for reasons of space, here we abbreviate **Meter** as **M** and **DekaMeter** as **DekaM**.

The expression [3 **DekaM** & 1 **M**] × [2 **DekaM** & 3 **M**] then represents on the board the ***tiles*** that we will need to tile the rectangle. We get:

				3 DekaM	&	1 M
			×	2 DekaM	&	2 M
				6 M×DekaM	&	2 M×M
6 DekaM×DekaM	&	2 DekaM×M				

Altogether, we thus find:

[3 **DekaM** & 1 **M**] × [2 **DekaM** & 2 **M**] = 6 **DekaM**×**DekaM**,
& 2 **DekaM**×**M**
& 6 **M**×**DekaM**
& 2 **M**×**M**

which is the (board) representation of the above.

4. The point now is to see what happens in the preceding example if we don't care about the way the tiles are striped or if they are striped the same way.

a. We begin with English denominators

♠ First, observe (Figure **??** on page ??) that a ***one-inch-by-one-foot rectangle*** and a ***one-foot-by-one-inch rectangle*** can both be tiled with TWELVE ***one-inch-by-inch-inch mosaics***.

common denominator

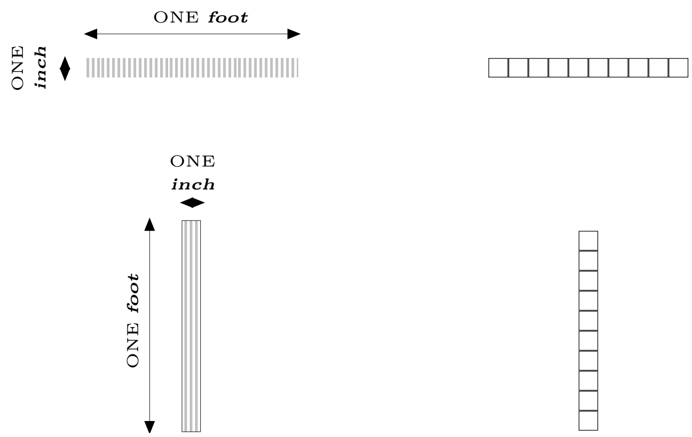


Figure 5.9: Both a ***one-inch-by-one-foot rectangle*** and a ***one-foot-by-one-inch rectangle*** can be tiled with TWELVE ***one-inch-by-inch-inch mosaics***

Thus, from that viewpoint, the SIX ***one-foot-by-one-inch tiles*** and the TWO ***one-inch-by-one-foot tiles*** in Figure 5.7 on page 78 are of the same kind and we can *aggregate* them.

❖ We set up in the way we learned in elementary school because it will make it easier to *add* **Inch**×**Foot** and **Foot**×**Inch**.

Which *denominator* to use, **Inch**×**Foot** or **Foot**×**Inch**, is up to us but we need to agree on it.

		3 **foot**	&	1 **inch**
	×	2 **foot**	&	2 **inch**
		6 **Inch**×**Foot**	&	2 **Inch**×**Inch**
6 **Foot**×**Foot**	&	2 **Foot**×**Inch**		
6 **Foot**×**Foot**	&	8 **Foot**×**Inch**	&	2 **inch** × **Inch**

if we agree on **Foot**×**Inch** as **common denominator**, or

6 **Foot**×**Foot**	&	8 **Inch**×**Foot**	&	2 **inch**×**Inch**

if we agree on **Inch**×**Foot** as common denominator.

b. We look at the "same" example but with metric denominators.

♠ First, observe (Figure 5.10 on the next page) that a ***one-meter-by-one-dekameter rectangle*** and a ***one-dekameter-by-one-meter rectangle*** can both be tiled with TEN ***one-meter-by-inch-meter tiles***.

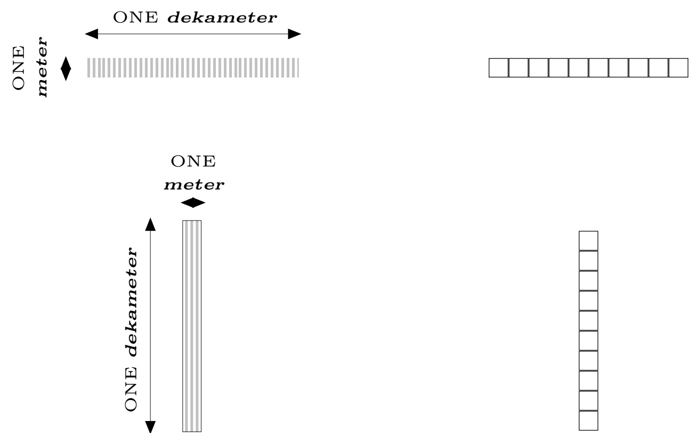


Figure 5.10: Both a ***one-meter-by-one-dekameter rectangle*** and a ***one-dekameter-by-one-meter rectangle*** can be tiled with TEN ***one-meter-by-one-meter tiles***

Thus, from that viewpoint, the SIX ***one-dekameter-by-one-meter tiles*** and the TWO ***one-meter-by-one-dekameter tiles*** in Figure **??** on page ?? are of the same kind and we can *aggregate* them.

❖ We set up again in the way we learned in elementary school because it will make it easier to *add* **Meter×DekaMeter** and **DekaMeter×Meter**.

		3 **DekaMeter**	&	1 **Meter**
	×	2 **DekaMeter**	&	2 **Meter**
		6 **Meter×DekaMeter**	&	2 **Meter×Meter**
6 **DekaMeter×DekaMeter**	&	2 **DekaMeter×Meter**		
6 **DekaMeter×DekaMeter**	&	8 **DekaMeter×Meter**	&	6 **Meter×Meter**

if we agree on **DekaMeter×Meter** as common denominator, or

6 **DekaMeter×DekaMeter**	&	8 **Meter×DekaMeter**	&	2 **Meter×Meter**

if we agree on **Meter×DekaMeter** as common denominator.

5. To see how multiplication works when we have "carryovers", we will only use metric denominators because, as we already saw in Section xxx, English denominators do not lend themselves easily to computation since the English exchange rates are not always the same. (For instance, 1 **Foot** = TWELVE **Inch** while 1 **Yard** = 3 **Feet**.)

a. First we look at an example where the carryover will occur in the *addition*.

♠ Given a THIRTY-TWO ***meter*** length and a TWENTY-THREE ***meter*** length, we look at them as being made-up as follows:

- We look at the THIRTY-TWO ***meter*** length as being made up of THREE ***dekameters*** and TWO ***meters***
- We look at the TWENTY-THREE ***meter*** length as being made up of TWO ***dekameters*** and THREE ***meters***

We then construct a THIRTY-TWO ***meter*** by TWENTY-THREE ***meter*** rectangle, that is a rectangle that is THREE ***dekameters*** and TWO ***meters*** long one way and TWO ***dekameters*** and THREE ***meters*** long the other way.

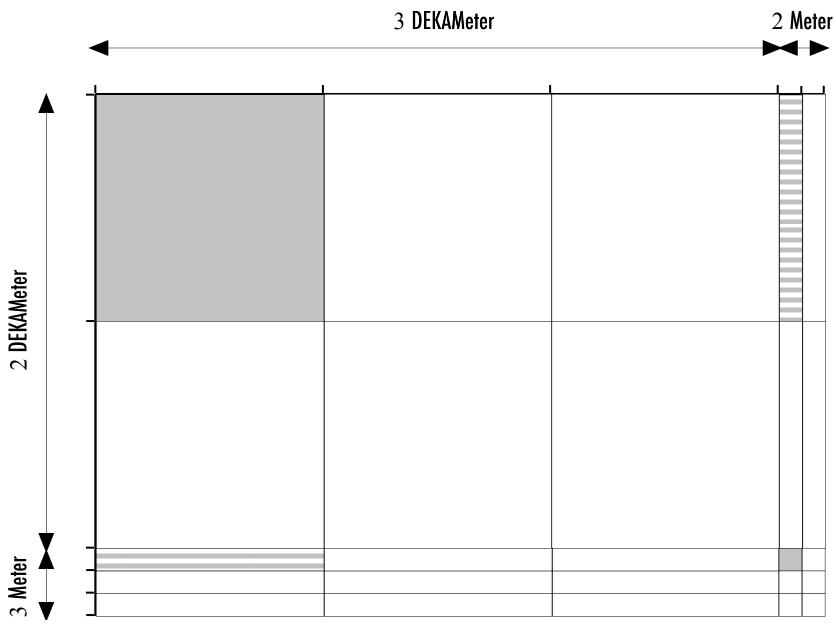


Figure 5.11: The tiling of a THIRTY-TWO ***meter*** by TWENTY-THREE ***meter*** rectangle.

Counting the ***tiles*** shows that we will need:

- SIX ***one-dekameter-by-one-dekameter tiles***
- FOUR ***one-dekameter-by-one-meter tiles***
- NINE ***one-meter-by-one-dekameter tiles***
- SIX ***one-meter-by-one-meter tiles***

Since we don't distinguish the ***one-dekameter-by-one-meter tiles*** from the ***one-meter-by-one-dekameter tiles***, we can aggregate them and we get

THIRTEEN ***one-dekameter-by-one-meter tiles***

or

THIRTEEN ***one-meter-by-one-dekameter tiles***

depending on how we want to see them.

However, since we are not going to be able to write THIRTEEN, we must *change* THIRTEEN ***one-dekameter-by-one-meter tiles*** (or THIRTEEN ***one-meter-by-one-dekameter tiles***) and the question is for what?

Figure 5.12 shows that TEN ***one-meter-by-one-dekameter tiles*** tile ONE ***one-dekameter-by-one-dekameter tiles***

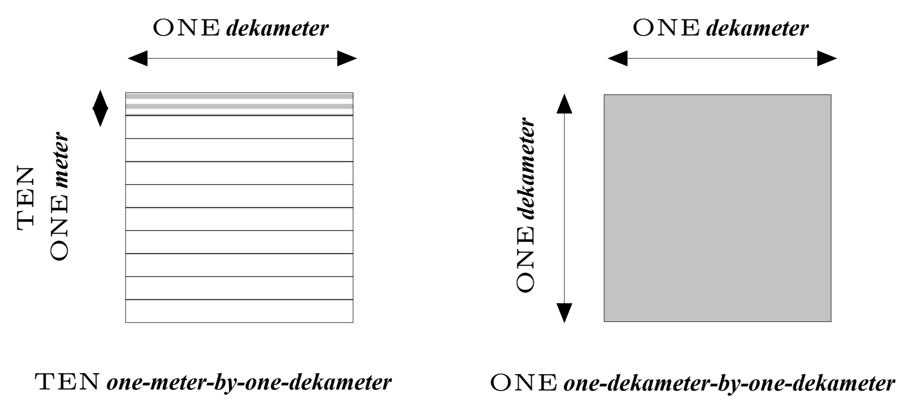


Figure 5.12: Changing TEN ***one-meter-by-one-dekameter tiles***

❖ Here again, the (board) procedure reflects what we just did.

		3 DekaM	&	2 M
	×	2 DekaM	&	3 M
		9 M×DekaM	&	6 M×M
6 DekaM×DekaM	&	4 DekaM×M		
6 DekaM×DekaM	&	THIRTEEN DekaM×M	&	6 M×M

or

6 DekaM×DekaM	&	THIRTEEN M×DekaM	&	6 M×M

And we complete the (board) procedure as follows

		3 DekaM	&	2 M
	×	2 DekaM	&	3 M
1 DekaM×DekaM				
		9 M×DekaM	&	6 M×M
6 DekaM×DekaM	&	4 DekaM×M		
7 DekaM×DekaM	&	3 DekaM×M	&	6 M×M

or

7 DekaM×DekaM	&	3 M×DekaM	&	6 M×M

b. Now we look at an example where the carryover will occur in one of the *multiplications*

♠

❖ First we proceed as we did before:

		1 DekaM	&	5 M
	×	1 DekaM	&	3 M
		3 M×DekaM	&	fifteen M×M
1 DekaM×DekaM	&	5 DekaM×M		
1 DekaM×DekaM	&	8 DekaM×M	&	fifteen M×M

if we agree on DekaM×M as common denominator, or

1 DekaM×DekaM	&	8 M×DekaM	&	fifteen M×M

if we agree on M×DekaM as common denominator.
Now we must change fifteen M×M for 1 DekaM & 5 M×M:

		1 DekaM	&	5 M
	×	1 DekaM	&	3 M
		1 M×DekaM		
		3 M×DekaM	&	5 M×M
1 DekaM×DekaM	&	5 DekaM×M		
1 DekaM×DekaM	&	9 DekaM×M	&	5 M×M

if we agree on DekaM×M as common denominator, or

1 DekaM×DekaM	&	9 M×DekaM	&	5 M×M

if we agree on M×DekaM as common denominator.
Of course, we shouldn't wait and we should do the change *immediately* rather than write fifteen M×M

6. We shall now see how the above multiplication looks under a *heading*.

square denominator

a. First, we recall that the metric heading for lengths is:

KILO Meter	HECTO Meter	DEKA Meter	Meter	DECI Meter	CENTI Meter	MILLI Meter

and that the rate of change is TEN for 1.

b. Corresponding to each of these denominators, we have the corresponding **square denominator**:

- **MILLIMeter×MILLIMeter** also called **Square MILLIMeter**
- **CENTIMeter×CENTIMeter** also called **Square CENTIMeter**
- **DECIMeter×DECIMeter** also called **Square DECIMeter**
- **Meter×Meter** also called **Square Meter**
- **DEKAMeter×DEKAMeter** also called **Square DEKAMeter**
- **HECTOMeter×HECTOMeter** also called **Square HECTOMeter**
- **KILOMeter×KILOMeter** also called **Square KILOMeter**

Square KILO Meter		Square HECTO Meter		Square DEKA Meter		Square Meter		Square DECI Meter		Square CENTI Meter		Square MILLI Meter

Note that the rate of change from one denominator to the next is still TEN to 1 and that the *empty* spaces correspond to the *non-square* denominators. For instance, the empty space between **Square Meter** and **Square DEKAMeter** is for

DEKAMeter×Meter

or

Meter×DEKAMeter

Note that the rate of change from one square denominator to the next is HUNDRED to 1.

HUNDRED ***one-meter-by-one-meter tiles*** ONE ***one-dekameter-by-one-dekameter tile***

Figure 5.13: Changing HUNDRED ***one-meter-by-one-meter tiles***

c. We now write the above multiplication under the heading:

Square DEKAMeter		Square Meter
1		
	3 2	2 3
 6	9 4	6
7	3	6

exchange
common denominator

Chapter 6

Fractions

We now deal with the case when the two collections consist of *different* kinds of objects.

1. In that case, we just *cannot* aggregate the two collections and, on the board, we can only write them as a *combination* and that is that!

For instance, given ***dollar, dollar*** and ***dime, dime, dime***, on the counter, we can only write the combination 2 **Dollars** & 3 **Dimes**. (Even if we were to "put together" ***dollar***, ***dollar*** and ***dime***, ***dime***, ***dime*** as ***dollar***, ***dollar***, ***dime***, ***dime***, ***dime***, we would not have a *collection*: what *denominator* would we use in the number-phrase?)

2. However, *if* we can **exchange** the objects in the two collections for collections of *identical* objects, *then* we have a **common denominator** and the combination can then be "turned" into an addition.

♠ For instance, if we can change ***quarter*** for ***nickel***, ***nickel***, ***nickel***, ***nickel***, ***nickel*** and ***dime*** for ***nickel***, ***nickel***, then we can change
quarter and ***dime***, ***dime***
for
nickel, ***nickel***, ***nickel***, ***nickel***, ***nickel*** and ***nickel***, ***nickel***, ***nickel***, ***nickel***,
that is for
nickel, ***nickel***, ***nickel***, ***nickel***, ***nickel***, ***nickel***, ***nickel***, ***nickel***, ***nickel***.

❖ On the board, we proceed exactly in the same manner as on the counter and we write:

If: 1 **Quarter** $=$ 5 **Nickels** and 1 **Dime** $=$ 2 **Nickels**,

then: 1 **Quarter** & 2 **Dime** $= 5$ **Nickels** $+ 2(2$ **Nickels**$)$

$= 9$ **Nickels**

However, while changing facilities will often be available, we should not expect that they always will.

1. We already saw above that the same ideas continue to apply even when the rate of exchange from one denominator to the next is not always the same or does not exist as for instance when we deal with goods in the British units. But, as we also saw already there, these ideas do not work as smoothly. We return to money because, even there,

Hundred-dollar-bill
Fifty-dollar-bill
Twenty-dollar-bill
Ten-dollar-bill
Five-dollar-bill

with ***Hundred-dollar-bill***, ***Fifty-dollar-bill***, ***Twenty-dollar-bill***, ***Ten-dollar-bill***, ***Five-dollar-bill***, ***One-dollar-bill***, we observe that 1 ***Hundred-dollar-bill*** exchanges for 2 ***Fifty-dollar-bill*** but that 1 **Fifty** does not exchange for any number of ***Twenty-dollar-bill***. However, both ***Fifty-dollar-bill*** and ***Twenty-dollar-bill*** exchange for any of the following common denominators: ***Ten-dollar-bill***, ***Five-dollar-bill***, ***One-dollar-bill***.

It is a mildly interesting exercise to learn how to add and subtract under such headings but we shall turn our attention to something more fruitful which is how to introduce ever-smaller units, that is going *downward*.

This is of course what we just did above, in Systematic Arithmetic, when we defined ***dime***, ***cent*** and ***mill***. For instance, starting with ***One-dollar-bill*** as origin, 1 ***quarter*** is defined as 1 of-which-4-will-exchange-for-1-***One-dollar-bill***, where the phrase of-which-4-will-exchange-for-1-***One-dollar-bill*** is the denominator , and, say, 3 ***quarter*** is read as 3 of-which-4-will-exchange-for-1-***One-dollar-bill***.

Unfortunately, such denominators are traditionally coded in a manner that, if historical, is not very evocative of what they represent. The best way to deal with this issue is to morph one into the other:

i. 3 ***quarter*** (In which ***quarters*** is the denominator.)

ii. 3 of-which-4-will-exchange-for-1-***One-dollar-bill*** (In which of-which-4-will-exchange-for-1-***One-dollar-bill*** is the denominator.)

iii. 3 4 ? 1-***One-dollar-bill*** (In which 4 ? 1-***One-dollar-bill*** is the denominator.)
iv. (in which is the denominator.)
v. (in which is the denominator.)
vi. ***One-dollar-bill*** (in which ***One-dollar-bill*** is the denominator.)

Note that, here, the horizontal line in iv, v, and vi is merely a separator called fraction bar. Note that in v and vi there ought to be a1 before ***One-dollar-bill*** but it goes without saying.

Unfortunately, two things happened that made the use of fractions a lot less easy than it should be. The first is that, as usual, the unit, here ***One-dollar-bill***, almost always goes without saying. The second is that the 4 in the above code is called the denominator while it is only part of the denominator.

2. However, from this we can now deal easily with the usual topics regarding fractions.

a. For some reason lost in history, a fraction whose numerator is more than the denominator is deemed to be improper and to some extent inadmissible. What one is supposed to do is to change a maximum of parts for units. For instance, given 7 of-which-4-will-exchange-for-1-***One-dollar-bill***, we can exchange 4 of-which-4-will-exchange-for-1-***One-dollar-bill*** for 1 ***One-dollar-bill*** after which we have the combination 1 ***One-dollar-bill*** & 3 of-which-4-will-exchange-for-1-***One-dollar-bill***. Similarly, given 23 of-which-4-will-exchange-for-1-***One-dollar-bill***, we want to exchange as many of them for ***One-dollar-bill***s. So we must find out how many ***One-dollar-bill***s we will be getting and how many parts will remain. In other words, we divide 23 by 4 to get the number of ***One-dollar-bill*** and the number of those of-which-4-will-exchange-for-1-***One-dollar-bill*** that will remain.

Since ***One-dollar-bill***s = 5 ***One-dollar-bill***s & 3 of-which-4-will-exchange-for-1 ***One-dollar-bill***, it is natural to exchange 5 ***One-dollar-bill***s for 20 of-which-4-will-exchange-for-1 ***One-dollar-bill*** after which they have 23 of-which-4-will-exchange-for-1-***One-dollar-bill***. And vice versa.

b. When the denominators are the same, they get ***One-dollar-bill*** + ***One-dollar-bill*** = ***One-dollar-bill*** in exactly the same manner as 7 ***nickel*** + 9 ***nickel*** = 16 ***nickel***.

c. When the denominators are not the same, they must change to a common denominator before they can add: ***One-dollar-bill*** & ***One-dollar-bill*** = ***One-dollar-bill*** + ***One-dollar-bill*** = ***One-dollar-bill*** in exactly the same manner as 7 ***dime*** & 3 ***quarter*** = 14 ***nickel*** + 15 ***nickel*** = 29 ***nickel***.

3. In the case of multiplication and division, the difficulty is, again, with the denominators but a picture readily shows that, for instance, = 24 1-inch x 1-inch & 12 1-inch x 1-of-which-5-will-exchange-for-1-inch & 6 1-of-which-2-will-exchange-for-1-inch x 1-inch & 3 1-of-which-2-will-exchange-for-1-inch x 1-of-which-5-will-exchange-for-1-inch, that is, more familiarly, = 24 + 12 + 6 + 3 = 27 + + .
Note. While one might want at this point to make the students realize that , it is of no use here and, by the time they will have to use it, that is when investigating functions, they will have forgotten. Anyhow, it will be much more satisfying to do with the notion of nearness, in conjunction with .

coefficient
original
exponent

Chapter 7

Multiplicative Powers

7.1 Repeated multiplication/division

Given a number a, we shall often have to multiply or divide it by a number of copies of some other number x

1. We begin by discussing the corresponding language.

- $a\ (x)^{+3}$ is to be read as a *multiplied* by 3 copies of x
- $a\ (x)^{-3}$ is to be read as a *divided* by 3 copies of x

  where
  - the number a is called the **coefficient**,
  - the number x is called the **original**
  - the number, $+3$ or -3, is called the **exponent** where
    - the counting-number 3 indicates the number of copies to be made of the original
    - the sign, $+$ or $-$, indicates whether the coefficient is to be *multiplied* or *divided* by the copies

Occasionally, the exponent will turn out to be 0, but, even in that case, we will continue to have

- $a\ (x)^{0}$ is to be read as a *multiplied/divided* by 0 copies of x

In this last case, we thus have

$$a\ (x)^0 = a$$

But then, by comparison with

$$a \cdot (+1) = a$$

we can conclude that

$$(x)^0 = +1$$

positive third power
negative third power
zeroth power

2. When replacing a and x by signed numbers, it is safer to enclose them within parentheses. For instance, with $a = +7$ and $x = +5$, we write:

$$\begin{aligned}(+7)(+5)^{+3} &= +7 \textit{ multiplied} \text{ by 3 copies of } +5 \\ &= (+7)\cdot(+5)\cdot(+5)\cdot(+5) \\ &= (+7)\cdot(+125) \\ &= +825\end{aligned}$$

and

$$\begin{aligned}(+7)(+5)^{-3} &= +7 \textit{ divided} \text{ by 3 copies of } +5 \\ &= \frac{+7}{(+5)\cdot(+5)\cdot(+5)} \\ &= \frac{+7}{+125} \\ &= +0.056\end{aligned}$$

and

$$\begin{aligned}(+7)(+5)^{0} &= +7 \textit{ multiplied} \text{ by 0 copies of } +5 \\ &= +7\end{aligned}$$

Altogether, we have

The code:	to be read as:	says to write:	gives:
$(+7)\ (+5)^{+3}$	+7 multiplied by 3 copies of +5	$(+7) \times (+5)(+5)(+5)$	$(+7)\cdot(+125)$ $= +875$
$(+7)\ (+5)^{-3}$	+7 divided by 3 copies of +5	$(+7) \div (+5)(+5)(+5)$	$\frac{+7}{+125}$ $= +0.056$
$(+7)\ (+5)^{0}$	+7 multiplied by 0 copy of +5	+7	$= +7$

7.2 Powers

Of course, + and +1 and, even more so, +1 tend to go without saying.

1. In particular, the *coefficient* +1 usually goes without saying and then

- $(x)^{+3}$ is called the **positive third power** of x
- $(x)^{-3}$ is called the **negative third power** of x
- $(x)^{0}$ is called the **zeroth power** of x

gauge power

In other words, powers can be looked upon as repeated multiplications involving the coefficient $+1$ with the latter going without saying.
However, it will be more convenient for us to use both the coefficient $+1$ and the coefficient -1. Thus,

- $+(+5)^{+3}$ will mean $(+1) \cdot (+5)^{+3}$
- $-(+5)^{+3}$ will mean $(-1) \cdot (+5)^{+3}$
- $+(+5)^{-3}$ will mean $(+1) \cdot (+5)^{-3}$
- $-(+5)^{-3}$ will mean $(-1) \cdot (+5)^{-3}$

and we will call these **gauge powers**.

2. When it is the *exponent* which is equal to $+1$, it is less of an issue to let it go without saying since

- $(+7)(+5)^{+1}$ is to be read as $+7$ multiplied by 1 copy of $+5$, that is $(+7)(+5)$,

while

- $(+7)(+5)$ is to be read as $+7$ multiplied by $+5$, that is $(+7)(+5)$

which is exactly the same as above.
On the other hand, the exponent -1 can never go without saying.

3. When replacing x by a negative number, for example -5, in a power, one should be careful that

-5^{+4} does *not* work out to the same number as $(-5)^{+4}$

Indeed,

- in -5^{+4}, the $-$ stands for the coefficient -1 and the 5 stands for $+5$ and is the original of which the copies are to be made. In other words, we have:

$$\begin{aligned} -5^{+4} &= (-1)\ (+5)^{+4} \\ &= (-1) \text{ multiplied by 4 copies of } (+5) \\ &= (-1) \cdot (+5) \cdot (+5) \cdot (+5) \cdot (+5) \\ &= -625 \end{aligned}$$

- in $(-5)^{+4}$ the coefficient is $+1$, going entirely without saying, and -5 is the orginal of which the copies are to be made. In other words, we have:

$$\begin{aligned} (-5)^{+4} &= (+1)\ (-5)^{+4} \\ &= (+1) \text{ multiplied by 4 copies of } (-5) \\ &= (+1) \cdot (-5) \cdot (-5) \cdot (-5) \cdot (-5) \\ &= +625 \end{aligned}$$

On the other hand,

-5^{+3} *does* work out to the same number as $(-5)^{+3}$

parity

Indeed,

- in -5^{+4} the $-$ stands for the coefficient -1 and 5 standing for $+5$ is the original of which the copies are to be made. In other words, we have:

$$\begin{aligned}-5^{+3} &= (-1)\,(+5)^{+3}\\ &= (-1) \text{ multiplied by 3 copies of } (+5)\\ &= (-1)\cdot(+5)\cdot(+5)\cdot(+5)\\ &= -125\end{aligned}$$

- in $(-5)^{+3}$ the coefficient is $+1$, going entirely without saying, and -5 is the original of which the copies are to be made. In other words, we have:

$$\begin{aligned}(-5)^{+3} &= (+1)\,(-5)^{+3}\\ &= (+1) \text{ multiplied by 3 copies of } (-5)\\ &= (+1)\cdot(-5)\cdot(-5)\cdot(-5)\\ &= -125\end{aligned}$$

This has nothing to do with the sign of the exponent and the same would hold true with a negative exponent.
What is involved here is the **parity** of the exponent, that is whether the exponent indicates an *odd* number or an *even* number of copies. This will turn out to be extremely important when we investigate power functions.

7.3 Roots

+ (...)

Chapter 8

Real Numbers And Their Graphic Representation

Engineers like to say that the only *real* Real Numbers are the Decimal Numbers [1].

8.1 Decimal Approximations

Since, other than counting-numbers that were defined *directly*, all other numbers were specified *indirectly*, that is as solutions of equations that we can usually solve only approximately, we will use real numbers only as code to designate collectively all the many Decimal Numbers that are approximate solutions of the equation[2].

After whatever decimal number we use, though, we shall have to write **+ (...)**. read "plus a little bit too small to matter in the current situation" because the equality will not be exact [3]. We look at a couple of examples.

Example 1. We view $\frac{11}{7}$ as a *specifying-phrase*, namely the solution of the equation $7x = 11$, that is $\frac{11}{7}$ stands for whatever decimal number 7 copies of which will *add* to 11 within the approximation required by the situation. So, we might use any one of the following.

[1] As opposed to Educologists, "Just plain folks", as well as engineers, have no particular reason to distinguish repeating from non-repeating decimals.

[2] In other words, we look at Real Numbers are just equivalence classes, of Dedekind cuts from the order point of view, or of Cauchy sequences from the metric point of view! Even non-standard infinitesimals, should one wish to develop them eventually, can be looked upon as an extension of the decimal numbers.

[3] In other words, (...) is a "precursor" of the little o notation.

- $\frac{11}{7} = 1 + (...)$
  because
  $7 \cdot \big[1 + (...)\big] = 7 \cdot 1 + 7 \cdot (...) = 7 + (...)$
- $\frac{11}{7} = 1.5 + (...)$
  because
  $7 \cdot \big[1.5 + (...)\big] = 7 \cdot 1.5 + 7 \cdot (...) = 10.5 + (...)$
- $\frac{11}{7} = 1.57 + (...)$
  because
  $7 \cdot \big[1.57 + (...)\big] = 7 \cdot 1.57 + 7 \cdot (...) = 10.99 + (...)$
- $\frac{11}{7} = 1.571 + (...)$
  because
  $7 \cdot \big[1.571 + (...)\big] = 7 \cdot 1.571 + 7 \cdot (...) = 10.997 + (...)$
- $\frac{11}{7} = 1.5714 + (...)$
  because
  $7 \cdot \big[1.5714 + (...)\big] = 7 \cdot 1.5714 + 7 \cdot (...) = 10.99998 + (...)$

Thus, in practice, depending on the circumstances, we might replace $\frac{11}{7}$ by any one of the following:

$1 + (...)$
$1.5 + (...)$
$1.57 + (...)$
$1.571 + (...)$
$1.5714 + (...)$

Example 2. Similarly, we view $\sqrt[3]{13}$ as a *specifying-phrase* standing for whatever *decimal number* is a solution of the equation $x^3 = 13$ that is, $\sqrt[3]{13}$ stands for whatever decimal number 3 copies of which will *multiply* to 13 within the approximation required by the situation. So, we might use any one of the following.

- $\sqrt[3]{13} = 2 + (...)$
  because
  $\big[2 + (...)\big]^3 = 8 + (...)$
- $\sqrt[3]{13} = 2.3 + (...)$
  because
  $\big[2.3 + (...)\big]^3 = 12.167 + (...)$
- $\sqrt[3]{13} = 2.4 + (...)$
  because
  $\big[2.4 + (...)\big]^3 = 13.824 + (...)$
- $\sqrt[3]{13} = 2.35 + (...)$
  because
  $\big[2.35 + (...)\big]^3 = 12.977875 + (...)$

Sign
Size
same-size

- $\sqrt[3]{13} = 2.351 + (...)$
  because
  $\left[2.351 + (...)\right]^3 = 12.994449551 + (...)$

Thus, in practice, depending on the circumstances, we might replace $\sqrt[3]{13}$ by any one of the following:

$2 + (...)$,
$2.4 + (...)$,
$2.35 + (...)$,
$2.351 + (...)$,
$2.3513 + (...)$,

And, as we shall see presently, when in need to record how small (...) is, we will use a power of h instead.

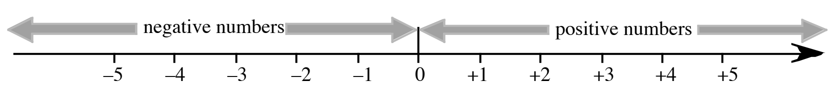
8.2 Sign and Size of a Number

Given a type of function, in order to find out under what conditions we can "join smoothly" a plot into a graph, we shall investigate the kind of outputs this type of functions returns for various kinds of inputs.

Thus, before we can start on our program, we need to look at numbers from a new point of view.

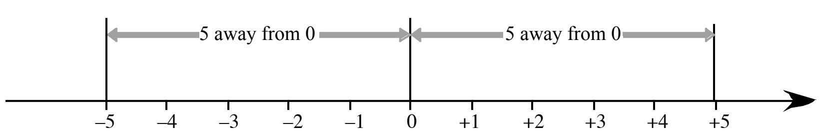
1. The **Sign** of a number is the side of 0 that the number is on the ruler. For instance,

- Sign of -7 is *negative* because -7 is *left of* 0 on the ruler.
- Sign of $+3$ is *positive* because $+3$ is *right of* 0 on the ruler.



Sign *means* Which Side of 0.

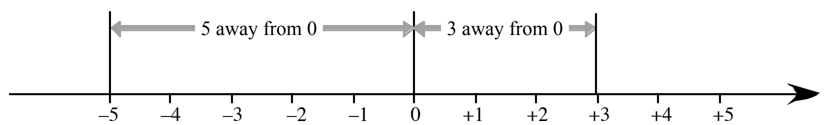
The **Size** of a number is *how far from* 0 the number is on the ruler. For instance, -5 and $+5$ are the **same-size**, namely 5, because they are both 5 away from 0:



Size *means* How Far From 0.

algebra-smaller
larger-in-size
algebra-larger
smaller-in-size
small-in-size
algebra-small
h
large-in-size
algebra-large
algebra-between

2. While −5 is *algebra-smaller* than +3 (see Section 3.7), −5 is **larger-in-size** than +3 because −5 is *further away from* 0 than +3: −5 is 5 away from 0 while +3 is only 3 away from 0.
Similarly, while +3 is *algebra-larger* than −5, (see Section 3.7), +3 is **smaller-in-size** than −5 because +3 is *closer from* 0 than −5: +3 is 3 away from 0 while −5 is 5 away from 0.



Larger/Smaller In Size *means* Farther from/Closer to 0.

3. We will have to distinguish:

- Numbers that are **small-in-size**, as opposed to numbers that are *algebra-small*. We shall use the letter h to stand for inputs that are *small-in-size*.
- Numbers that are **large-in-size**, as opposed to numbers that are *algebra-large*. Unfortunately, there is no generally accepted letter to stand for inputs that are *large-in-size*.

To an extent, *small-in-size* and *large-in-size* are relative concepts. For instance, whether a *gain* or a *loss*, ten-thousand dollars is a small sum of money for people like Bill Gates or George W. Bush while to "the rest of us" ten-thousand dollars is large enough. However, the words have one definite meaning that is the same for everybody. For instance, nobody likes losing a large sum of money. Bill Gates or George W. Bush wouldn't like to lose a million dollars just the way "the rest of us" wouldn't like to lose ten-thousand dollars. It is just the *cutoff point* that varies from people to people.

4. Still, for our purpose, it will be necessary to make the concepts a bit more precise. We observe that:

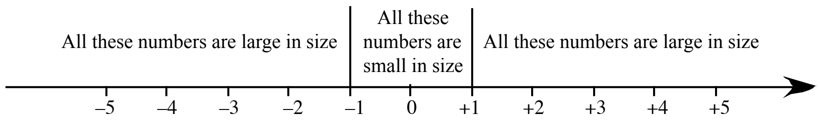
- Any number of copies of an original whose Size is *equal* to 1, and that is only either the number −1 or the number +1, multiply to a result which is the exact *same-size* as the original.

On the other hand,

- Any number of copies of an original whose Size is *larger* than 1, that is numbers either *algebra-smaller* than −1 or *algebra-larger* than +1, multiply to a result which is *larger-in-size* than the original.
- Any number of copies of an original whose Size is *smaller* than 1, that is numbers that are **algebra-between** −1 and +1, multiply to a result which is *smaller-in-size* than the original.

in-the-ones
order of magnitude
in-the-hundreds

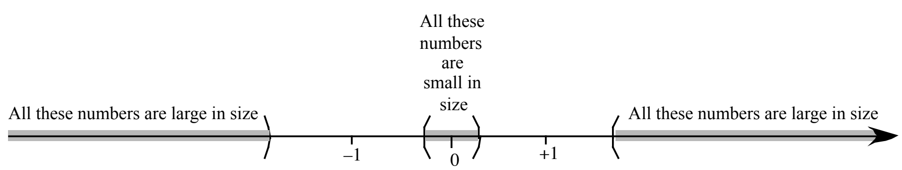
We thus have:



Small In Size and Large In Size (Strictly).

For instance, we could treat -1.1 as a *large-in-size* number because, say, three copies of -1.1 multiply to -1.331 which is *larger-in-size* than the original, -1.1.
Similarly, we could treat that $+0.2$ as a *small-in-size* number because, say, three copies of $+0.2$ multiply to $+0.008$ which is *smaller-in-size* that the original, $+0.2$.
Strictly speaking, this is the only thing we will need to know about a number to decide whether to treat it as *large-in-size* or *small-in-size*.

5. To make things easier, though, we will stay way away from -1 and $+1$ as in



Small In Size and Large In Size (Safely).

where the parentheses indicate that the endpoints are "open to change", according to the circumstances.

6. In fact, practically, we will "think" of the endpoints as being -10, -0.1, $+0.1$, $+10$ and this for the following reason.
Consider -7 and $+2$. They are both **in-the-ones** and therefore of the same **order of magnitude**. Multiplying, say, three copies of each shows that they are both *large-in-size*.
However,

- $+8$, the third power of $+2$, is *in-the-ones* and therefore of the *same* order of magnitude as the original.
- -343, the third power of -7, is **in-the-hundreds** and therefore of a *larger* order of magnitude than the original.

So, there is a *qualitative* difference between -7 and $+2$: While both are *large-in-size*, their third powers are *not* of the same order of magnitude. We

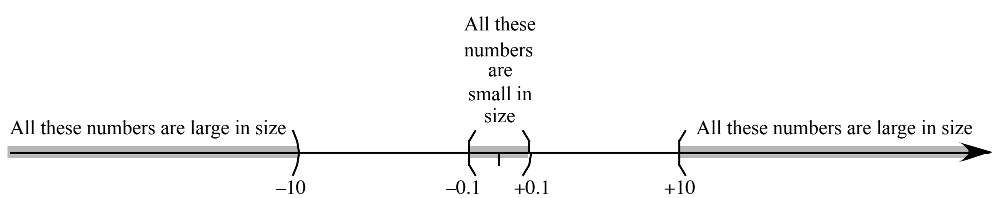
by ... order(s) of magnitude
in-the-tenths
in-the-tens
in-the-thousands
in-the-ten-thousandths

shall say that while -7 and $+2$ are both *large-in-size*, they are so *by different orders of magnitudes*.

Similarly, consider -0.7 and -0.2. They are both **in-the-tenths** and therefore of the same *order of magnitude*. Multiplying, say, two copies of each shows that they are both *small-in-size*.
However,

- $+0.49 = +0.5 + (...)$, the second power of -0.7 is *in-the-tenths* and therefore of the same order of magnitude as the original.
- $+0.04$, the second power of -0.2, is *in-the-hundredths* and therefore of a *smaller* order of magnitude than the original.

So, there is a *qualitative* difference between -0.7 and -0.2: While both are *small-in-size*, their second powers are *not* of the same order of magnitude. We shall say that while -0.7 and -0.2 are both *small-in-size*, they are so *by different orders of magnitudes.*
Thus, finally, the endpoints that we shall use in practice are:



Small In Size and Large In Size (Practically).

What this will do is to ensure that:

- Copies of a large-in-size original will multiply to results that are not only larger-in-size than the original but are so *by an order of magnitude.*
- Copies of a small-in-size original will multiply to results that are not only smaller-in-size than the original but are so *by an order of magnitude.*

Thus for instance, consider $+20$ and -70 which are both **in-the-tens**. When we multiply 2 copies, we get $(+20)(+20) = +400$ which is *in-the-hundreds* while $(-70)(-70) = +4900$ which is **in-the-thousands**. Of course, we still have a qualitative difference but, at least, both are *larger* than the originals *by an order of magnitude.*
Similarly, consider -0.02 and -0.07 which are both *in-the-hundredths.* When we multiply 2 copies, we get $(-0.02)(-0.02) = +0.0004$ which is **in-the-ten-thousandths** while $(-0.07)(-0.07) = +0.0049 = +0.005 + (...)$ which is *in-the-thousandths.* Again, we still have a qualitative difference but, at least, both are *smaller* than the originals *by an order of magnitude.*

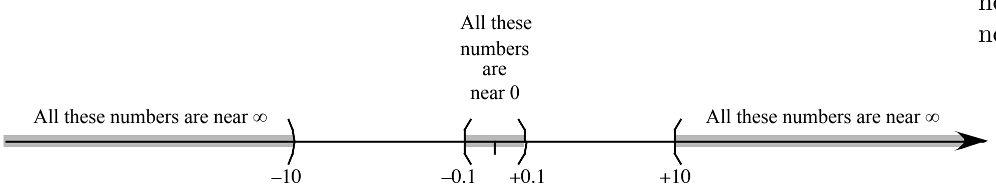
From now on, we shall use:

small
large
near 0
near ∞
near 0^+
near 0^-
near $+\infty$
near $=\infty$

- **small** as short for *small-in-size*
- **large** as short for *large-in-size*

We shall also use the following language:

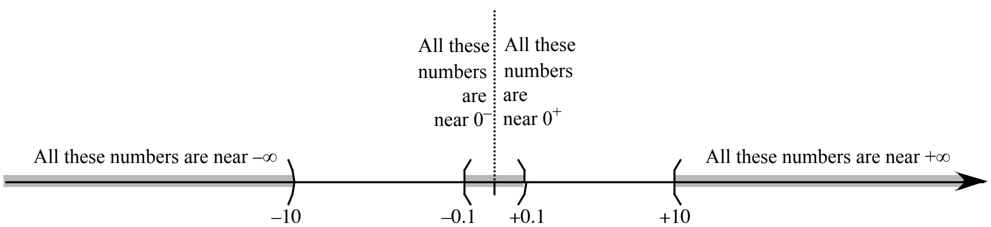
- x is **near** 0 will mean the same as x is *small* (in size).
- x is **near** ∞ will mean the same as x is *large* (in size).



Near 0 and Near ∞

and, when we want to include the sign,

- x is **near** 0^+ will mean the same as x is *small* (in size) and *positive*.
- x is **near** 0^- will mean the same as x is *small* (in size) and *negative*.
- x is **near** $+\infty$ will mean the same as x is *large* (in size) and *positive*.
- x is **near** $=\infty$ will mean the same as x is *large* (in size) and *negative*.



Near 0^+ and Near 0^- and Near $+\infty$ and Near $-\infty$

For most practical purposes, we can think of

- large numbers as positive powers of 10, possibly dilated by a single digit number such as, for instance, $3 \cdot 10^{+5} = 300000$ but also even $3 \cdot 10^{+1} = 30$,
- small numbers as negative powers of 10, possibly dilated by a single digit number such as, for instance, $3 \cdot 10^{-5} = 0.00003$ but also even $3 \cdot 10^{-1} = 0.3$,
- finite numbers as the zero power of 10 such as, for instance, $3 \cdot 10^{+0} = 3$.

8.3 The Arithmetic of Signs and Sizes

It will be crucial to be able to operate with numbers in relation to multiplication and division on the sole basis of their sign or size.

1. Signs behave according to the so-called "rule of signs":

$$+ \text{ times } + = +$$
$$+ \text{ times } - = -$$
$$+ \text{ times } + = -$$
$$- \text{ times } - = +$$

and

$$\frac{+}{+} = +$$
$$\frac{+}{-} = -$$
$$\frac{-}{+} = -$$
$$\frac{-}{-} = +$$

2. Sizes behave logically and we must be careful about that logic.

a. Sizes behave obviously in the following cases:

$\text{finite} \times \text{large} = \text{large}$

$\dfrac{\text{finite}}{\text{large}} = \text{small}$

$\text{finite} \times \text{small} = \text{small}$

$\dfrac{\text{finite}}{\text{small}} = \text{large}$

$\text{large} \times \text{large} = \text{large}$

$\dfrac{\text{large}}{\text{small}} = \text{large}$

$\text{small} \times \text{small} = \text{small}$

$\dfrac{\text{small}}{\text{large}} = \text{small}$

b. However, sizes do not behave that simply in the following cases:

$\text{large} \times \text{small} = ???$

$\dfrac{\text{large}}{\text{large}} = ???$

$\dfrac{\text{small}}{\text{small}} = ???$

This is because here the matter of how small is small and how large is large comes in. For instance, the following three are all instances of large $\times$ small. picture ruler

We have

$$2000000. \times 0.0003 = 600.$$

but

$$20000. \times 0.00000 = 0.06$$

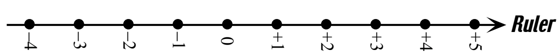
and

$$200000 \times 0.00003 = 6.$$

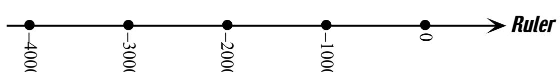
That is, this is a matter of order of magnitude and we will deal with this before we start investigating RATIONAL FUNCTIONS.

8.4 Rulers

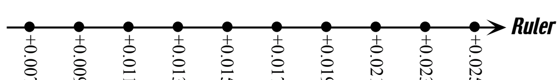
Up until now, we have *represented* numbers by writing *signed-number-phrases* but, quite often, it will be very convenient also to **picture** numbers. To this purpose, we shall use **rulers** by which we mean essentially what goes by that name in the real world, like, for instance,



or

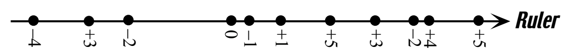


or



as opposed, for instance, to :

number lines
window
screen
first ruler
second ruler
grey-space
first-number point



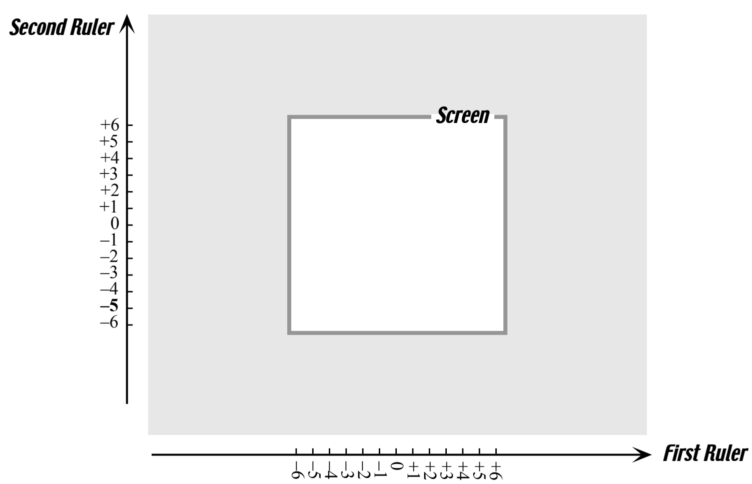
which, for a variety of reasons, is *not* a ruler.

Note. In school environments, rulers are usually called **number lines**.

8.5 Windows

In order to *picture* a *number-pair*, we will need a **window** which consists of:

- a **screen**
- a **first ruler** placed under the *screen*,
- a **second ruler** placed left of the *screen*,
- **grey-space** between the *screen* and the *rulers.* We shall see in what it is for.)



Note. The above arrangement is quite *arbitrary.* In particular, the rulers need not go from left to right and from bottom to top. More generally, for instance, the rulers need not be at a 90 degree angle.

Then, for instance, to represent the *number-pair* $(-2, +5)$, (see Figure below),

i. we represent the *first number* in the pair, -2, by a **first-number point** on the *first ruler*,

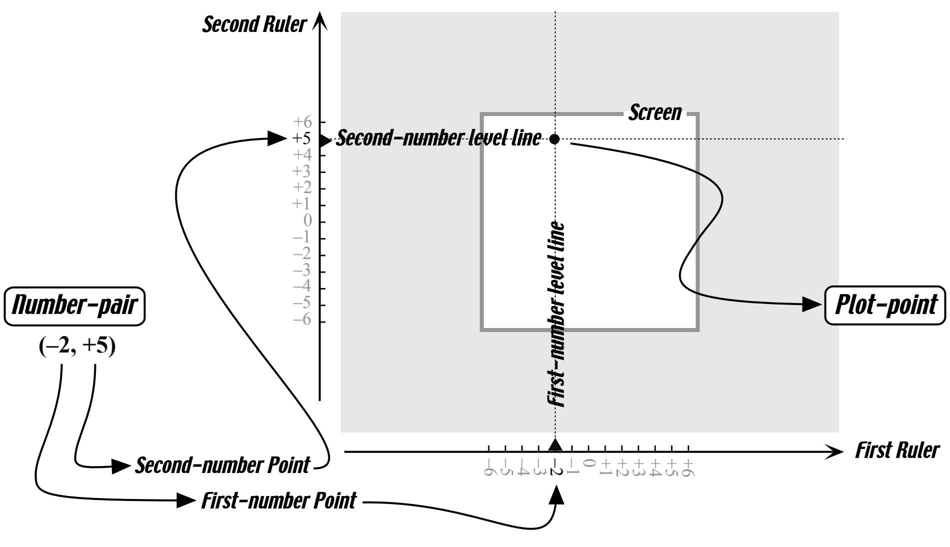
first-number level line
second-number point
second number level line
plot point
good picture

ii. we draw the **first-number level line**—a *vertical* line, through the *first-number point*,

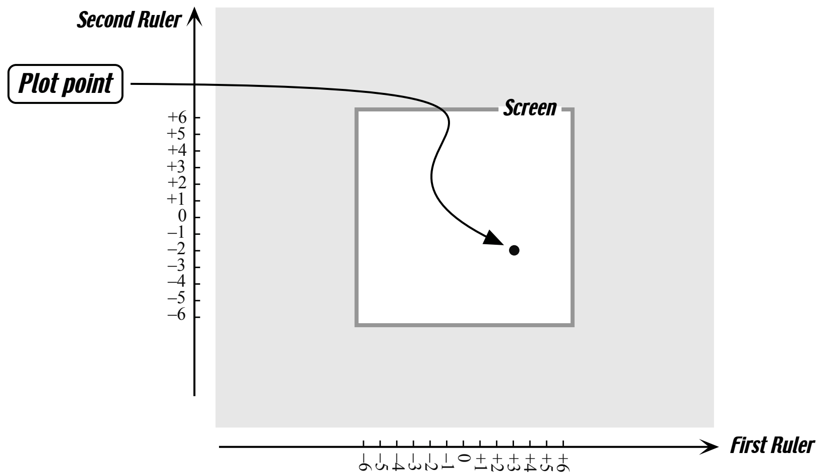
iii. we represent the *second number* in the pair, +5 by a **second-number point** on the *second* ruler,

iv. we draw the **second number level line**—a *horizontal* level line, through the *second-number point*,

v. Then, the **plot point**, that is the point that represents the *pair* $(-2, +5)$ on the *screen* is where the first number level line and the second-number level level line intersect.



Observe that a *plot-point* is a **good picture** of a *number-pair* because, once we have drawn the plot-point, we can erase the two number points as well as the two level lines *without loss of information.* In other words, given, for instance, the following *plot-point*



we can recover the *number-pair* of which this plot-point is the picture. All we have to do is to go backwards through the above steps:

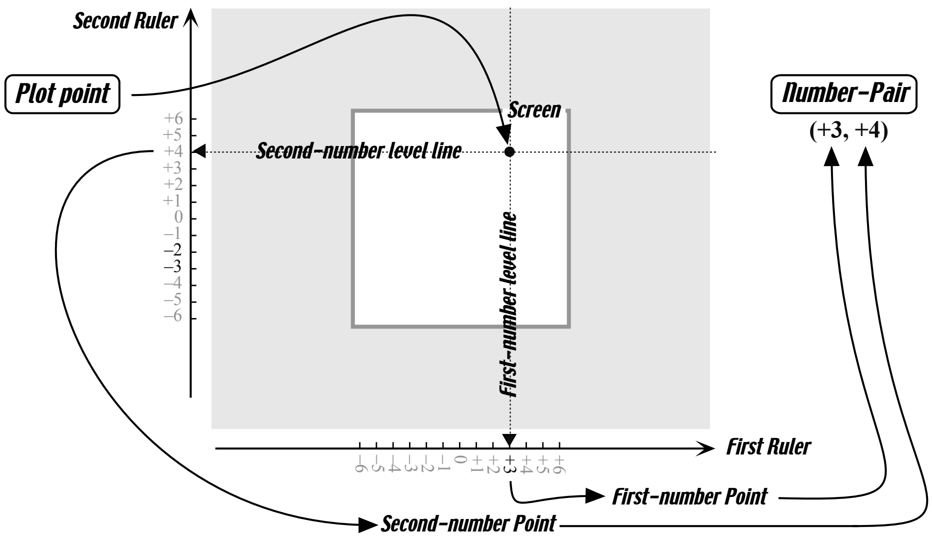
i. we draw the *first-number level line* (vertical) through the given plot point,

ii. the point that represent the *first number* in the pair is where the first-number level line intersects the first-number ruler,

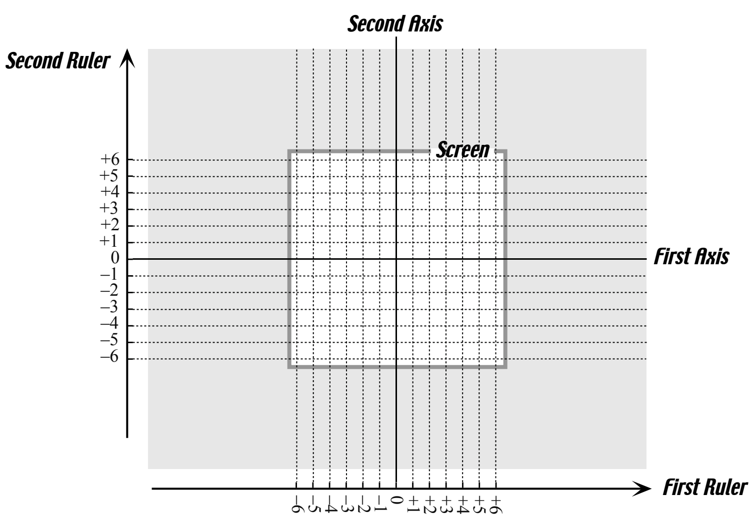
iii. we draw the *second-number level line* (horizontal) through the given plot point,

iv. the point that represent the *second number* in the pair is where the second-number level line intersects the second-number ruler.

graph-paper



Sometimes, to facilitate plotting, the window comes as **graph-paper**, that is already equipped with level lines that form a grid:



first-axis
second-axis
infinite
finite
infinitesimal

Note. Very often, the level lines for 0 are singled out as for instance on the graph paper and are called **first-axis** and **second-axis**. Sometimes, only the *axes* are drawn without the rest of the graph paper and/or the rulers. We shall always draw the rulers but, when doing *qualitative* investigations, we will only draw those levels lines that are relevant to the investigation.

We will distinguish:

- **infinite** is what cannot be seen because it is outside the window regardless of the *extent* of the ruler. This is most of the ruler.
- **finite** is what can be seen. So, first it has to be in the window and then it has to appear as a tick-mark on the corresponding ruler. Thus, whether something is finite depends on the *extent* of the ruler and then on the *resolution* of the ruler. See Section 8.4
- **infinitesimal** is what cannot be seen in the window regardless of the *resolution* of the ruler

Chapter 9

Introduction to Functions

Recall that, from the ARITHMETIC viewpoint, *numbers* were specified *directly* while, from the ALGEBRA viewpoint, we looked for numbers that were specified *indirectly* by being solutions of given *equations.*

We will now investigate the *processes, devices, procedures, agencies, converters, translators,* etc, by which *new* numbers can be *obtained* from *given* numbers.

In this PART II. ALGEBRAIC FUNCTIONS, the processes, devices, procedures, agencies, converters, exchangers, translators, etc,, ... , etc, that we will investigate will be specified *directly*, just like numbers were in ARITHMETIC. We shall do this by way of *input-output rules.*

In PART III. TRANSCENDENTAL FUNCTIONS, the processes, devices, procedures, agencies, converters, exchangers, translators, etc, that we shall investigate will be specified *indirectly*, just like numbers were in ALGEBRA, that is by being required to be solutions of given *equations.*

9.1 What are relations and functions?

To perceive that something is *changing*, it is necessary to look at it against something else that either does *not* change at all or that changes *differently.* For instance, the amount of income tax changes *in terms of* income, the amount of property tax changes *in terms of* assets, the amount of sales tax changes *in terms of* expenses.

More precisely, in order to observe something changing, we must pair each of the stages that it goes through in terms of the stages that some reference thing goes through, if only a clock or a calendar!

For instance, we might say that, in 2003, someone's income tax was

input
output
relation
input number
output number
input-output pair
function
return

$6,753. Just to have said that someone's income tax was $6,753 would not say much since $6,753 is a lot less money now than in 1913 when income tax was first created.

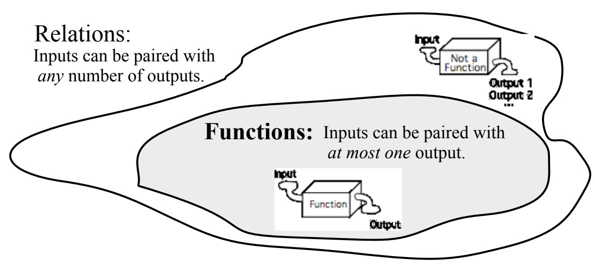
The reference stages will be called **inputs** and the stages of what we are investigating will be called **outputs**.

We will call **relation** the *pairing* of **input numbers** with **output numbers** that results from any process, device, procedure, agency, converter, exchanger, translator, etc. An input number together with an output number it is paired with make up what we shall call an **input-output pair**. (Note that we will be using parentheses to enclose input-output pairs and that this is yet *another* use of parentheses.)

Note. Eventually, we shall just use the word "input" instead of the phrase "input number" and the word "output" instead of the phrase "output number".

After this INTRODUCTION, we shall investigate *only* those relations, called **functions**, that meet the *single* requirement that *no input be paired with more than one output.* In terms of something that we are observing, this means that once a reference stage has been picked, no more than one stage can be observed.

In other words, given *any* input, a *function* may **return** either *no* output at all or *one* output but no more than one output



Beyond that functions need not meet any other requirement and, in particular, a function may return the same output for different inputs.

For example, a *parking meter* is a function because, given an input, say 1 **Quarter**, the parking meter returns a definite amount of parking time, say 30 **Minutes**. We would then say that (1 **Quarter**, 30 **Minutes**) is an input-output pair. Note that any amount of money above the maximum will return the same amount of parking time.

On the other hand, a *slot machine* is not a function because, given an input, say 1 **Quarter**, a slot machine could return *any* number of **Quarters**.

9.2 Relations specified by an input-output table.

specify
input-row
input-column
output-row
output-column
input-output table

The simplest way to **specify** a relation is just to write out all the input-output pairs. This is often done by writing the inputs in an **input-row** (or in an **input-column**) and the outputs *returned* by the relation directly underneath the inputs in an **output-row** (or directly next to the inputs in an **output-column**). We shall then say that the relation is specified by an **input-output table**.

For example, consider the input-output table

Inputs	-6	-5	-3	-2	0	$+1$	$+2$	$+6$
Outputs	-4		$+5$	$+5$	-1	0	-5	-5

in which, for instance, the input -3 is paired with the output $+5$. Since, here, no input is paired with more than one output, the input-output table specifies in fact a *function.*

Note that this *function* does *not* return any output for -5. For that matter, neither does it return any output for, say, -4, -1 or for $+1.7$ or -647 or, indeed, for any input other than those listed. Also, observe that for the inputs -3, -2 and $+6$, the function returns the *same* output -5.

On the other hand, consider the following input-output table

Inputs	-42	-20	$+2$	$+24$	$+57$	$+173$	$+345$	$+359$
Outputs	$+765$	$+56$	$+3, -7$	$+56$	$+56$		-111	0

It pairs the input $+2$ with *more than one* output, namely the two outputs $+3$ and -7, and, even though it pairs all the other inputs with *at most one* output each, this is enough to prevent this table from specifying a *function.*

However, note that, other than for the input $+2$, this input-output table *would* specify a function. In particular, the fact that the inputs -20, $+24$ and $+57$ all give the same output $+56$ would not, by itself, prevent the table from specifying a *function.*

9.3 Plot of a relation specified by an input-output table

There are many ways to represent a relation by a *picture* but since, here, input-output pairs are number-pairs, we shall use the *windows* that we introduced in Section 8.5.

1. For instance, to represent the *input-output pair* $(-2, +5)$, (see Figure below),

input-point
input ruler
input level line
output-point
output ruler
output level line
plot-point

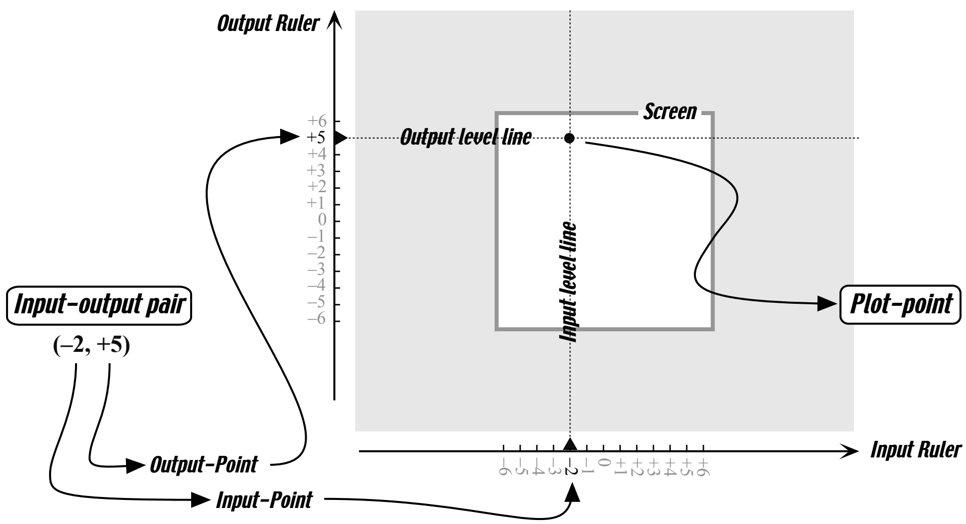
i. we represent the *input-number* -2, by an **input-point** on the **input ruler**,

ii. we draw the **input level line**—a *vertical* line, through the *input-point*,

iii. we represent the *output-number*, $+5$ by an **output-point** on the **output ruler**,

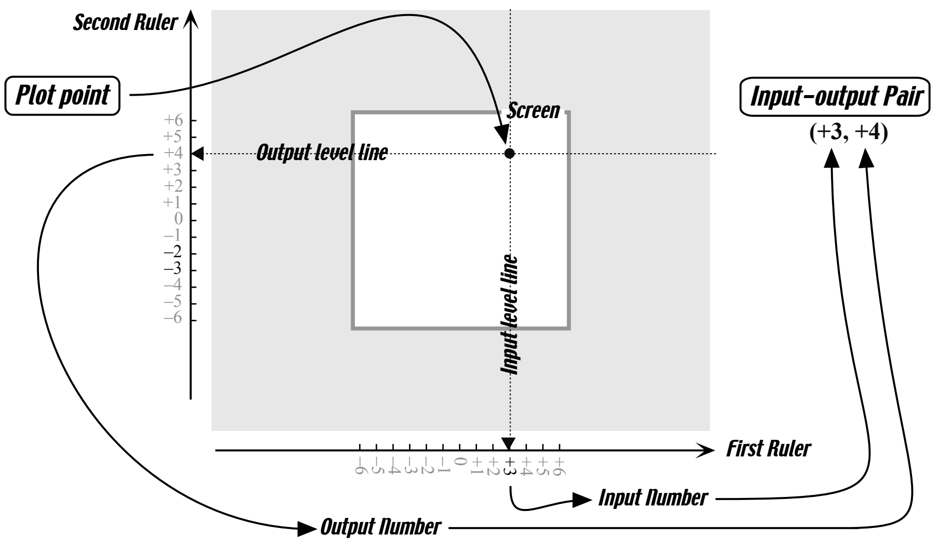
iv. we draw the **output level line**—a *horizontal* line, through the *output-point*

v. Then, the **plot-point**, that is the point that represents the *input-output pair* $(-2, +5)$ on the *screen* is at the intersection of the *input level line* and the *output level line.*



2. Observe that a plot point is a *good* picture of an input-output pair because, given a plot-point, we can erase the input-point, the output-point as well as the level lines without loss of information. In other words, given a plot-point we can recover the input-output pair of which the plot point is the picture. All we have to do is to go backwards through the above steps as in Section 8.5:

plot

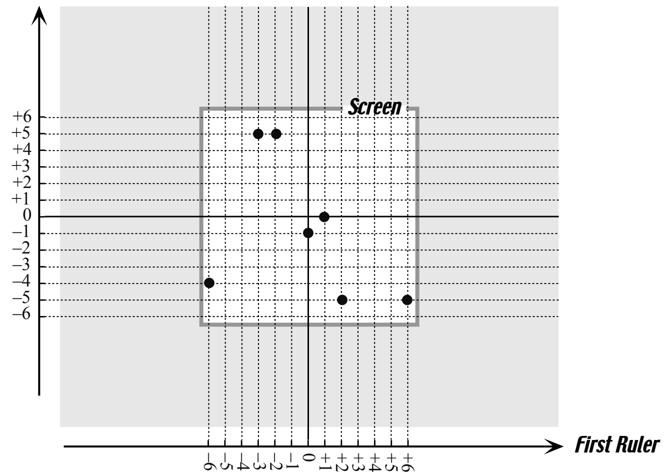


3. Now we define a **plot** as a collection of plot points for all the input-output pairs in the table.

For example, given the table

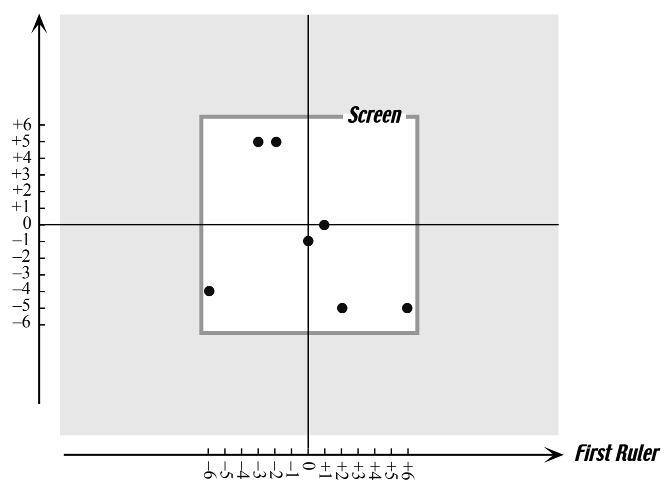
Inputs	-6	-5	-3	-2	0	$+1$	$+2$	$+6$
Outputs	-4		$+5$	$+5$	-1	0	-5	-5

we can plot each and every input-output pair which gives us the *plot* of the above *table*:



Observe that, since each plot-point is a *good* picture of an input-output pair, the plot is a good picture of the input-output table. In other words, the plot contains exactly the same information as the input-output table so that a relation can be specified by a *plot* just as well as by an *input-output table* and we can go back and forth between the two.

Moreover, most of the time, only the plot-points are shown since the input level-lines and the output level-lines can always be recovered and thus can "go without saying".

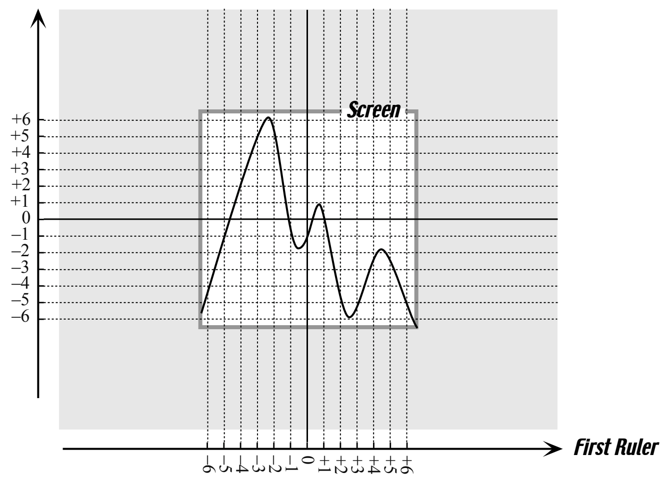


9.4 Relations specified by a graph

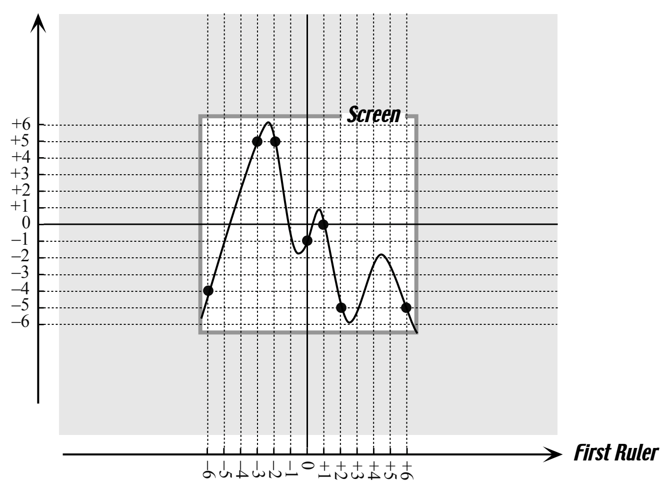
graph

We saw above that, when we represent input-output pairs by a plot-point in a window, the picture is a *good* one because, from the plot-point, we can recover the input-output pair. So, a **graph**, that is any line drawn on the screen, whether curved or not, will specify a *relation*.

For instance, given the following graph,



we can pick the following graph-points

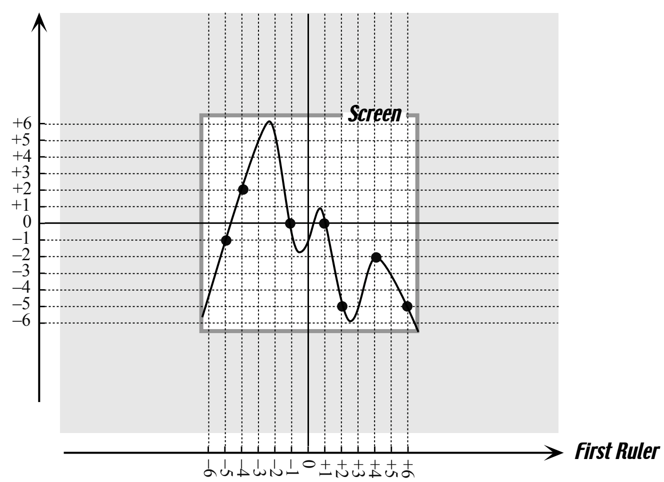


input-output rule
place holder

and, proceeding as in Section 9.3 , get the following input-output table

Inputs	-6	-3	-2	0	$+1$	$+2$	$+6$
Outputs	-4	$+5$	$+5$	-1	0	-5	-5

However, by picking the following graph-points on the same graph



we would get a different input-output table

Inputs	-5	-4	-1	$+1$	$+2$	$+4$	$+6$
Outputs	-1	$+2$	0	0	-5	-2	-5

In other words, since we can go on for ever picking graph points, a *graph* carries a lot more information than we can put in any *input-output table.*

9.5 Functions specified by an input-output rule.

While in some sciences, such as PSYCHOLOGY and SOCIOLOGY, functions are usually specified by input-output tables, in other sciences, such as PHYSICS and ELECTRONICS, *functions* are usually specified by **input-output rules** as follows:

i. We use a letter, usually x, as **place holder** for *inputs.* In other words, we will be able to replace x by any number we want.

ii. We must either have or give a name to the function. In the absence of any given name, we shall usually use the letter f.

iii. Then, $f(x)$ will stand for the output returned for the input x by the function f.

For instance, if a function called, say, FUN doubles the input and adds 5 to give the output, then the input-output rule of FUN is:

$$x \xrightarrow{FUN} FUN(x) = 2x + 5$$

Then, to find the output for a *specific* input, say 7, we just replace all the occurrences of x by 7:

$$7 \xrightarrow{FUN} FUN(7) = 2 \cdot 7 + 5$$

and then we compute the output underneath:

$$\begin{aligned} 7 \xrightarrow{FUN} FUN(x) &= 2 \cdot 7 + 5 \\ &= 14 + 5 \\ &= 19 \end{aligned}$$

We can then write

$$7 \xrightarrow{FUN} 19$$

which says that $(7, 19)$ is an input-output pair under the function FUN.

However, instead of the above and because it will make it easier later on to keep track of what we are doing, we shall use the input-output rule as follows:

$$\begin{aligned} \text{When } x = 7 \xrightarrow{FUN} FUN(x)|_{\text{when } x=7} &= 2x + 5|_{\text{when } x=7} \\ &= 2 \cdot 7 + 5 \\ &= 14 + 5 \\ &= 19 \end{aligned}$$

9.6 The Fundamental Problem

We now address the problem of picturing by a graph a function given by an input-output rule.

The first idea that comes to mind is to pick a number of inputs, create an input-output table and then plot the input-output table. Students are then usually advised to "join smoothly" the plot-points to get the graph of the function but this clearly cannot work because, on the basis of the outputs we have already computed, we cannot find all the other outputs which is what the graph would do.

Consider, for example, the function given by the input-output rule

$$x \xrightarrow{\ RAT\ } RAT(x) = \frac{3x+6}{x^2-10}$$

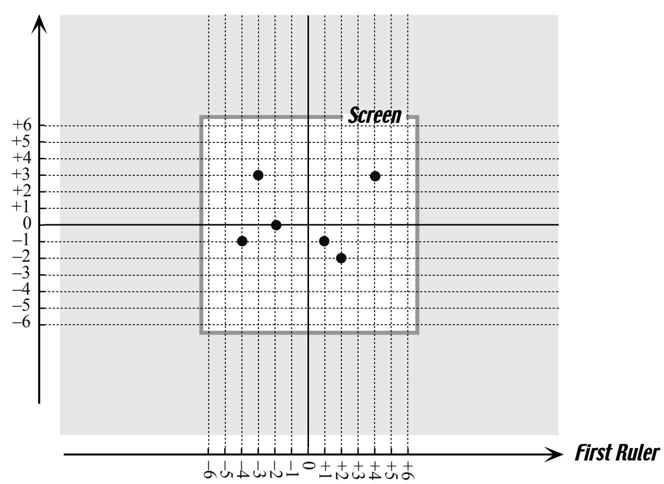
We can find the output for, say, the input $+2$

$$\begin{aligned}\text{When } x=+2 \xrightarrow{\ RAT\ } RAT(x)\big|_{\text{when } x=+2} &= \left.\frac{3x+6}{x^2-10}\right|_{\text{when } x=+2}\\ &= \frac{3\cdot(+2)+6}{(+2)^2-10}\\ &= \frac{12}{-6}\\ &= -2\end{aligned}$$

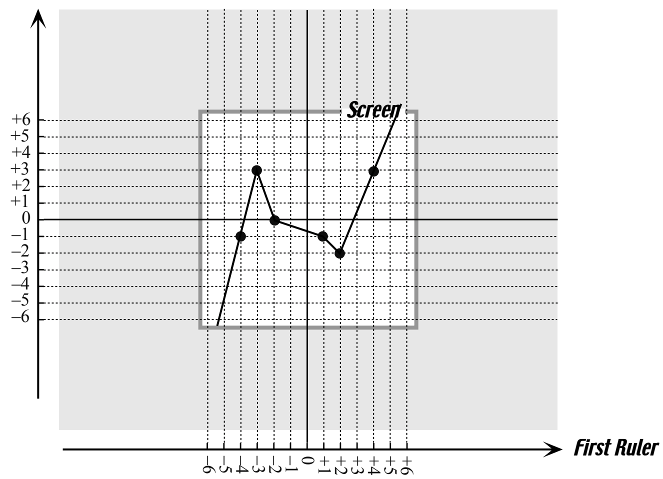
Proceeding as above, we can get, for instance, the following input-output table:

Inputs	-4	-3	-2	$+1$	$+2$	$+4$
Outputs	-1	$+3$	0	-1	-2	$+3$

and, from it, the following plot:

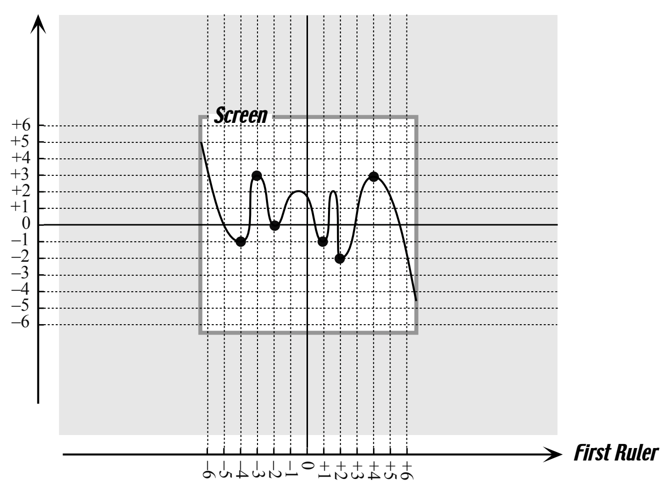


But now, how are we to "join" the plot points? The first thing that usually comes to the mind is something like this



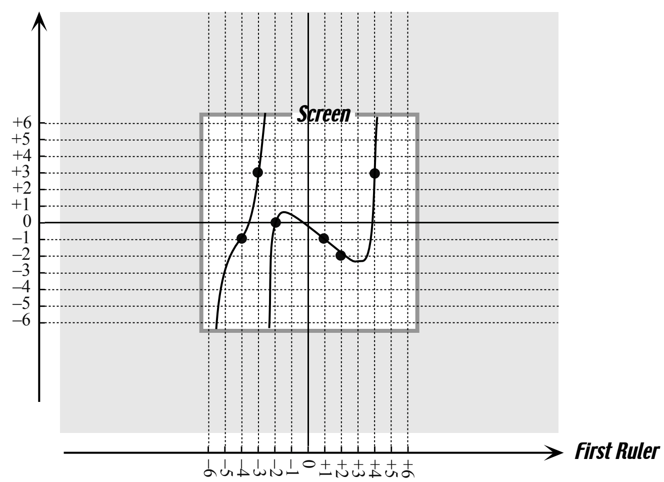
But now the *graph* gives the output $+\frac{1}{2}$ for the input +3 and it is not likely that this is what the *input-output rule* would give. Beyond that, it is not likely that the extrapolation given by the input-output rule is just as given by the graph. Nor that the plot-points we picked should happen to be precisely the peaks and valleys of the graph. Etc.

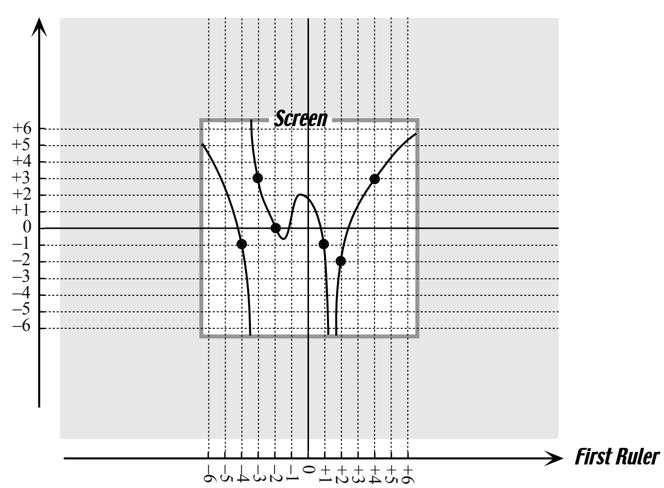
On second thought then, we might "join the plot-points smoothly", somewhat as below.

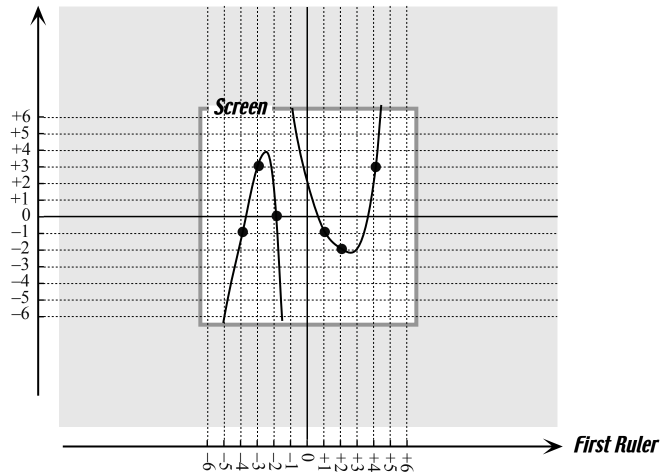


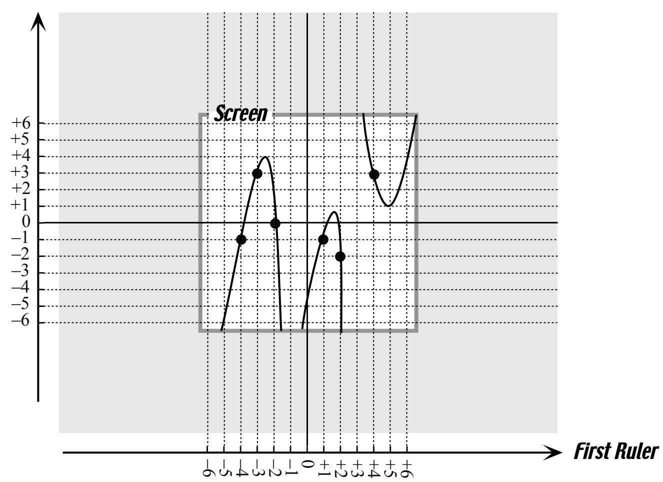
But again, since we drew the graph more or less at random, it is very unlikely that for, say, the input +3 the *input-output rule* would give the output +1 which is the output given by the *graph*. Neither is it likely that the plot should happen to be precisely the highs and lows of the graph.

In fact, given the above plot, anyone of the following is just as likely or unlikely to be the graph:





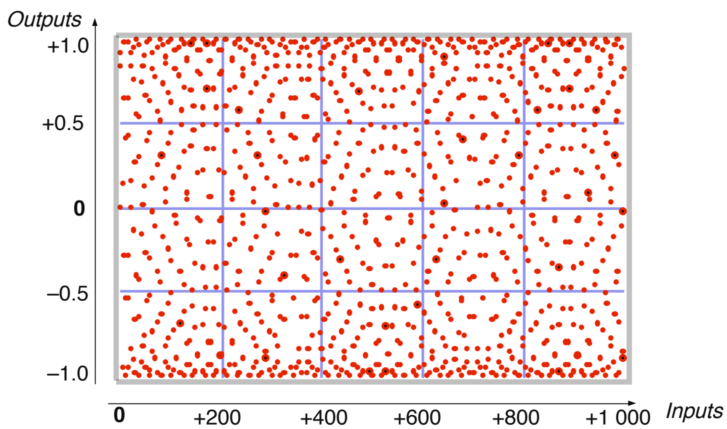




The advice usually given at this point is "just get more plot points" but the next question is then "how many plot-points is enough to guarantee that we have the graph?"

In fact, having very many plot points may make it impossible to "join them smoothly" as demonstrated by Strang's Famous Example:

types of functions



Yet this plot turns out to be a plot of a well-known function, called $SINE$, whose "smooth" graph will be found in the next course.

The following then will be our "program" to cope with this "fundamental problem":

i. Rather than try to find a universal way to find the graph of any function, we shall investigate **types of functions**, one type at a time, and find out for each type under what conditions we can "join smoothly" a plot into a graph. Here are the types of functions that we shall investigate in this part.

- **a.** Power Functions
- **b.** Polynomial Functions
  - **i.** Affine Functions
  - **ii.** Quadratic Functions
  - **iii.** Cubic Functions
- **c.** Rational Functions

ii. By the way, the reverse problem of going from a *graph* to an *input-output rule* can be a surprisingly difficult one and, other than in a very few cases, we shall not deal with here.

iii. On the other hand, the problem of going from a *plot* to an *input-output rule* is completely different in nature and central to the subject called Data Analysis.

Power functions
coefficient
exponent
features (of the input-output rule)
parity
even
odd

Chapter 10

Power Functions

Power functions are functions that multiply or divide a *finite* **coefficient** by a number of copies of the *input* according to the **exponent**:

- The counting number in the *exponent* is the number of copies of the input that are to be used.
- If the sign of the exponent is $+$, the coefficient is to be *multiplied* by the copies of the input,
- If the sign of the exponent is $-$, the coefficient is to be *divided* by the copies of the input.

The input-output rule of a power function has three **features**:

- The *sign* of the *coefficient*,
- The *sign* of the *exponent*,
- The **parity** of the *exponent* (the fact that it is **even** or **odd**.)

For instance, the function $BLIP$ whose input-output rule is

$$x \xrightarrow{BLIP} BLIP(x) = (-7)x^{+3}$$
$$= (-7) \cdot x \cdot x \cdot x$$

is a power function whose input-output rule has the following features:

- The *coefficient* is *negative*,
- The *exponent* is *positive*,
- The *exponent* is *odd*.

and the function $FLOP$ whose input-output rule is

$$x \xrightarrow{FLOP} FLOP(x) = (+6)x^{-4}$$

is a power function whose input-output rule has the following features normalize

- The *coefficient* is *postive*,
- The *exponent* is *negative*,
- The *exponent* is *even*.

The reasons we start with the investigation of power function are that:

i. Power function are the simplest possible functions,

ii. Power function are the simplest embodiment of fundamental features,

iii. Power function are the building blocks in terms of which we will de-construct other functions.

10.1 Size of Outputs

Given a power function and given the size of an input, we want to find out if we can determine the *size* of the output in terms of just the *features* of the input-output rule.

Note. We shall represent *large* numbers by $large$ and *small* numbers by $small$ but it will be most important to be completely specific and not to confuse the size of the *input* and the size of the *output*.

We begin with an example. Given the function KIT whose input-output rule is

$$x \xrightarrow{KIT} KIT(x) = (-8)x^{+5}$$

we want to find out the size of the outputs for a *large* input that we will represent by $large$.

Since the coefficient -8 is "in the ones" and since we are multiplying the coefficient by copies of inputs that are *large*, the actual coefficient does not matter so we **normalize** the coefficient -8 to -1, that is we represent its size by 1.

Altogether, we have

$$\begin{aligned} large \xrightarrow{KIT} KIT(large) &= (1)(large)^{+5} \\ &= (1) \cdot \underbrace{(large) \cdot (large) \cdot (large) \cdot (large) \cdot (large)}_{5 \text{ copies of } large} \end{aligned}$$

and, since 5 copies of $large$ multiply to $large$,

$$\begin{aligned} &= (1) \quad \cdot \quad (large) \\ &= large \end{aligned}$$

Next, proceeding as in the example, we shall consider the following question:

QUESTION 1. Which features of the input-output rule control the size of the output in terms of the size of the input?

1. Given a function $POSEXP$ whose input-output rule is of the form

$$x \xrightarrow{POSEXP} POSEXP(x) = (-7)x^{+\text{number}}$$

we want to find out the size of the outputs (since -7 is "in the ones", we will represent its size by 1).

- When the inputs are *large*, we have

$$\begin{aligned} large \xrightarrow{POSEXP} POSEXP(large) &= (1)(large)^{+\text{number}} \\ &= (1) \cdot \underbrace{(large) \cdot (large) \cdot ... \cdot (large)}_{\text{number copies of } large} \end{aligned}$$

  and, since any number of copies of *large* multiply to *large*,

$$\begin{aligned} &= (1) \quad \cdot \quad (large) \\ &= large \end{aligned}$$

  We see that the size of the output is *large*, that is the *same as* the size of the *input*.

- When the inputs are *small* we have

$$\begin{aligned} small \xrightarrow{POSEXP} POSEXP(small) &= (1)(small)^{+\text{number}} \\ &= (1) \cdot \underbrace{(small) \cdot (small) \cdot ... \cdot (small)}_{\text{number copies of } small} \end{aligned}$$

  and, since any number of copies of *small* multiply to *small*,

$$\begin{aligned} &= (1) \quad \cdot \quad (small) \\ &= small \end{aligned}$$

  We see that the size of the output is *small*, that is the *same as* the size of the *input*.

So, when the exponent is *positive*, the size of the output is the *same* as the size of the input.

2. Given a function $NEGEXP$ whose input-output rule is of the form

$$x \xrightarrow{NEGEXP} NEGEXP(x) = (-7)x^{-\text{number}}$$

we want to find out the size of the outputs.

- When the inputs are *large*, we have

$$\begin{aligned} large \xrightarrow{NEGEXP} NEGEXP(large) &= (1)(large)^{-\text{number}} \\ &= \frac{1}{\underbrace{(large)\cdot(large)\cdot\ldots\cdot(large)}_{\text{number copies of } large}} \end{aligned}$$

  and, since any number of copies of *large* multiply to *large*,

$$\begin{aligned} &= \frac{1}{large} \\ &= small \end{aligned}$$

  We see that the size of the output is *small*, that is the *opposite from* the size of the input.

- When the inputs are *small*, we have

$$\begin{aligned} small \xrightarrow{NEGEXP} NEGEXP(small) &= (1)(large)^{-\text{number}} \\ &= \frac{1}{\underbrace{(small)\cdot(small)\cdot\ldots\cdot(small)}_{\text{number copies of } small}} \end{aligned}$$

  and, since any number of copies of *small* multiply to *small*,

$$\begin{aligned} &= \frac{1}{small} \\ &= large \end{aligned}$$

So, when the exponent is *negative*, the size of the output is the *reciprocal* of the size of the input.

3. Altogether then, the answer to **QUESTION 1** is given by

THEOREM 1. The *size* of the outputs in terms of the size of the inputs depends only on the *sign* of the *exponent*:

- When the *exponent* is *positive*, the size of the output is the *same* as the size of the input.
- When the *exponent* is *negative*, the size of the output is the *reciprocal* of the size of the input.

10.2 Sign of Outputs

Given a power function and given the sign of an input, we want to find out if we can determine the *sign* of the output in terms of just the *features* of the input-output rule.

Note. We shall represent *positive* numbers by + and *negative* numbers by − but it will be most important to be completely specific and not to confuse the sign of the *input* and the sign of the *output*.

We begin with an example. Given the function KIT whose input-output rule is

$$x \xrightarrow{KIT} KIT(x) = (-8)x^{+5}$$

we want to find out the sign of the outputs for a *positive* input that we will represent by +. We have

$$\begin{aligned} + \xrightarrow{KIT} KIT(+) &= (-8)(+)^{+5} \\ &= (-) \cdot \underbrace{(+) \cdot (+) \cdot (+) \cdot (+) \cdot (+)} \end{aligned}$$

and, since 5 copies of + multiply to +,

$$\begin{aligned} &= (-) \qquad \cdot \qquad (+) \\ &= - \end{aligned}$$

Next, proceeding as in the example, we shall investigate separately

- the sign of the outputs for *positive* inputs,
- the sign of the outputs for *negative* inputs.

Sign of the outputs for *positive* inputs

We consider here the following question:

QUESTION 2. Which features of the input-output rule control the sign of the output for a *positive* input.

1. We start with the case where the *exponent* is *positive* and we investigate separately the sub-case where the exponent is *even* and the sub-case where the exponent is *odd*.

a. Given a function $POSEVEXP$ whose input-output rule is of the form

$$x \xrightarrow{POSEVEXP} POSEVEXP(x) = (-7)x^{+\text{even}}$$

we want to find out the sign of the outputs for *positive* inputs. We have

$$\begin{aligned} + \xrightarrow{POSEVEXP} POSEVEXP(+) &= (-)(+)^{+\text{even}} \\ &= (-) \cdot \underbrace{(+) \cdot (+) \cdot \ldots \cdot (+)}_{\text{even number of } +} \\ &= (-) \quad \cdot \quad (+) \\ &= - \end{aligned}$$

We see that, since an *even* number of + multiply to +, the sign of the output is equal to the sign of the coefficient multiplied by + and so the sign of the output is the *same as* the sign of the *coefficient*.

b. Given a function $POSODEXP$ whose input-output rule is of the form

$$x \xrightarrow{POSODEXP} POSODEXP(x) = (-7)x^{+\text{odd}}$$

we want to find out the sign of the outputs for *positive* inputs. We have

$$\begin{aligned} + \xrightarrow{POSODEXP} POSODEXP(+) &= (-)(+)^{+\text{odd}} \\ &= (-) \cdot \underbrace{(+) \cdot (+) \cdot \ldots \cdot (+)}_{\text{odd number of } +} \\ &= (-) \quad \cdot \quad (+) \\ &= - \end{aligned}$$

We see that, since an *odd* number of + multiply to +, the sign of the output is equal to the sign of the coefficient multiplied by + and so the sign of the output is the *same as* the sign of the *coefficient*.

So, we see that when the exponent is *positive* the *parity* of the exponent does not matter and we have that, for *positive* inputs, the sign of the *output* is the *same as* the sign of the *coefficient*.

2. We now continue with the case where the *exponent* is *negative* and we investigate again separately the sub-case where the exponent is *even* and the sub-case where the exponent is *odd*.

a. Given a function $NEGEVEX$ whose input-output rule is of the form

$$x \xrightarrow{NEGEVEXP} NEGEVEXP(x) = (+4)x^{-\text{even}}$$

we want to find out the sign of the outputs for *positive* inputs. We have

$$\begin{aligned} + \xrightarrow{NEGEVEXP} NEGEVEXP(+) &= (+)(+)^{+\text{even}} \\ &= \frac{(+)}{\underbrace{(+)\cdot(+)\cdot\ldots\cdot(+)}_{\text{even number of }+}} \\ &= \frac{(+)}{(+)} \\ &= + \end{aligned}$$

We see that, since an *even* number of + multiply to +, the sign of the output is equal to the sign of the coefficient divided by + and so the sign of the output is the *same as* the sign of the *coefficient*.

b. Given a function $NEGODEXP$ whose input-output rule is of the form

$$x \xrightarrow{NEGODEXP} NEGODEXP(x) = (-4)x^{-\text{odd}}$$

we want to find out the sign of the outputs for *positive* inputs. We have

$$\begin{aligned} + \xrightarrow{NEGODEXP} NEGODEXP(+) &= (-)(+)^{-\text{odd}} \\ &= \frac{(-)}{\underbrace{(+)\cdot(+)\cdot\ldots\cdot(+)}_{\text{odd number of }+}} \\ &= \frac{(-)}{(+)} \\ &= - \end{aligned}$$

We see that, since an *odd* number of + multiply to +, the sign of the output is equal to the sign of the *coefficient* divided by + and so is the *same as* the sign of the *coefficient*.

So, we see that when the exponent is *negative* the *parity* of the exponent does not matter and we have that, for *positive* inputs, the sign of the *output* is the *same as* the sign of the *coefficient*.

3. Altogether, we see that neither the *sign* nor the *parity* of the exponent matters and we have that, for *positive* inputs, only the sign of the *coefficient* matters since the sign of the *output* is always the same as the sign of the *coefficient*.

THEOREM 2. For *positive* inputs, the sign of the *output* of a power function is the same as the sign of the *coefficient* regardless of everything else.

Sign of the outputs for *negative* inputs

We consider here the following question:

QUESTION 3. Which features of the input-output rule control the sign of the output for a *negative* input.

1. We start with the case where the *exponent* is *positive* and we investigate separately the sub-case where the exponent is *even* and the sub-case where the exponent is *odd*.

a. Given a function $POSEVEXP$ whose input-output rule is of the form

$$x \xrightarrow{POSEVEXP} POSEVEXP(x) = (-7)x^{+\text{even}}$$

we want to find out the sign of the outputs for *negative* inputs. We have

$$\begin{aligned} - \xrightarrow{POSEVEXP} POSEVEXP(-) &= (-)(-)^{+\text{even}} \\ &= (-) \cdot \underbrace{(-) \cdot (-) \cdot \ldots \cdot (-)}_{\text{even number of } -} \\ &= (-) \quad \cdot \quad (+) \\ &= - \end{aligned}$$

We see that, since an *even* number of $-$ multiply to $+$, the sign of the output is equal to the sign of the coefficient multiplied by $+$ and so the sign of the output is the *same as* the sign of the *coefficient*.

b. Given a function $POSODEXP$ whose input-output rule is of the form

$$x \xrightarrow{POSODEXP} POSODEXP(x) = (-7)x^{+\text{odd}}$$

we want to find out the sign of the outputs for *negative* inputs. We have

$$+ \xrightarrow{POSODEXP} POSODEXP(+) = (-)(-)^{+\text{odd}}$$
$$= (-) \cdot \underbrace{(-) \cdot (-) \cdot \ldots \cdot (-)}_{\text{odd number of } -}$$
$$= (-) \quad \cdot \quad (-)$$
$$= +$$

We see that, since an *odd* number of − multiply to −, the sign of the output is equal to the sign of the coefficient multiplied by − and so the sign of the output is the *opposite of* the sign of the *coefficient*.

So, we see that when the exponent is *positive* the *parity* of the exponent does matter and we have that, for *negative* inputs,

- when the exponent is *even*, the sign of the *output* is the *same as* the sign of the *coefficient*.
- when the exponent is *odd*, the sign of the *output* is the *opposite of* the sign of the *coefficient*.

2. We now continue with the case where the *exponent* is *negative* and we investigate again separately the sub-case where the exponent is *even* and the sub-case where the exponent is *odd*.

a. Given a function $NEGEVEXP$ whose input-output rule is of the form

$$x \xrightarrow{NEGEVEXP} NEGEVEXP(x) = (+4)x^{-\text{even}}$$

we want to find out the sign of the outputs for *negative* inputs. We have

$$+ \xrightarrow{NEGEVEXP} NEGEVEXP(+) = (+)(-)^{+\text{even}}$$
$$= \frac{(+)}{\underbrace{(-) \cdot (-) \cdot \ldots \cdot (-)}_{\text{even number of } -}}$$
$$= \frac{(+)}{(+)}$$
$$= +$$

We see that, since an *even* number of − multiply to +, the sign of the output is equal to the sign of the coefficient divided by + and so the sign of the output is the *same as* the sign of the *coefficient*.

b. Given a function $NEGODEXP$ whose input-output rule is of the form

$$x \xrightarrow{NEGODEXP} NEGODEXP(x) = (+4)x^{-\text{odd}}$$

qualitative graph

we want to find out the sign of the outputs for *negative* inputs. We have

$$
\begin{aligned}
+ \xrightarrow{NEGODEXP} NEGODEXP(+) &= (+)(-)^{+\text{odd}} \\
&= \frac{(+)}{\underbrace{(-)\cdot(-)\cdot\ldots\cdot(-)}_{\text{odd number of } -}} \\
&= \frac{(+)}{(-)} \\
&= -
\end{aligned}
$$

We see that, since an *odd* number of −s multiply to −, the sign of the output is equal to the sign of the coefficient divided by − and so the sign of the output is the *opposite of* the sign of the *coefficient*.

So, we see that when the exponent is *negative* the *parity* of the exponent does matter and we have that, for *negative* inputs,

- when the exponent is *even*, the sign of the *output* is the *same as* the sign of the *coefficient*.
- when the exponent is *odd*, the sign of the *output* is the *opposite of* the sign of the *coefficient*.

3. Altogether then, for *negative* inputs, it is the *parity* of the exponent that controls the sign of the outputs and we have:

THEOREM 3. For *negative* inputs, the sign of the *output* of a power function relative to the coefficient depends on the *parity* of the exponent:

- When the exponent is *even*, the sign of the *output* is the *same as* the sign of the coefficient,
- When the exponent is *odd*, the sign of the *output* is the *opposite of* the sign of the coefficient,

10.3 Qualitative graphs

The information we are thus able to get at this point from the input-output rule of a power function is already enough to translate into what we shall call a **qualitative graph**, by which, for the time being, we shall mean the *simplest*, *most regular*, *most natural* graph that is consistent with the above type of information. After we shall have described another two features of a function, we shall make the concept somewhat more precise.

We look at two examples.

1. Given, for instance, a function $POSEVEXP$ whose input-output rule is local area

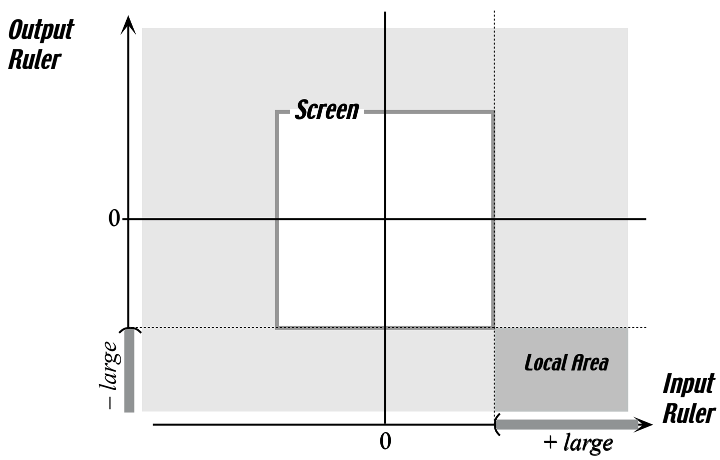
$$x \xrightarrow{POSEVEXP} POSEVEXP(x) = (-7)x^{+\text{even}}$$

a. We begin by looking for the **local areas**, that is the areas in which the graph is in when the inputs are *small* and when the inputs are *large*. For that purpose, we shall need the *size* and the *sign* of the outputs and, while we shall essentially proceed as in the previous sections, we shall now do so in a more compact manner:

- When the inputs are $+\,large$, we have

$$\begin{aligned} +\,large \xrightarrow{POSEVEXP} POSEVEXP(+\,large) &= (-1)(+\,large)^{+\text{even}} \\ &= (-1)\cdot\underbrace{(+\,large)\cdots(+\,large)}_{\text{even number of }(+\,large)} \\ &= (-1) \quad\cdot\quad (+\,large) \\ &= -\,large \end{aligned}$$

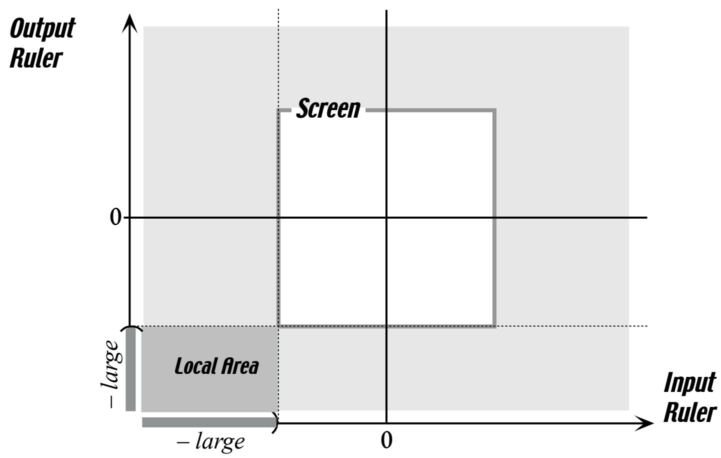
From the graphic viewpoint, this means that the part of the graph for $+\,large$ inputs is in the following *local area*:



- When the inputs are $-\,large$, we have

$$\begin{aligned} -\,large \xrightarrow{POSEVEXP} POSEVEXP(-\,large) &= (-1)(-\,large)^{+\text{even}} \\ &= (-1)\cdot\underbrace{(-\,large)\cdots(-\,large)}_{\text{even number of }(-\,large)} \\ &= (-1) \quad\cdot\quad (+\,large) \\ &= -\,large \end{aligned}$$

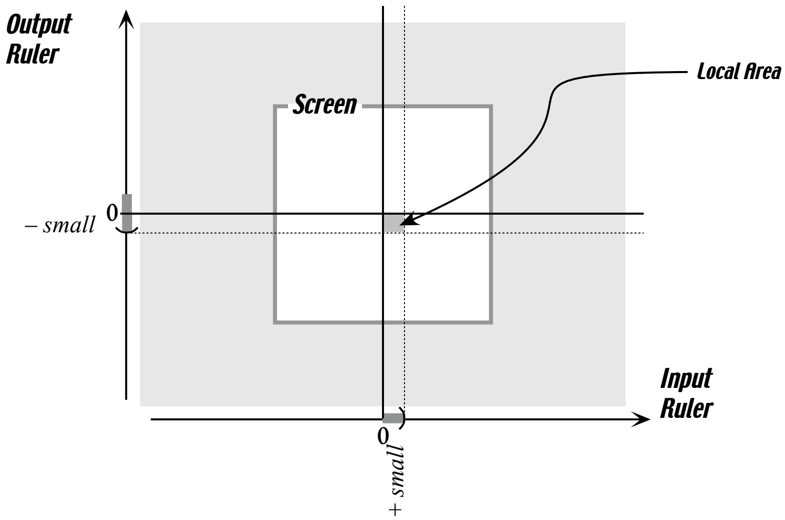
From the graphic viewpoint, this means that the part of the graph for $-$ *large* inputs is in the following *local area*:



- When the inputs are $+$ *small*, we have

$$
\begin{aligned}
+\,small \xrightarrow{POSEVEXP} POSEVEXP(+\,small) &= (-1)(+\,small)^{+\text{even}} \\
&= (-1)\cdot \underbrace{(+\,small)\cdots(+\,small)}_{\text{even number of } (+\,small)} \\
&= (-1) \quad\cdot\quad (+\,small) \\
&= -\,small
\end{aligned}
$$

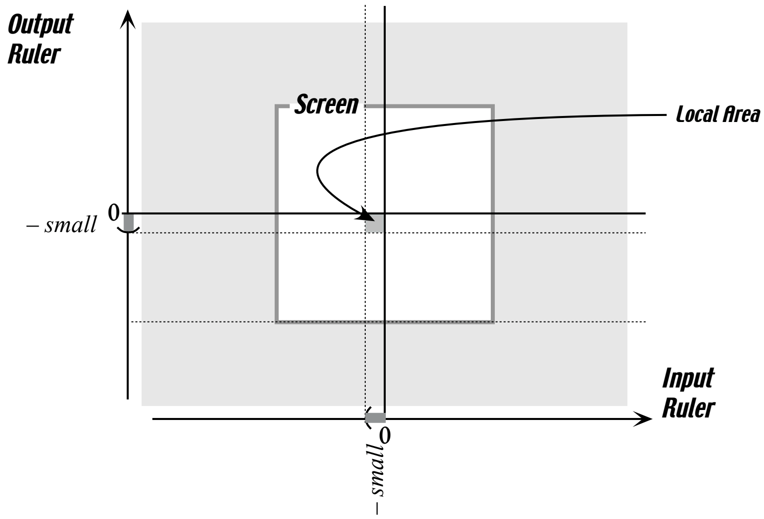
From the graphic viewpoint, this means that the part of the graph for $-$ *small* inputs is in the following *local area*:



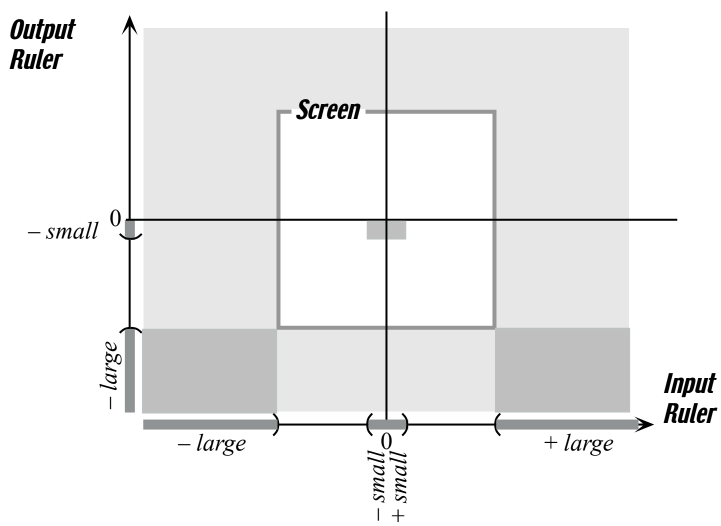
- When the inputs are $-$ *small*, we have

$$\begin{aligned}
-\,small \xrightarrow{POSEVEXP} POSEVEXP(-\,small) &= (-1)(-\,small)^{+\text{even}}\\
&= (-1)\cdot\underbrace{(-\,small)\cdots(-\,small)}_{\text{even number of }(-\,small)}\\
&= (-1)\quad\cdot\quad(+\,small)\\
&= -\,small
\end{aligned}$$

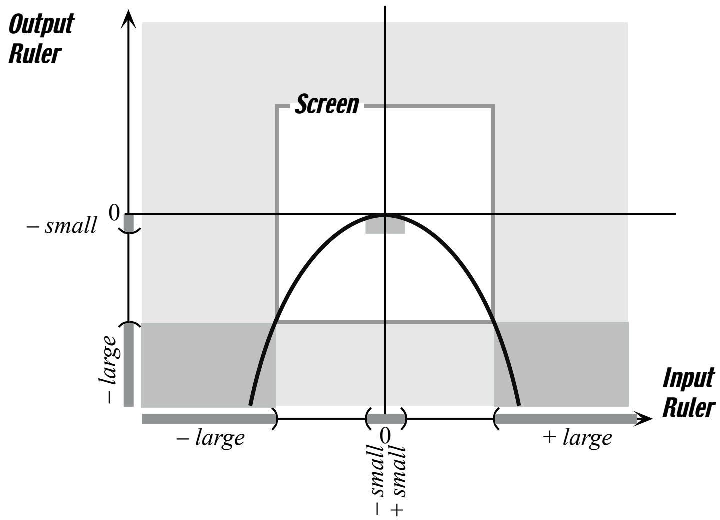
From the graphic viewpoint, this means that the part of the graph for $-$ *small* inputs is in the following *local area*:



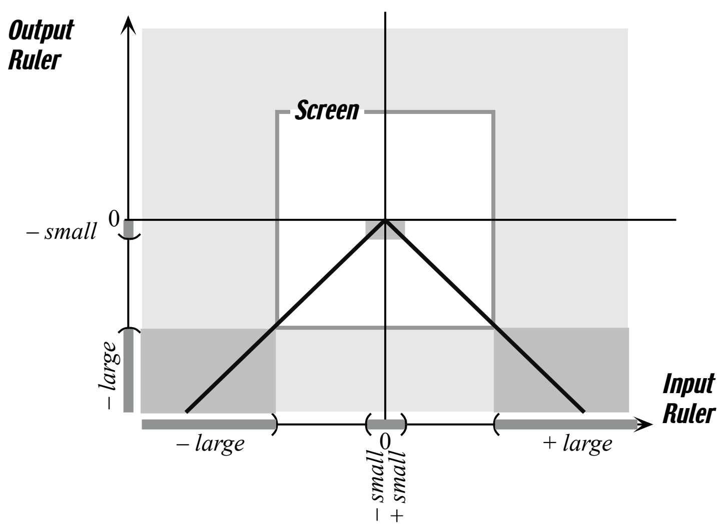
b. Altogether, then, the graph of the function $POSEVEXP$ will be going through the following four *local areas*:



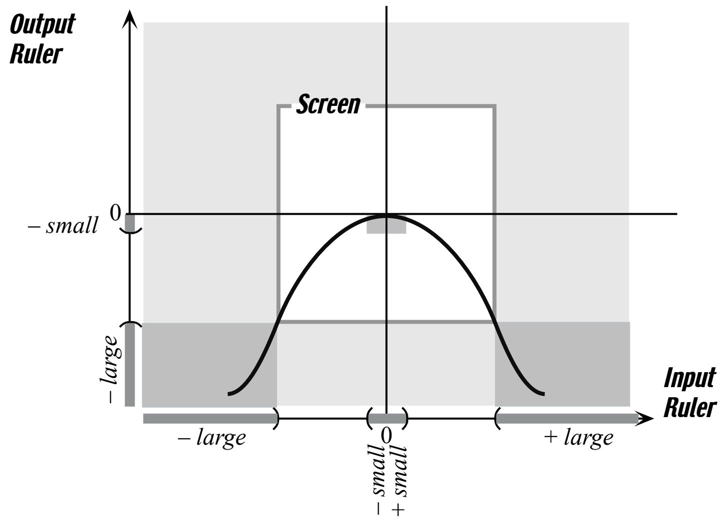
as follows



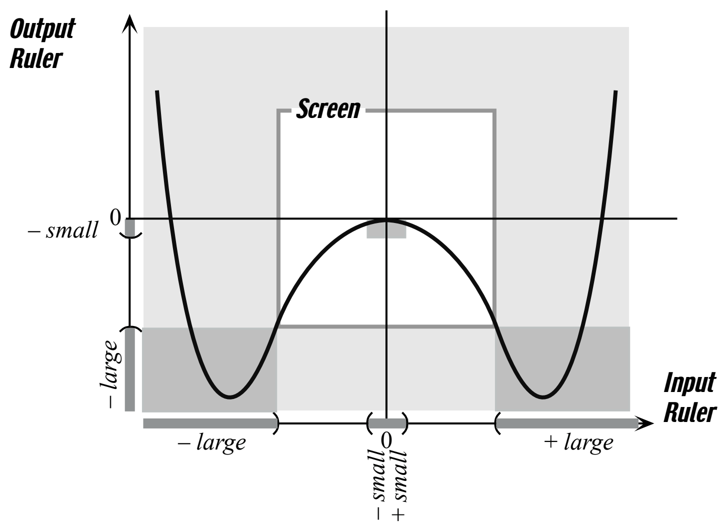
c. Note that while the following graph, for instance,



might appear simpler, it really isn't since straight lines and sharp angles are not something one often finds in nature. More about this later. The following graph too would not be very likely

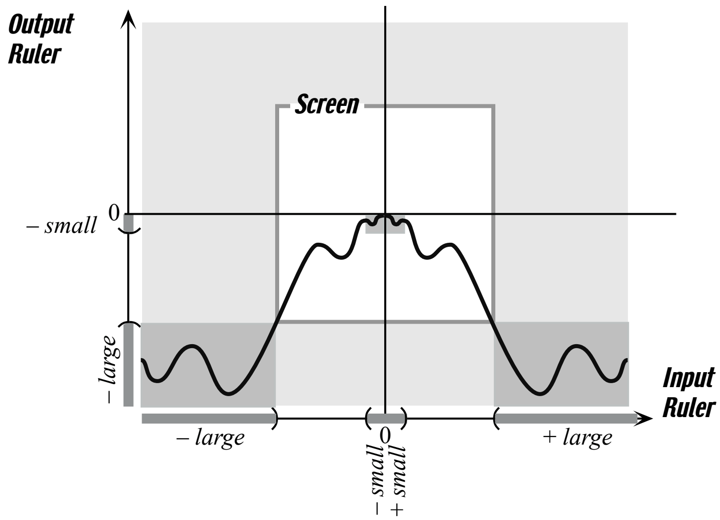


because it gives the wrong impression about the outputs for sufficiently large inputs in that, for instance, the graph could be thought to continue as follows



which of course cannot be the case since $POSEVEXP$ gives negative outputs for all large inputs.

Or, the graph might oscillate as in



but, up front, this graph looks awfully complicated for a function with as simple an input-output rule as $POSEVEXP$.

2. Given, as another example, a function $NEGODEXP$ whose input-output rule is

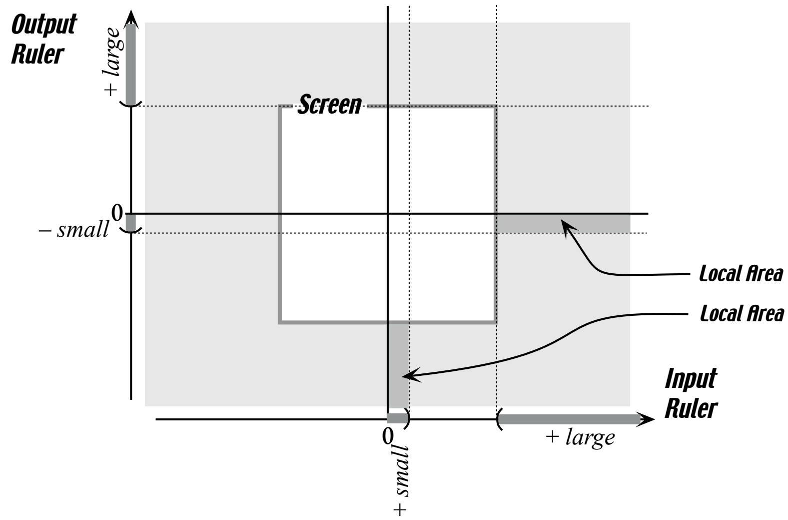
$$x \xrightarrow{NEGODEXP} NEGODEXP(x) = (-4)x^{-\text{odd}}$$

We want to find out the *local areas* when the inputs are *small* and when the inputs are *large*.

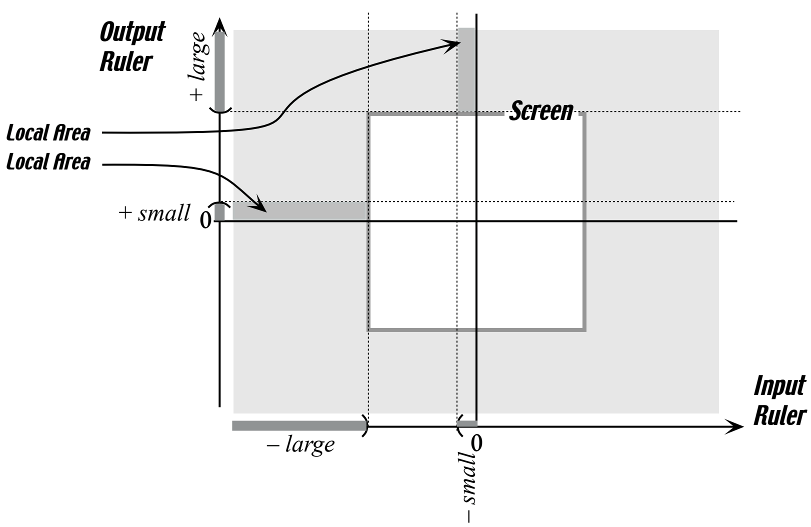
a. While we shall essentially proceed as before, we shall now do so in an even more compact manner:

- We start with *positive* inputs.
  - Since the sign of the outputs for *positive* inputs is always the same as the sign of the coefficient, the sign for *positive* inputs is $-$.
  - Since the exponent is *negative*, the coefficient will be *divided* by the copies of the inputs and
    the outputs for + *large* inputs will be − *small*
    the outputs for + *small* inputs will be − *large*

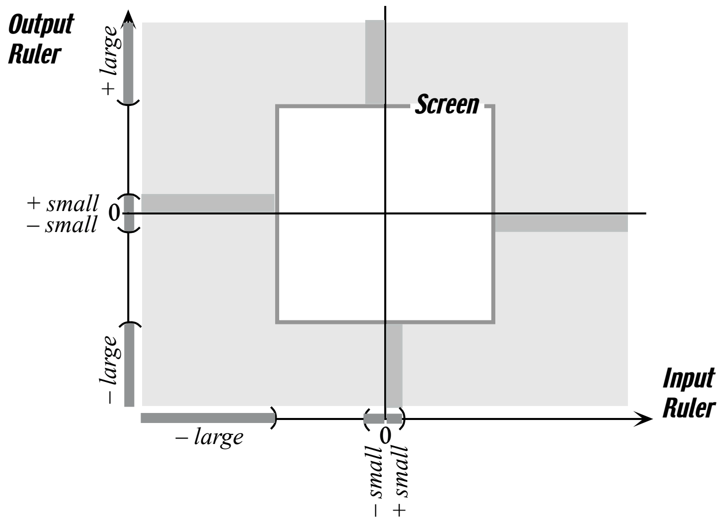
  From the graphic viewpoint, this means that the local areas for + *large* and + *small* inputs are



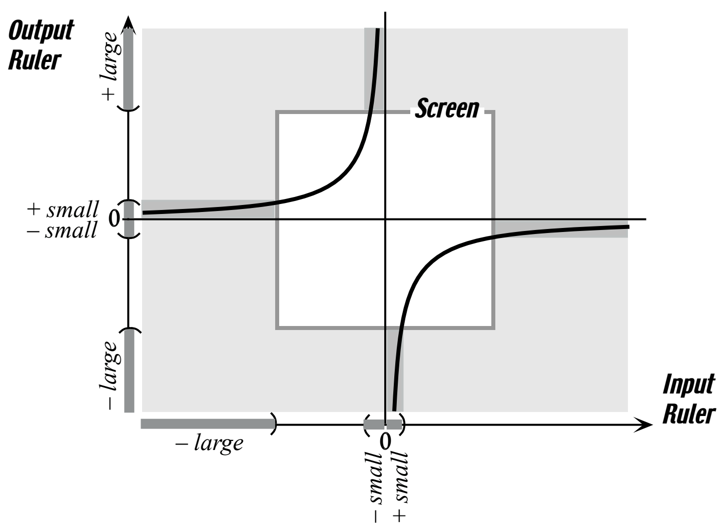
- Looking at *negative* inputs, since the exponent is *odd*, the sign of the outputs for *negative* inputs will be the *opposite* of the sign of the coefficient and will thus be +.
  From the graphic viewpoint, this means that the local areas for − *large* and − *small* inputs will be the *opposite* of those for + *large* and + *small* inputs



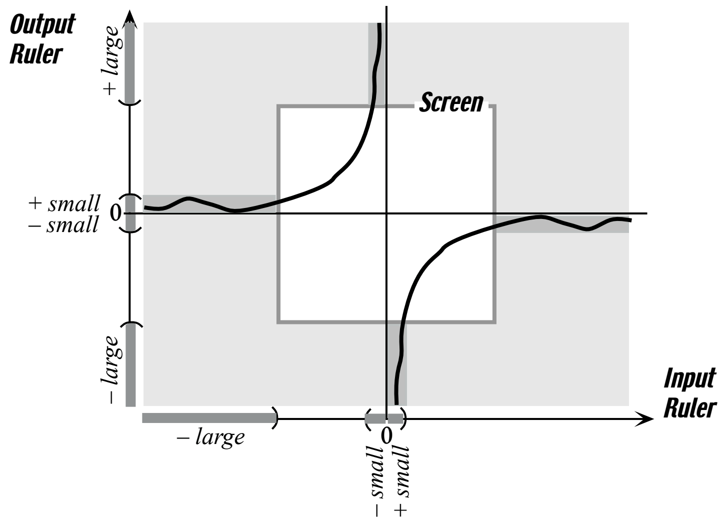
b. Altogether, then, the graph of the function $NEGODEXP$ will be going through the following four *local areas*:



as follows



c. Here again, for all we really know, the graph could be, for instance, wavering as in



but again, this graph looks awfully complicated for a function with as simple an input-output rule as $NEGODEXP$.

In any case, we shall discuss and settle this kind of issue after we have described a couple of features that a function can have.

10.4 Slope

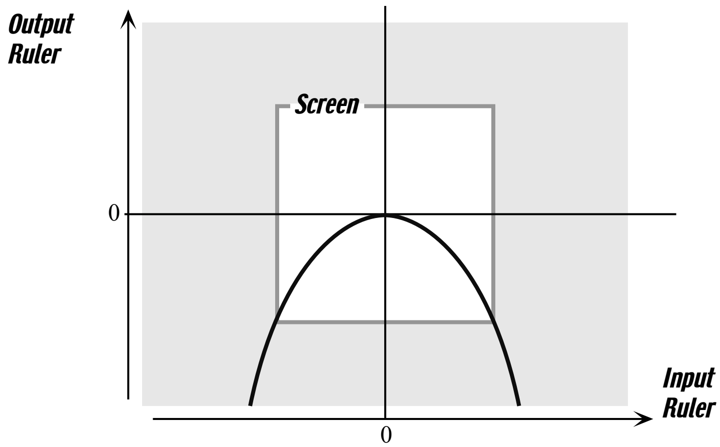
We now use

$$x \xrightarrow{POSEVEXP} POSEVEXP(x) = (-7)x^{+\text{even}}$$

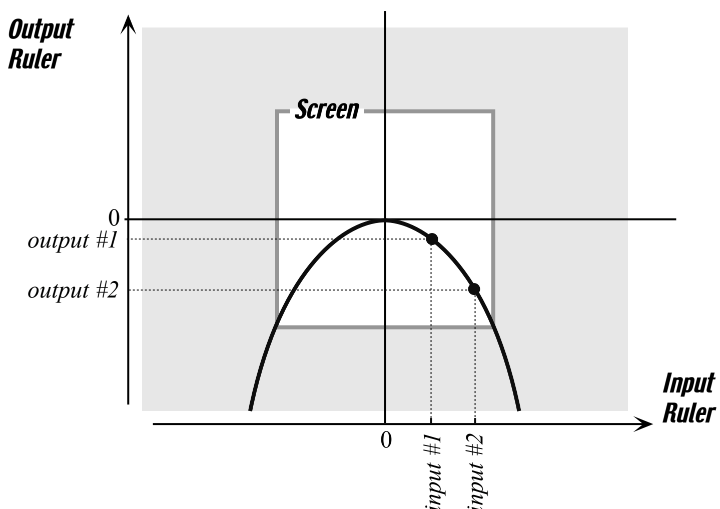
to describe a *feature* which is not readily apparent from the input-output rule but is quite clear on its qualitative graph

1. From the graphic point of view, we use the qualitative graph

input change
run
output change
rise
sloping down



a. Take two *positive* inputs such as in the picture below, call them *input* #1 and *input* #2, and get their outputs, *output* #1 and *output* #2:



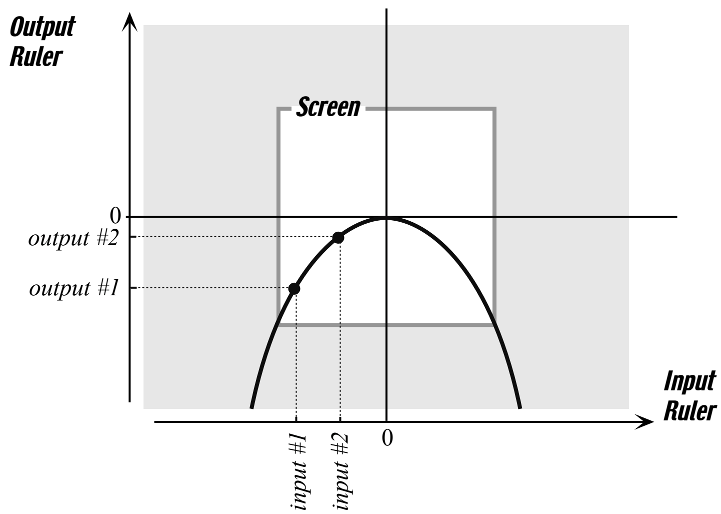
We observe that

- the **input change** (also know as **run**) from *input* #1 to *input* #2 is *positive*
- the **output change** (also known as **rise**) from *output* #1 to *output* #2 is *negative*

that is, the sign of the *output change* is the *opposite* from the sign of the *input change*. We shall say that the function $POSEVEXP$ is **sloping down** from *input* #1 to *input* #2.

sloping up

b. Similarly, take two *negative* inputs such as in the picture below, call them *input* #3 and *input* #4, and get their outputs, *output* #3 and *output* #4:



We observe that

- the *input change* from *input* #1 to *input* #2 is *positive*
- the *output change* from *output* #1 to *output* #2 is *positive*

that is, the sign of the *output change* is the *same* as the sign of the *input change*. We shall say that the function $POSEVEXP$ is **sloping up** from *input* #3 and *input* #4.

2. From the input-output rule point of view, we must take a specific instance of $POSEVEXP$ in order to compute the outputs and their changes and we take

$$x \xrightarrow{JANE} JANE(x) = (-7)x^{+4}$$

a. We pick *input* #1 = +3 and *input* #2 = +5,

i. We compute the outputs:

$$\begin{aligned} input\ \#1 = +3 \xrightarrow{JANE} JANE(+3) &= (-7)(+3)^{+4} \\ &= (-7)\cdot(+3)\cdot(+3)\cdot(+3)\cdot(+3) \\ &= -567 = output\ \#1 \end{aligned}$$

and

$$\begin{aligned} input\ \#2 = +5 \xrightarrow{JANE} JANE(+5) &= (-7)(+5)^{+4} \\ &= (-7)\cdot(+5)\cdot(+5)\cdot(+5)\cdot(+5) \\ &= -4375 = output\ \#2 \end{aligned}$$

ii. We compute the *changes*. slope

$$\begin{aligned} \textit{input change} &= \textit{input } \#2 - \textit{input } \#1 \\ &= (+5) - (+3) \\ &= (+5) + (-3) \\ &= +2 \end{aligned}$$

and

$$\begin{aligned} \textit{output change} &= \textit{output } \#2 - \textit{output } \#1 \\ &= (-4375) - (-567) \\ &= (-4375) + (+567) \\ &= -3808 \end{aligned}$$

iii. We compute the **slope**

$$\begin{aligned} \textit{slope} &= \frac{\textit{output change}}{\textit{input change}} \\ &= \frac{-3808}{+2} \\ &= -1904 \end{aligned}$$

b. We pick *input* #3 $= -2$ and *input* #4 $= -5$,

i. We compute the outputs:

$$\begin{aligned} \textit{input } \#3 = -2 \xrightarrow{JANE} JANE(-2) &= (-7)(-2)^{+4} \\ &= (-7)\cdot(-2)\cdot(-2)\cdot(-2)\cdot(-2) \\ &= -112 = \textit{output } \#3 \end{aligned}$$

and

$$\begin{aligned} \textit{input } \#4 = -6 \xrightarrow{JANE} JANE(-6) &= (-7)(-5)^{+4} \\ &= (-7)\cdot(-6)\cdot(-6)\cdot(-6)\cdot(-6) \\ &= -9072 = \textit{output } \#4 \end{aligned}$$

ii. We compute the *changes*.

$$\begin{aligned} \textit{input change} &= \textit{input } \#2 - \textit{input } \#1 \\ &= (-6) - (-2) \\ &= (-6) + (+2) \\ &= -4 \end{aligned}$$

and

$$\begin{aligned} \textit{output change} &= \textit{output } \#2 - \textit{output } \#1 \\ &= (-9072) - (-112) \\ &= (-9072) + (+112) \\ &= -8960 \end{aligned}$$

iii. We compute the *slope*

$$\begin{aligned} \textit{slope} &= \frac{\textit{output change}}{\textit{input change}} \\ &= \frac{-8960}{-4} \\ &= +2240 \end{aligned}$$

3. Comparing what we got from the qualitative graph with what we got from the input-output rule, we conclude that

- the graph is sloping *up* when the *slope* $= \frac{\textit{output change}}{\textit{input change}}$ is *positive*
- the graph is sloping *down* when the *slope* $= \frac{\textit{output change}}{\textit{input change}}$ is *negative*

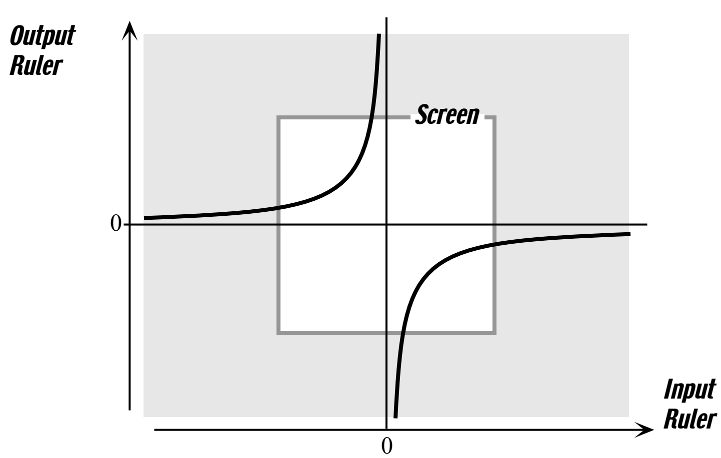
10.5 Concavity

We now use

$$x \xrightarrow{\ NEGODEXP\ } NEGODEXP(x) = (-4)x^{+\text{odd}}$$

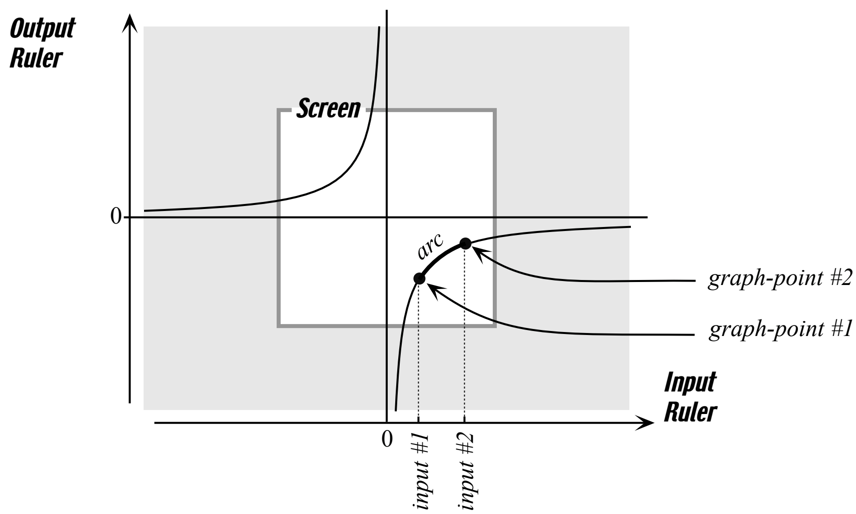
to describe another *feature* of a function which is also not readily apparent from the input-output rule but is quite clear on its qualitative graph.

1. From the graphic point of view, we use its qualitative graph



a. Take two *positive* inputs, call them *input* #1 and *input* #2, and get their graph-points, $graph - point$ #1 and $graph - point$ #2, as in the following picture.

arc
concave down



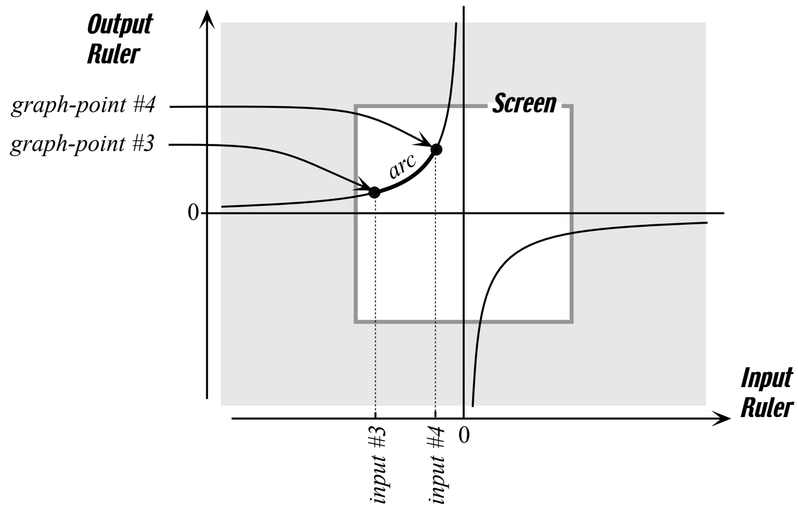
We observe that the **arc**  can be seen as part of the "cap"  but cannot be seen as part of the "cup" .

When the arc can be seen as part of a "cap" , we shall say that the function is **concave down**.

So, the function $NEGODEXP$ is concave *down* for all *positive* inputs.

b. Similarly, take two *negative* inputs, call them *input* #3 and *input* #4, and get their graph-points, $graph - points$ #3 and $graph - points$ #4, as in the following picture:

concave up



We observe that the *arc*  can be seen as part of the “cup”  but cannot be seen as part of the “cap” .

When the arc can be seen as part of a “cup” , we shall say that the function $NEGODEXP$ is **concave up**.

So, the function $NEGODEXP$ is concave *up* for all *negative* inputs.

2. From the input-output rule point of view, we must take a specific instance of $NEGODEXP$ in order to compute the outputs and their changes and we take

$$x \xrightarrow{JILL} JILL(x) = (-4)x^{-3}$$

a. We pick *input* #1 = +3 and *input* #2 = +5,

i. We compute the outputs:

$$\begin{aligned} input\ \#1 = +3 \xrightarrow{JILL} JILL(+3) &= (-4)(+3)^{-3} \\ &= \frac{-4}{(+3)\cdot(+3)\cdot(+3)} \\ &= \frac{-4}{+27} \\ &= -0.148 + (...) = output\ \#1 \end{aligned}$$

and

$$\begin{aligned} \textit{input } \#2 = +5 \xrightarrow{JILL} JILL(+5) &= (-4)(+5)^{-3} \\ &= \frac{-4}{(+5)\cdot(+5)\cdot(+5)} \\ &= \frac{-4}{+125} \\ &= -0.032 = \textit{output } \#2 \end{aligned}$$

ii. We compute the changes:

$$\begin{aligned} \textit{input change} &= \textit{input } \#2 - \textit{input } \#1 \\ &= (+5) - (+3) \\ &= (+5) + (-3) \\ &= +2 \end{aligned}$$

and

$$\begin{aligned} \textit{output change} &= \textit{output } \#2 - \textit{output } \#1 \\ &= (-0.032) - (-0.148 + (...)) \\ &= (-0.032) + (+0.148 + (...)) \\ &= +0.116 + (...) \end{aligned}$$

b. Now, we take again *input* #2 = +5 and we pick *input* #3 = +7

i. We compute *output* #3

$$\begin{aligned} \textit{input } \#3 = +7 \xrightarrow{JILL} JILL(+7) &= (-4)(+7)^{-3} \\ &= \frac{-4}{(+7)\cdot(+7)\cdot(+7)} \\ &= \frac{-4}{+343} \\ &= -0.012 + (...) = \textit{output } \#3 \end{aligned}$$

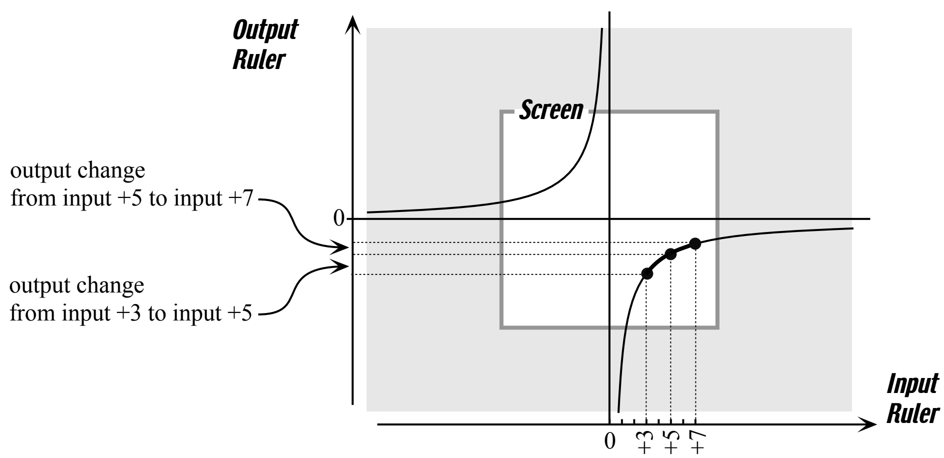
ii. We compute the changes.

$$\begin{aligned} \textit{input change} &= \textit{input } \#3 - \textit{input } \#2 \\ &= (+7) - (+5) \\ &= (+7) + (-5) \\ &= +2 \end{aligned}$$

and

$$\begin{aligned} \textit{output change} &= \textit{output } \#3 - \textit{output } \#2 \\ &= (-0.012 + (...)) - (-0.032) \\ &= (-0.012) + (+0.032) \\ &= +0.020 + (...) \end{aligned}$$

3. Here we will use the qualitative graph to keep track of things,



a. We observe that, even though

the second *input* change (from input +5 to input +7)

is the same as

the first *input* change (from input +3 to input +5)

the corresponding *output* changes are not the same:

the second *output* change (from input +5 to input +7), equal to +0.020, is *algebra*-smaller than

the first *output* change (from input +3 to input +5), equal to +0.116 + (...)).

This fits the fact that the graph is concave *down* for all positive inputs.

b. Since the second *input* change (from input +5 to input +7) is the same as the first *input* change (from input +3 to input +5), the fact that the second *output* change (from input +5 to input +7) is *algebra*-smaller than the first *output* change (from input +3 to input +5) means that the second *slope* (from the graph-point for input +5 to the graph-point for input +7) is *algebra*-smaller than the first *slope* (from the graph-point for input +3 to the graph-point for input +5).

So, the fact that the *graph* is *concave down* corresponds to the fact that the *slope* gets *algebra*-smaller when the inputs get *algebra*-larger.

10.6 Types of Power Functions

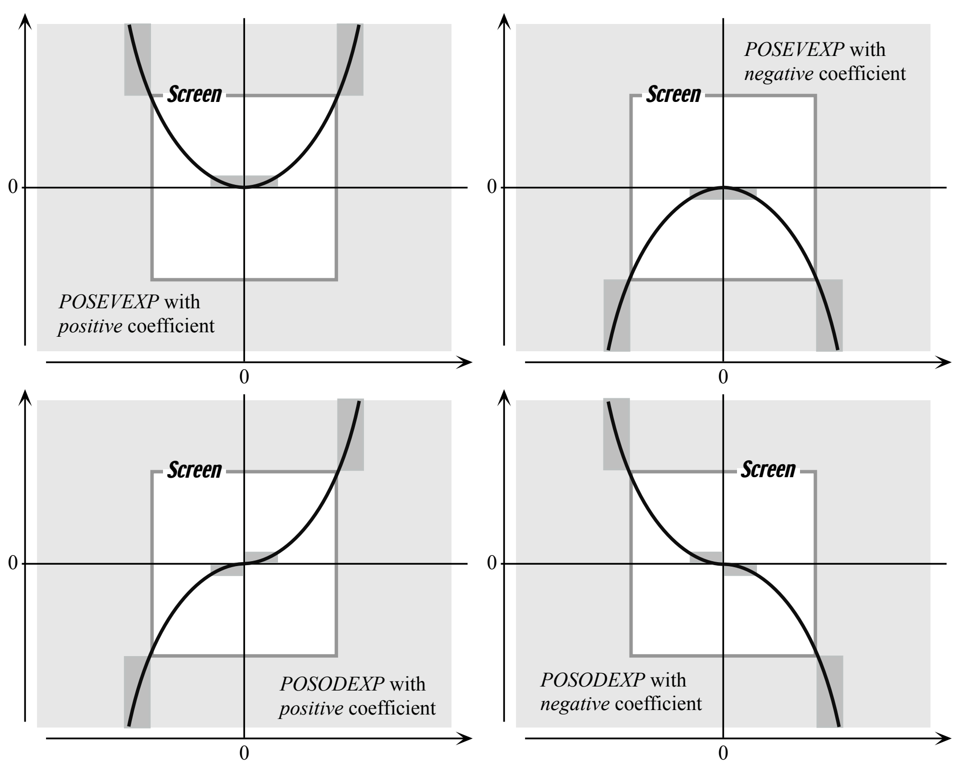
This is mostly a *systematic* putting-together of the information we have gathered so far about power functions.

1. There are two main types of power functions:

- The power functions whose *exponent* is positive
- The power functions whose *exponent* is negative

a. When the *exponent* of a power function is *positive*, so that the coefficient is *multiplied* by the copies of the input, then regardless of everything else,

- the smaller the inputs are in size, then even smaller the outputs are in size, and the local areas are *horizontal* rectangles
- the larger the inputs are in size, then even larger the outputs are in size, and the local areas are *vertical* rectangles

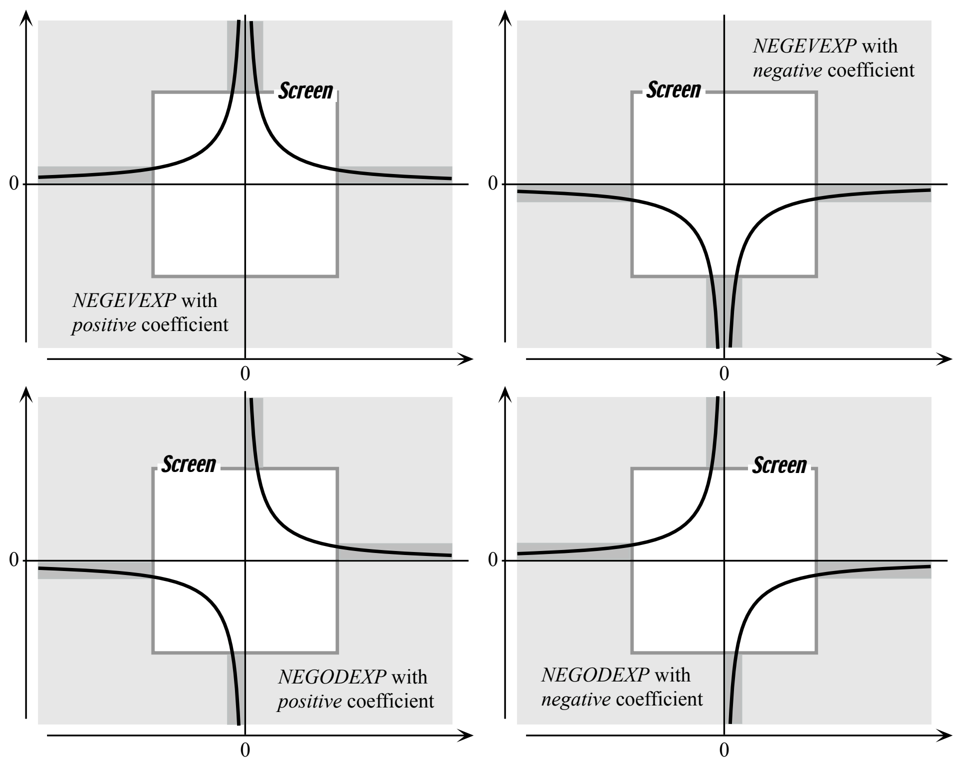


From a graphic viewpoint, this means that with *positive* exponent power functions,

- when the inputs are near 0, then the slope of the graph is nearly 0
- when the input are near ∞, then the slope of the graph is nearly ∞

b. When the *exponent* of a power function is *negative*, so that the coefficient is *divided* by the copies of the input, then regardless of everything else,

- the smaller the inputs are in size, then even the larger the outputs are in size, and the local areas are *vertical* rectangles
- the larger the inputs are in size, then even the smaller the outputs are in size, and the local areas are *horizontal* rectangles



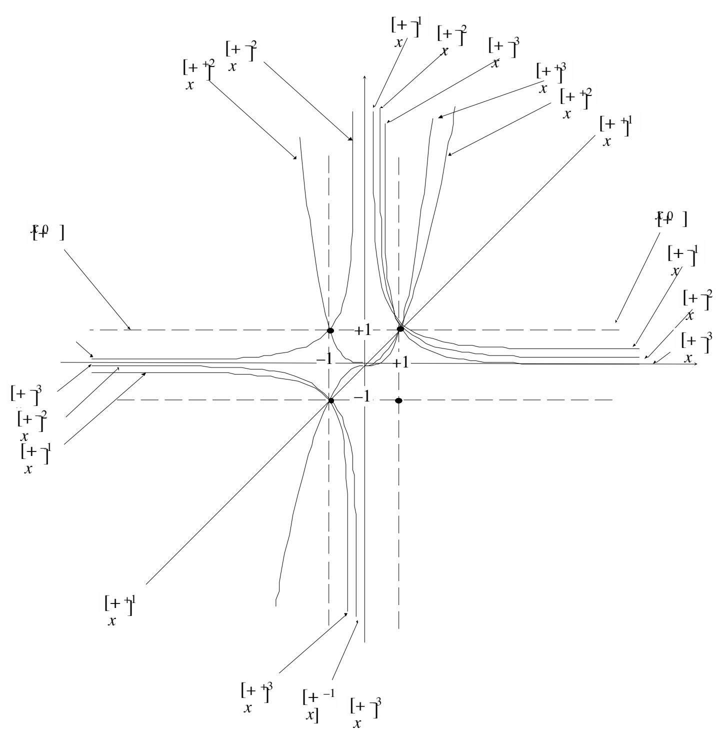
From a graphic viewpoint, this means that with *negative* exponent power functions,

- when the inputs are near 0, then the slope of the graph is nearly ∞
- when the input are near ∞, then the slope of the graph is nearly 0

=== reciprocal
===
===
===
===

c. There are a number of very nice "symmetries" worth noting:

- The behaviors relative to size near 0 and near ∞ are **reciprocal**, that is the size of the outputs when the inputs are near 0 is the reciprocal of the size of the outputs when the inputs are near ∞: when either ones are *small*, the others are *large*.
- When the *sign* of the exponent changes, then, relative to size, the behavior near 0 and the behavior near ∞ are exchanged regardless of everything else.
- All power functions whose coefficient is $+1$ give output $+1$ for the input $+1$. For the input -1, the output is $+1$ or -1 depending on whether the parity of the exponent is *even* or *odd*.
- All power functions whose coefficient is -1 give output -1 for the input $+1$. For the input -1, the output is -1 or $+1$ depending on whether the parity of the exponent is *even* or *odd*.



2.
3.
We look systematically at the local areas for each type of power functions.
1. • When the inputs are $+$ *large*, we have
- When the inputs are $-$ *large*, we have
- When the inputs are $+$ *small*, we have
- When the inputs are $-$ *small*, we have

2. • When the inputs are $+$ *large*, we have
- When the inputs are $-$ *large*, we have
- When the inputs are $+$ *small*, we have
- When the inputs are $-$ *small*, we have

3. • When the inputs are $+$ *large*, we have
- When the inputs are $-$ *large*, we have
- When the inputs are $+$ *small*, we have
- When the inputs are $-$ *small*, we have

4. • When the inputs are $+$ *large*, we have
- When the inputs are $-$ *large*, we have
- When the inputs are $+$ *small*, we have
- When the inputs are $-$ *small*, we have

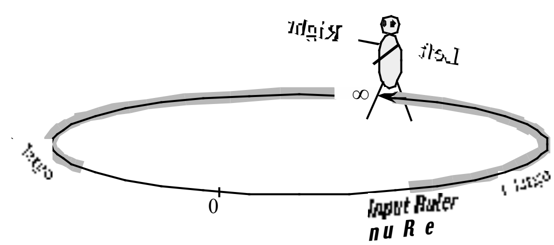


Figure 10.1: $+\infty$ is left of ∞ and $-\infty$ is right of ∞.

Most of the time, we shall not concern ourselves with *specific* inputs but only with inputs having certain *features.*

For instance, we take the power function whose input-output rule is as an example and we will investigate the SIZE and SIGN of the outputs that returns when we use inputs of a given SIGN and SIZE. We could of course look at SIGN and SIZE separately,

1. We use small positive inputs: + small $(-1)(+\text{small})$ $-3 = = -\text{large}$ where the small rectangle is where the correspondent plot-points will have to be.

2. We use small negative inputs: $-$ small $(-1)(-\text{ small})$ $-3 = = +$ large

3. Altogether, this gives us the local box when x is near 0 which is made up of the two separate pieces shaded outside the window
The graph when x is near 0 (when x is small) will be in this local box. We code this information as follows:

- SIGN height near 0 = (+, −)
- SIGN height near 0 = (+, −)

Note. The code left of the comma refers to the feature of the output when the input is left of the center-point which, here, is 0 and the code right of the comma refers to the feature of the output when the input is right of the center-point.

4. We take large inputs: + large (−1)(+ large) −3 = = − small − large (−1)(− large) 3 = = + small
The graphic representation is the local box when x is near made up of the two pieces shaded outside the window:
The graph when x is near (when x is large) will be in this local box. We code this information as follows:

- SIGN height near = (−, +)
- SIZE height near = (small, small)

Note. The code left of the comma refers to the feature of the output when the input is left of the center-point which, here, is and the code right of the comma refers to the feature of the output when the input is right of the center-point. Be careful though: even though large positive inputs (+ large) are right of 0, this is irrelevant because 0 is the center-point for small inputs. The center-point for large inputs is and large positive inputs are left of .

5. Altogether, the plot points for small inputs and for large inputs will be in the two local boxes:

10.7 Sign of *Slope*

- SIGN slope near 0 = (,) (The center point is 0 and 0_+ is the right half of the neighborhood of 0 and 0_- is the left half of the neighborhood of 0.)
- SIGN slope near $\infty.)$ = (,) (The center point is $\infty.)$ so that $+\infty.)$ is left of $\infty.)$ and $\infty.)$ is right of $\infty.)$

10.8 Sign of *Concavity*

10.9 Local Features

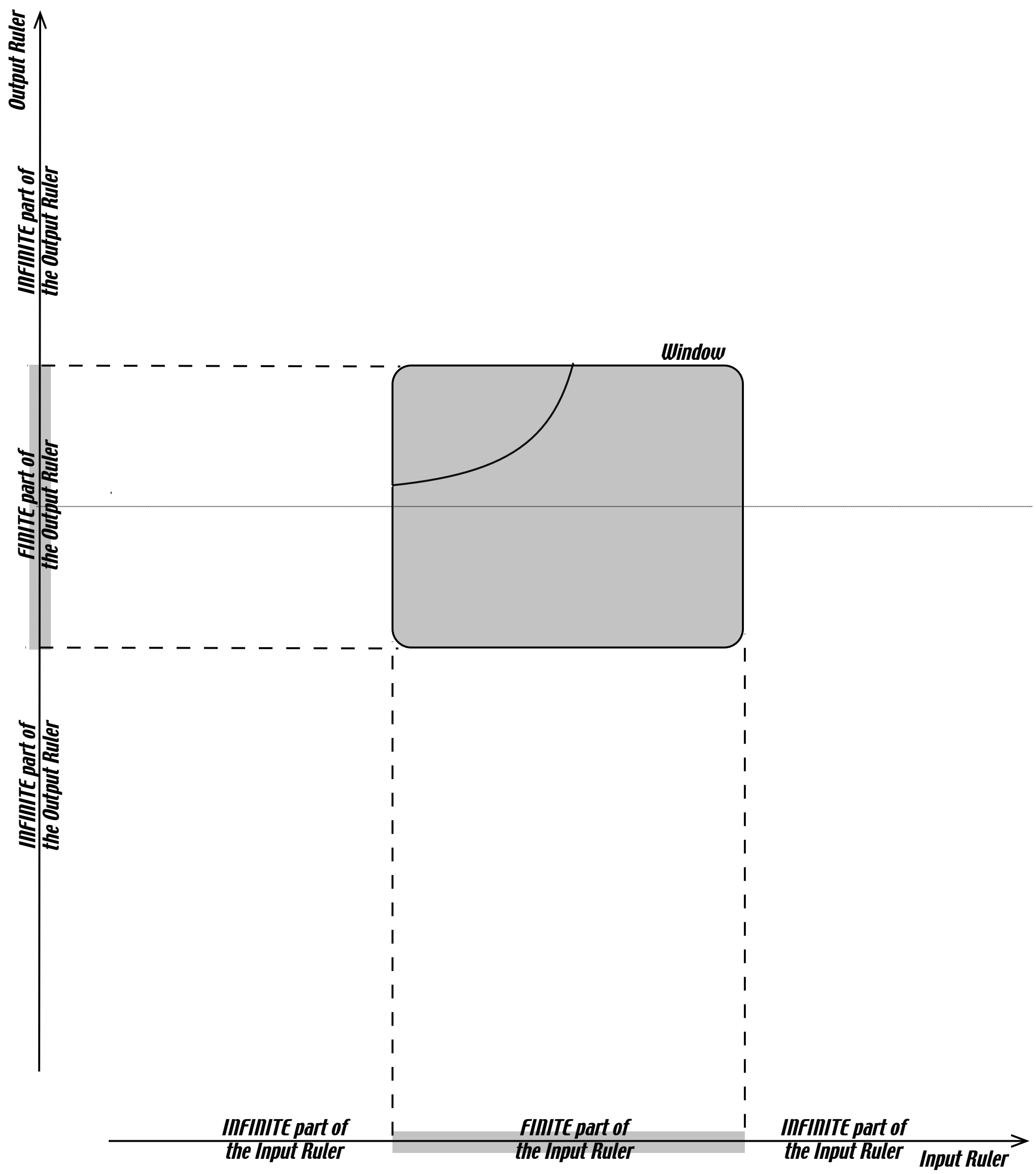


Figure 10.2: Finite Part of Graph.

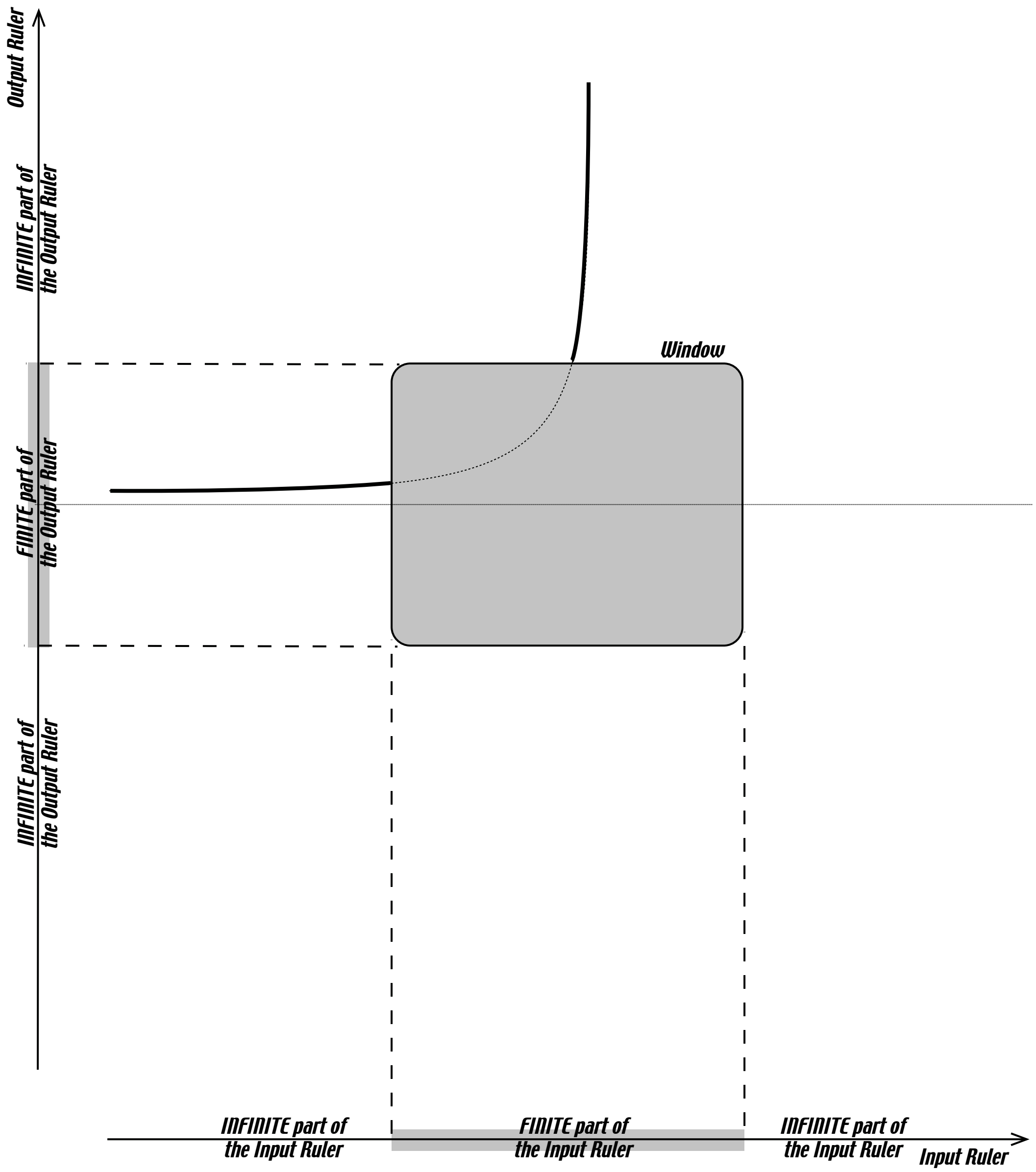


Figure 10.3: Graph near infinity.

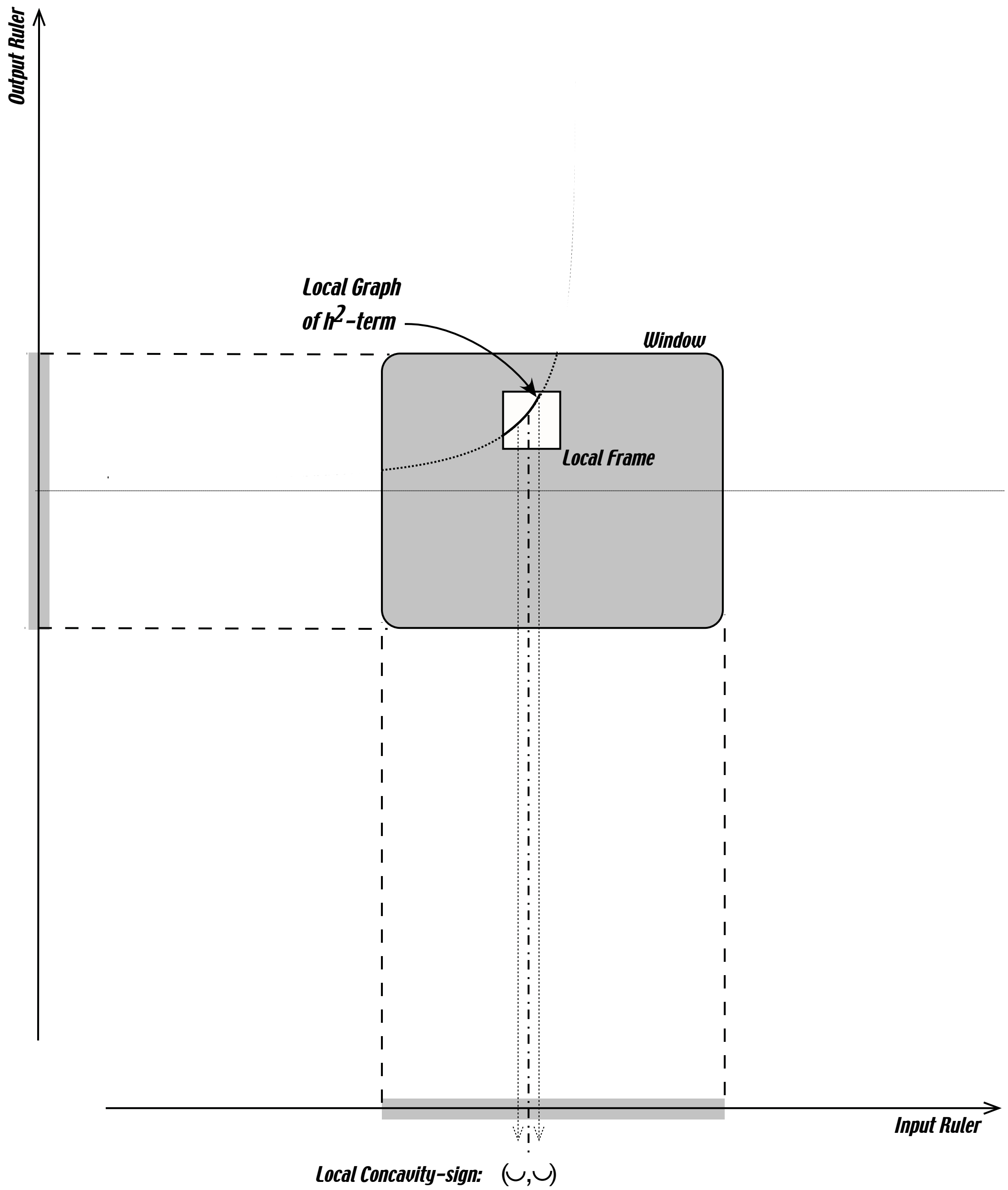


Figure 10.4: Local Concavity-sign.

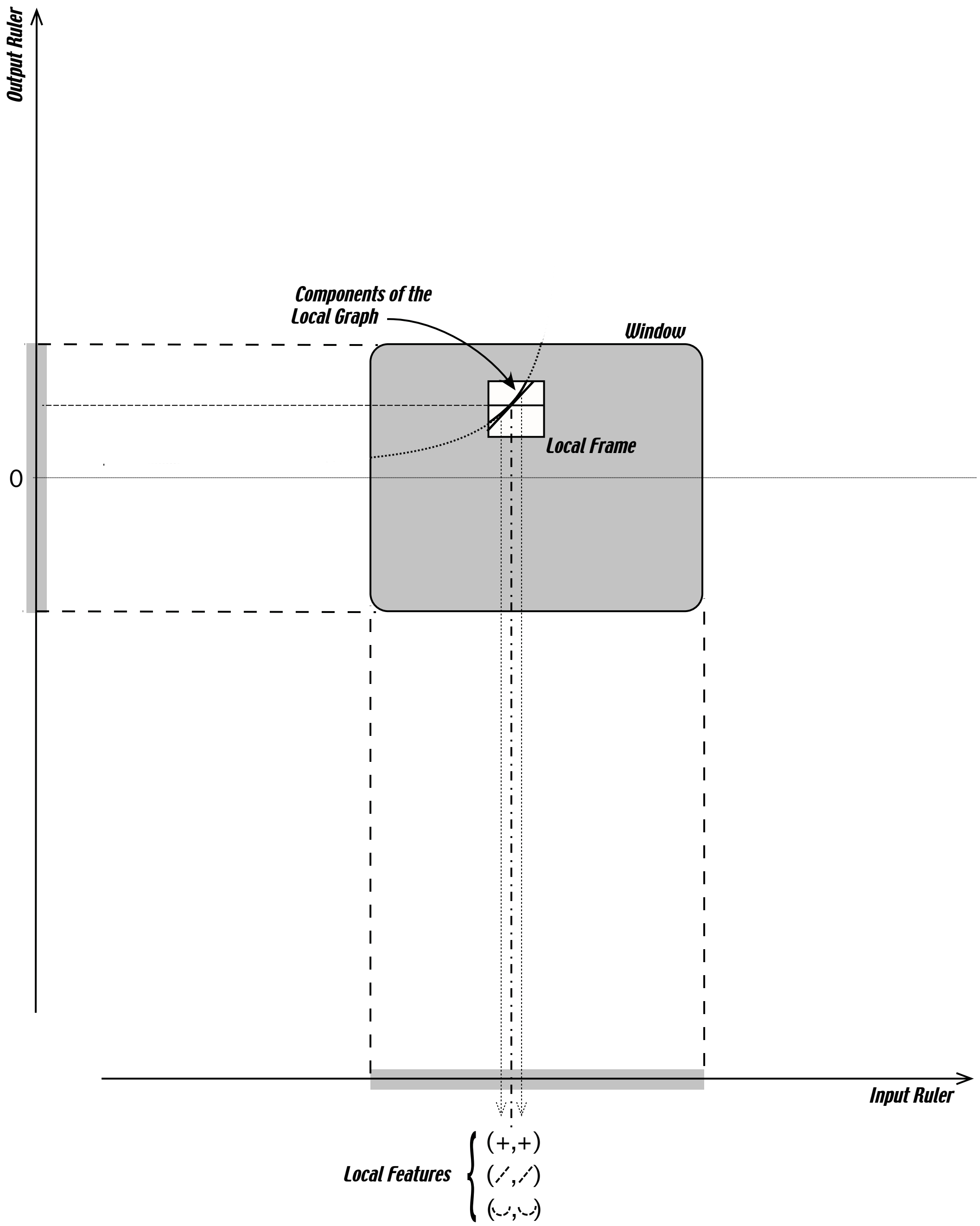


Figure 10.5: Local Features.

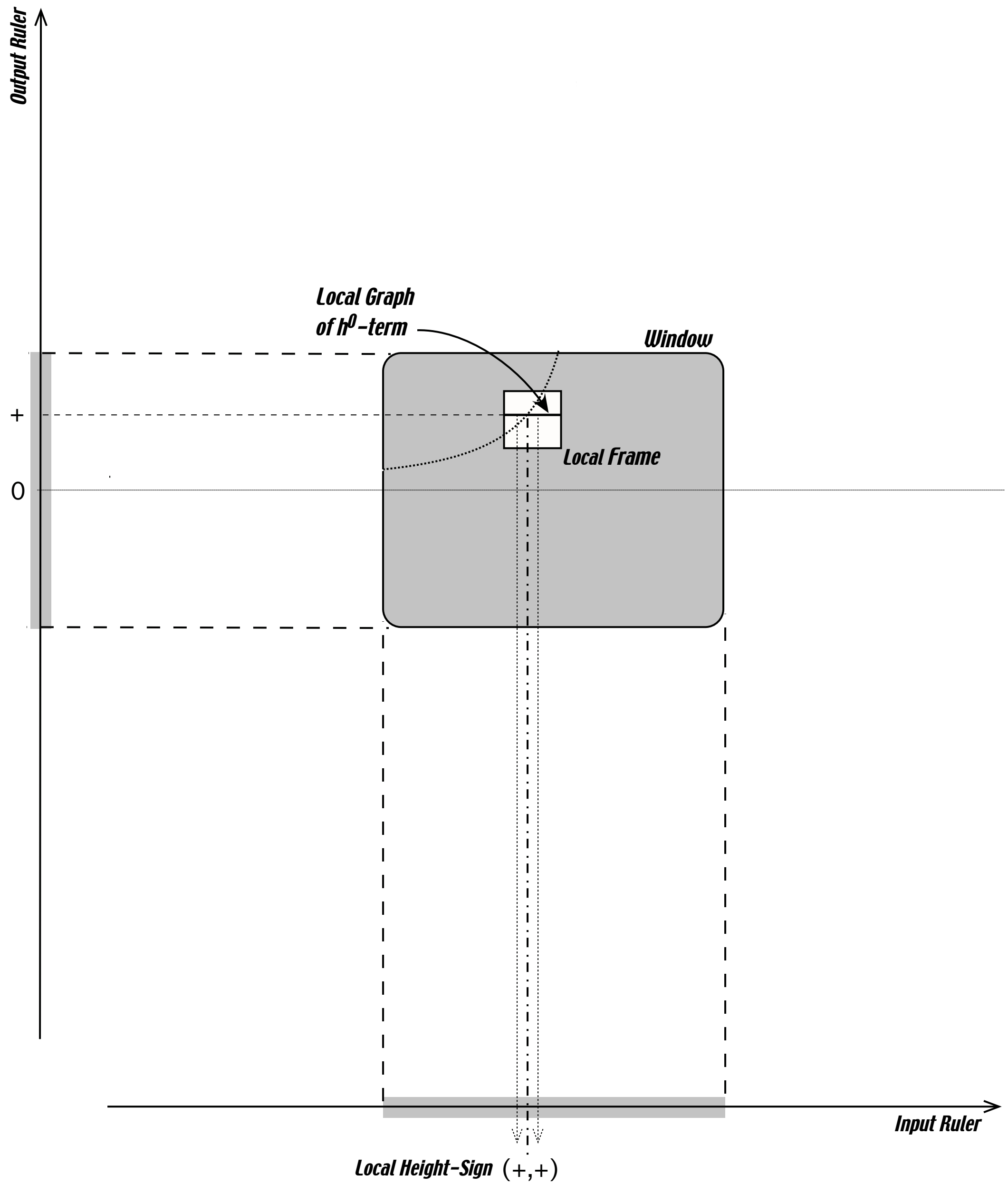


Figure 10.6: Local Height-sign.

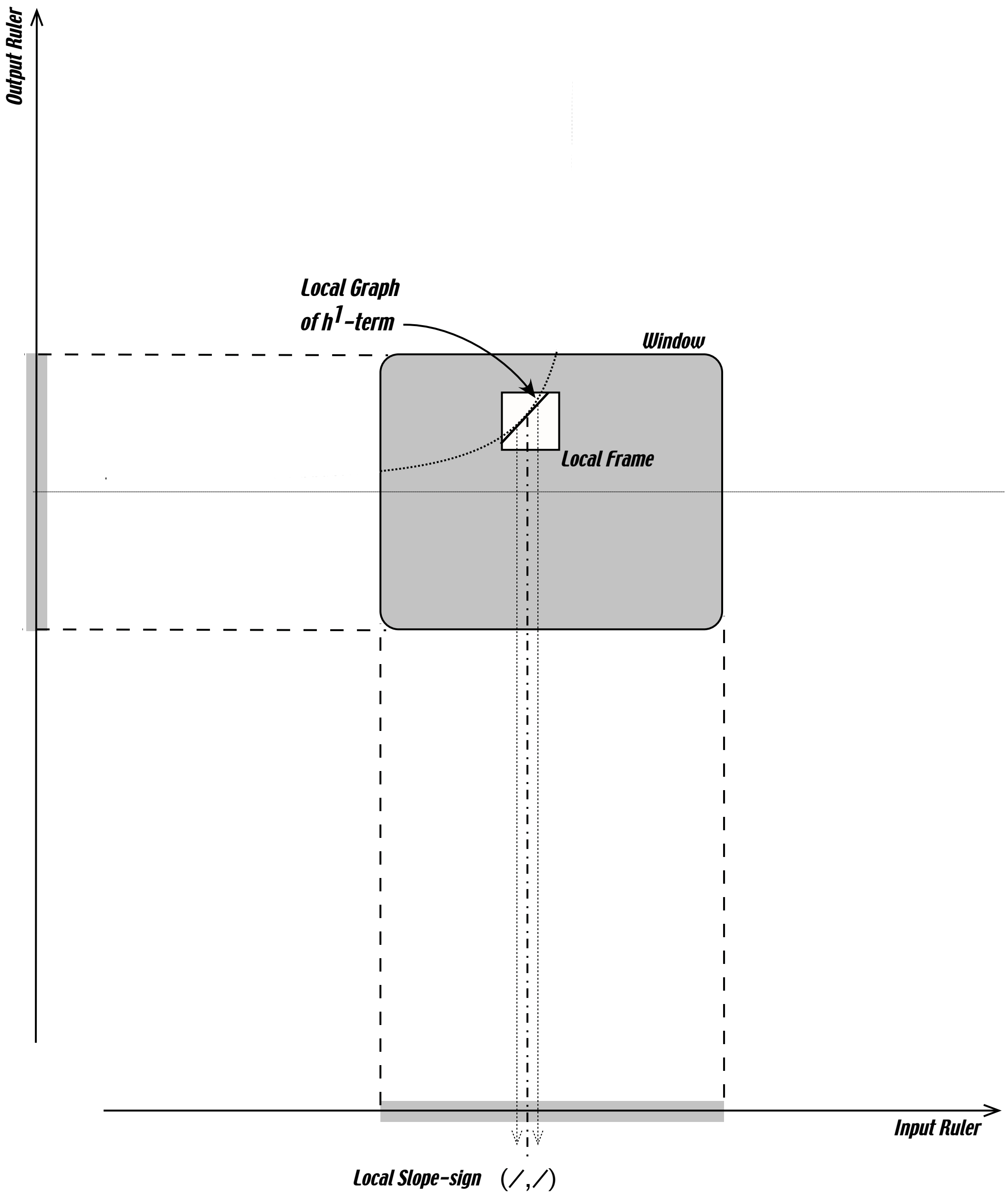


Figure 10.7: Local Slope-sign.

10.10 From Local to Global

From just the local graph near either 0 or ∞, we can get the qualitative global graph. From the local graph from one side of either 0 or ∞, we can get the qualitative global graph if we know the parity.

10.11 Global Graphs

10.12 From Rule to Graph

From the features of the Input-Output rule to the features of the (qualitative global) graph.

1. **a.** We compute the outputs, keeping in mind that when we multiply an even number of copies of a negative number the result is a negative number. We take small positive inputs: + small (1)(+small) +even = (1) = (1) (+ small) = small We take small negative inputs: small (1)(small) +even = (1) = (1) (+ small) = small We take large positive inputs: + large (1)(+ large) +even = (1) = (1) (+ large) = large We take large negative inputs: large (1)(large) +even = (1) = (1) (+ large) = large

b. In graphic terms, this gives us the following local boxes
Figure 1.

c. We code the results as follows SIGN height near 0 = (,) SIZE height near 0 = (small , small) SIGN height near = (,) SIZE height near = (large , large)

2. **a.** a. However, looking closer at the above computations, we see that the outputs are not just small (or large), but they are smaller (or larger) than the inputs that they come from.

b. b. In graphic terms, the local boxes are not squares but rectangles in a manner that corresponds to the relative sizes of the inputs and outputs as in Figure 2.

3. **a.** a. Now we dont just look at one (undisclosed) input at a time but at inputs that change and see how the corresponding outputs change.

b. b. Graphically this translates to the following run arrows, rise arrows and slope segments:

c. We code this as follows SIGN slope near 0 = (,) SIGN slope near = (,) (Remember that the center point is .)
0+ is right of 0

a. With small/large inputs, the size of the outputs determines the sign of the power. b. With positive inputs, the sign of the outputs determines the sign of the coefficient c. With negative inputs, the sign of the outputs determines the parity of the power

10.13 From Graph to Rule

From the features of the (qualitative global) graph to the features of the Input-Output rule. Conversely, we ask what feature of the graph entails what feature of the input-output rule. For in-stance, to say of a power function f that small large means that SIGN power = and to say that + means that SIGN coefficient = and then to say that means that PARITY power = even. Information about slope or concavity is a bit more difficult to utilize as, in practice, it requires that we already know the local graphs of the power functions and that we consider the two possibilities for SIGN power and check if they yield the same conclusion. Eventually however, we observe that, very fortunately, THE LOCAL GRAPHS OF A POWER FUNCTION BOTH NEAR 0 AND NEAR ARE DETERMINED BY JUST ITS LOCAL BOX NEAR EITHER 0 OR and that, therefore, so are its type of input-output rule and its global qualitative graph :

10.14 Comparing degrees

The higher the degree, the boxier the graph.

10.15 Comparing types

10.16 Exceptional Power Functions

local input-output rule

Chapter 11

Polynomial Functions

11.1 Introduction

Starting with this chapter and until the end of PART TWO: FUNCTIONS SPECIFIED DIRECTLY, for each *type* of functions, our goal will be:

- To find *general statements* about the *qualitative features* of this *type* of functions, given by an *input-output rule*,.
- To find a *global qualitative graph* for this *type* of functions (to serve as a basis for *interpolating-extrapolating plots* into actual *global quantitative graphs*):

- To find the *conditions* that a *plot* of a function of this *type* must meet in order to be *interpolated-extrapolated* into a global quantitative graph.

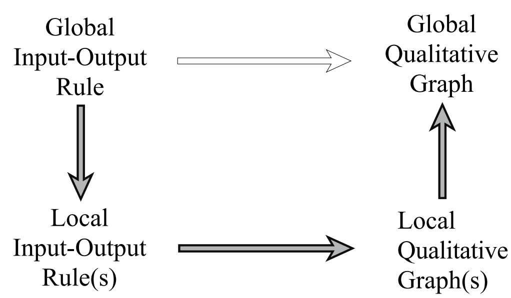
After we have dealt with *constant* functions, our *approach* will be based on the fact that, while, no matter how many plot points we have, they can never be *interpolated-extrapolated* into a *global quantitative graph*, *local graphs* can be, at least to a "certain" extent. In particular, in the presence of "certain" information, local graphs near "certain" input points can be *interpolated-extrapolated* a long way. Thus, for each given *kind* of functions, we shall proceed as follows:

- We shall find out how to get the **local input-output rule** near a given input point,
- We shall see how to *construct* from the *local input-output rule* the *local graph* near a given input point,

essential
operations on functions
construct
dilation
dilation factor

- We shall determine near which "certain" input points we need to find a local graph, how far these local graphs can be interpolated-extrapolated and what *information* is needed to decide that.
- We shall see how to *join smoothly* the local graphs near these "certain" points into an **essential** *global qualitative graph.*

In other words, other than with the *constant* functions, our general strategy will be the *long* way in



11.2 Operations On Functions

Before we can start on this program, though, we need to develop a couple of tools.

Indeed, since, earlier on, we defined polynomials as combinations of powers, and since polynomial functions will be functions whose output is a polynomial, it should not be surprising that we first need **operations on functions** to **construct** new functions out of given functions so as to look upon polynomial functions as constructed from power functions, that is as linear combinations of power functions.

How we look at these operations will depend on whether we look at them from the *input-output rule* viewpoint or from the *graphic* viewpoint.

To keep things simple from the graphic point of view, we will be looking here at functions specified by *plots* rather than by *graphs.*

11.2.1 Dilations Of Functions

The **dilation** of a given function f with a given **dilation factor** a, where a is a *signed* number, is a new function whose name is $[a \odot f]$ and whose outputs are the outputs of the given function f multiplied by the given number a.

dilated function
plain stretching

1. For instance, say some function called QG_{34} is given by the table

x	-6	-5	-3	-2	0	$+1$	$+2$	$+3$
$QG_{34}(x)$	-4	$+4$	$+5$	$+5$	-1	0	-5	$+5$

and that we want to look at $[-3 \odot QG_{34}]$, the *dilation* of the function QG_{34} by a *factor* of -3.
Then, for instance, given the input -2, we get its output under the **dilated function** $[-3 \odot QG_{34}]$ as follows

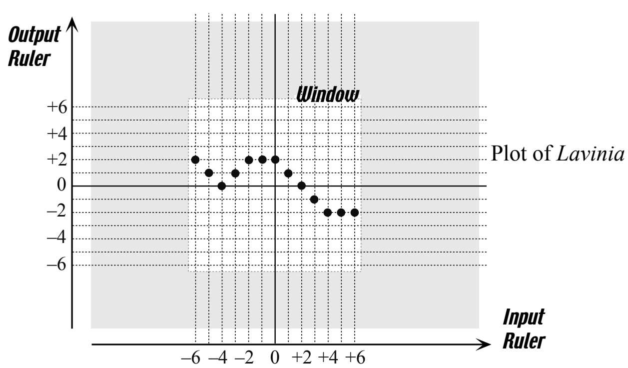
$$\begin{aligned} x \xrightarrow{[-3\odot QG_{34}]} [-3 \odot QG_{34}]\,(x) &= (-3)\cdot QG_{34}(-2) \\ &= (-3)\cdot(+5) \\ &= -15 \end{aligned}$$

In that manner, we can obtain the *table* for the *dilated function* $[-3 \odot QG_{34}]$:

x	-6	-5	-3	-2	0	$+1$	$+2$	$+3$
$[-3 \odot QG_{34}]\,(x)$	$+12$	-12	-15	-15	$+3$	0	$+15$	-15

2. From the *graphic* viewpoint and while, when we *computed* the outputs, the "rule of signs for multiplication" took automatically care of *negative* dilation factors as well as *positive* dilation factors, here we will have to look at the two cases *separately*.

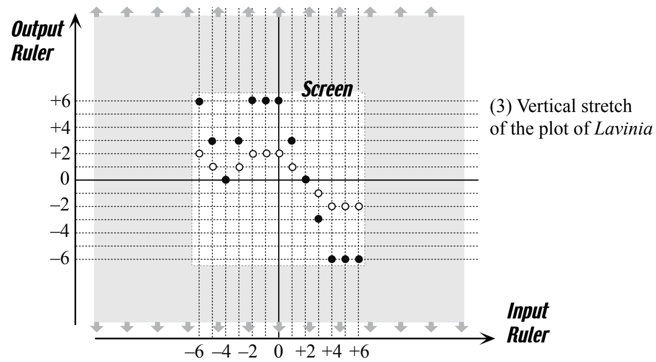
a. We thus begin by explaining what we mean by a **plain stretching**. For example, given some function called ZW_{009} whose plot is



say we want the plot of $[(+3) \odot ZW_{009}]$ or $[(-3) \odot ZW_{009}]$.
Imagine the plot of ZW_{009} printed of a sheet of stretchable, transparent rubber lying on top of the window but attached to the 0-output level line with a copy of the plot printed on the window itself with white circles just underneath the black circles so that they are not visible.

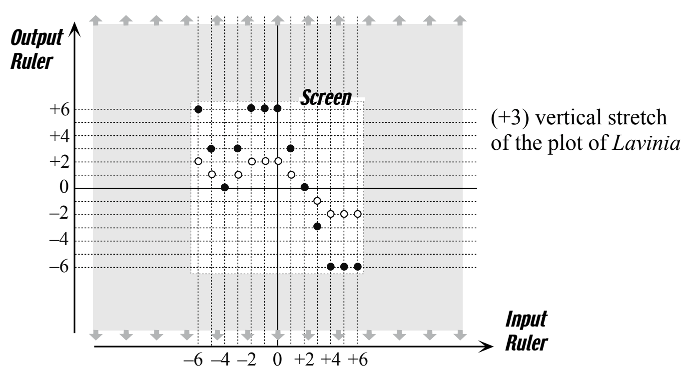
positive vertical stretching

Now grab the rubber sheet on its top and bottom outer edges and stretch it vertically with a *stretch factor* equal to SIZE(+3)=SIZE(−3)= 3, that is equal to the SIZE of the *dilation factor*:



The black circles show the plot of $[(3) \odot ZW_{009}]$ which is the *plain vertical stretching* of ZW_{009} with the coefficient 3.
We can now see (at least some of) the white circles of the copy of the plot of ZW_{009} and the white circles that we do not see are the plot points for the inputs −4 and +2 because the output $ZW_{009}(-4) = -4 \cdot 0 = 0$ and the output $ZW_{009}(+2) = +2 \cdot 0 = 0$ and since the rubber sheet is attached to the 0 output level line, the plot points for $[(3) \odot ZW_{009}]\,(-4)$ and $[(3) \odot ZW_{009}]\,(+2)$ stay right on top of them.

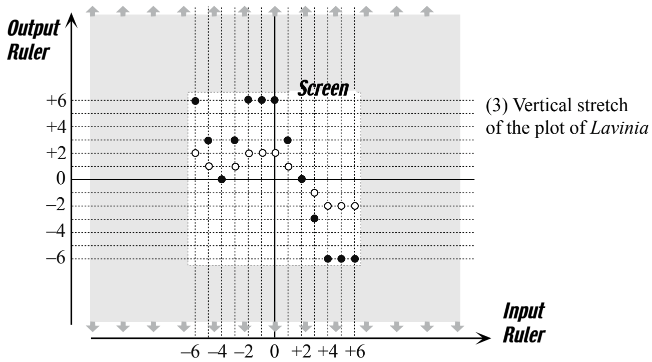
b. Since, when a is *positive*, we *identify* a with SIZE a, the **positive vertical stretching** $[(+3) \odot ZW_{009}]$ is the same as the *plain vertical stretching* $[(3) \odot ZW_{009}]$.



c. Since, when a is *negative*, we *identify* a with −SIZE a, we obtain the plot of $[(-3) \odot ZW_{009}]$ as follows

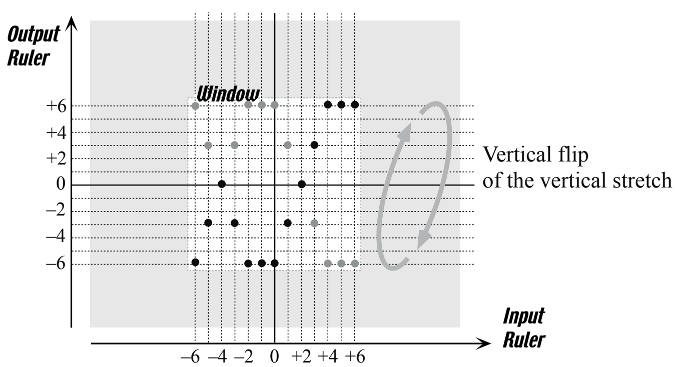
We start as before and do a *vertical stretch* with a *stretch factor* equal to SIZE$(-3) = 3$, that is again equal to the SIZE of the *dilation factor*.

vertical flip
negative stretching
followed by
expansion
contraction



in which the white circles show the plot of ZW_{009} and the grey circles show the plot of $[(+3) \odot ZW_{009}]$.

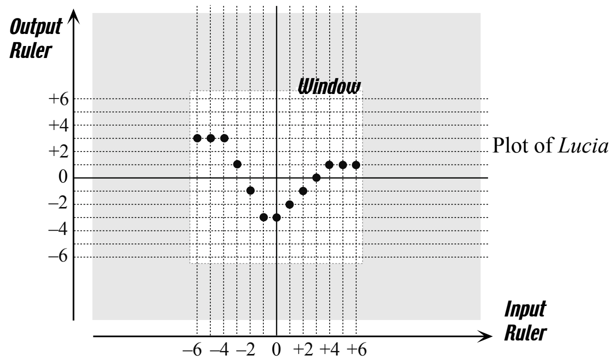
But, because the stretch factor is *negative*, we need an extra step to get the plot of $[(-3) \odot ZW_{009}]$ from the plot of the plain stretch, $[(3) \odot ZW_{009}]$, namely a **vertical flip**:



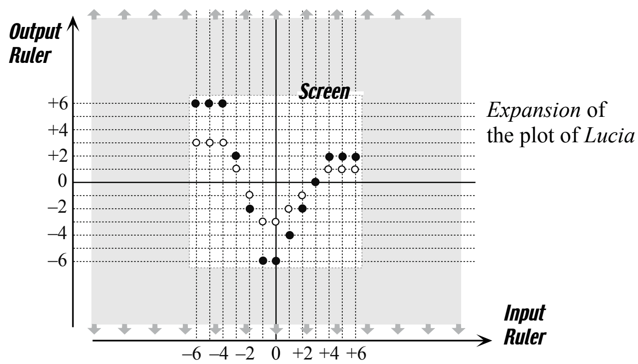
By a **negative stretching**, we shall thus mean a *plain vertical stretching* **followed by** a *vertical flip*.

d. Depending on whether the dilation factor is larger than 1 or smaller than 1, we shall speak of an **expansion** or a **contraction**.

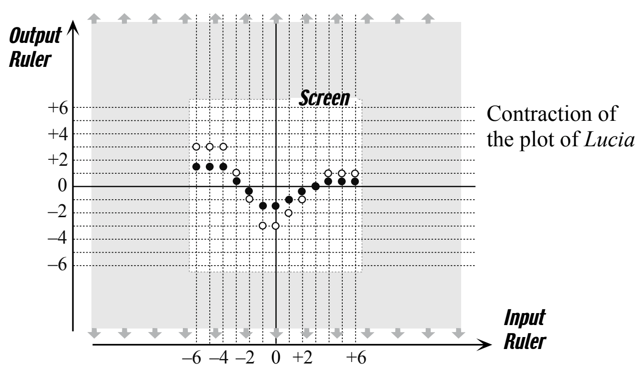
For instance, given the plot of XQ_{111}



the plain stretching $[(2) \odot XQ_{111}]$ is an *expansion*



while the plain stretching $\left[(\frac{1}{2}) \odot XQ_{111}\right]$ is a *contraction*.



3. From the *input-output rule* viewpoint, if the output of a function f is given by some given expression involving the input

$$x \xrightarrow{f} f(x) = [\text{some expression involving } x]$$

then the output of the function $[a \odot f]$, its dilation by a factor a, is given by a multiplied by the same given expression involving the input addition

$$x \xrightarrow{[a\odot f]} [a \odot f]\,(x) = a \cdot [\text{same expression involving } x]$$

Thus, for instance, the (-3) dilation of the power function whose input-output rule is

$$x \xrightarrow{f} f(x) = (-1)x^{-5}$$

will be the function whose input-output rule is

$$\begin{aligned} x \xrightarrow{[(-3)\odot f]} [(-3) \odot f]\,(x) &= (-3) \cdot \left[(-1)x^{-5}\right] \\ &= (+3)x^{-5} \end{aligned}$$

4. In all of the above, we used the symbol $\odot$ and the brackets [ ] for the sake of clarity but from now on, in the course of our investigations, we will almost conform to the general usage:

- We shall reuse the symbol $\cdot$ instead of using the symbol $\odot$,
- We shall use brackets *only* when writing *outputs*.

Thus, later on for example,

we will write	rather than	or the usual	but write the *outputs*
$3 \cdot f$	$[3 \odot f]$	$3f$	$[3 \cdot f]\,(-4)$
$m \cdot L$	$[m \odot L]$	mL	$[m \cdot L]\,(+51)$
$(+2) \cdot A$	$[(+2) \odot A]$	$+2A$	$[(+2) \cdot A]\,(\frac{7}{3})$
$a \cdot COS$	$[a \odot COS]$	$a\,COS$	$[a \cdot COS]\,(\frac{-2\pi}{3})$
$\sqrt{+5} \cdot EXP$	$\left[\sqrt{+5} \odot EXP\right]$	$\sqrt{+5}EXP$	$\left[\sqrt{+5} \cdot EXP\right]\,(x)$

11.2.2 Addition Of Functions

The **addition** of two functions f and g is a new function whose name is $f \oplus g$ and whose output is that of the addition of the two *given* functions.

Actually, here we will use the symbol $\oplus$ for the sake of clarity but, later on, when we use addition of functions on actual functions, we will conform to the usual practice of reusing the symbol $+$.

Thus, later on,

we will write	rather than
$COS + SIN$	$COS \oplus SIN$
?	?
?	?

constant function
constant term

Etc.

we will write $COS + SIN$ rather than .

11.2.3 Linear Combinations Of Functions

Thus, later on,

we will write	rather than
$3COS - 5SIN$	$(+3) \odot COS \oplus (-5) \odot SIN$
?	?
?	?

11.3 Constant Functions

Functions whose *input-output rule* is of the form:

$$x \xrightarrow{C_a} C_a(x) = ax^0$$

where a stands for a given *signed* number are called **constant functions**. C_a will be the name of the *constant* function *specified* by the signed number a. The given signed number a is called the **constant term** of C_a.
For example, C_{-3} is the *constant function* whose input-output rule is

$$x \xrightarrow{C_{-3}} C_{-3}(x) = (-3)x^0$$

that is the function whose *output* is equal to -3 multiplied by *zero* copy of the *input*.
Then, when given an input, say $+4$, we write

$$\begin{aligned} +4 \xrightarrow{C} C(x)|_{\text{when x=+4}} &= (-3)x^0\big|_{\text{when x=+4}} \\ &= (-3)(+4)^0 \\ &= -3 \end{aligned}$$

Note. However, unless there is need for the above complete way of writing things, as there will be when, for instance, we have to deal with more than one *constant function* at a time, we shall usually write a simpler form such as

$$x \xrightarrow{C} C(x) = a$$

In particular, the power x^0 in the term ax^0 will often "go without saying". For instance, instead of writing, as in the above example,

$$x \xrightarrow{C_{-3}} C_{-3}(x) = (-3)x^0$$

we shall usually write

$$x \xrightarrow{C} C(x) = -3$$

11.3.1 Graphing Constant Functions

We can proceed in either one of two ways.

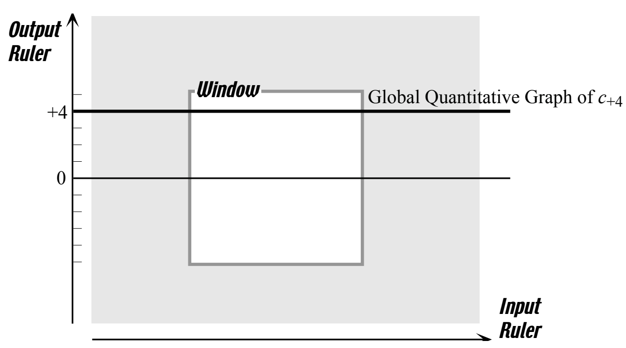
1. We can deal with a *constant* function in exactly the same manner as we dealt with the *unit* functions, saying that since $ax^0 = a$ the input-output rule of a constant function boils down to

$$x \xrightarrow{C_a} C_a(x) = a$$

the input does *not* appear in the output so that, no matter what the input x is, the output of the *constant* function C_a will remain equal to the number a. As a result, the *global quantitative graph* of the *constant* function C_a will be the *output level line* for the output a.
For instance, given the *constant* function C_{+4}, its *input-output rule* is

$$\begin{aligned} x \xrightarrow{C_{+4}} C_{+4}(x) &= (+4)x^0 \\ &= +4 \end{aligned}$$

and its *global quantitative graph* is



The *disadvantage* of getting the graph this way is that it depends too much on the fact that the function is a *constant function* so that it will not carry over to *other* functions.

2. The other way is based on the similarity of *constant* functions with *unit* functions.

a. First, we observe that *constant functions* are very similar to the two *unit functions*

$$\begin{aligned} x \xrightarrow{\quad UNIT_+ \quad} UNIT_+(x) &= (+1)x^0 \\ &= +1 \end{aligned}$$

and

$$\begin{aligned} x \xrightarrow{\quad UNIT_- \quad} UNIT_-(x) &= (-1)x^0 \\ &= -1 \end{aligned}$$

namely the two "exceptional power functions" whose input-output rule involve *zero* copy of the input.

b. To take advantage of this similarity, we need to use the *dilation* of a function.

For instance, the $(+4)$ *dilation* of the unit function $UNIT_+$ is the function whose name is $(+4)UNIT_+$ and whose *input-output rule* is

$$\begin{aligned} x \xrightarrow{\quad (+4)\,UNIT_+ \quad} (+4)UNIT_+(x) &= (+4)\cdot(+1)x^0 \\ &= (+4)x^0 \end{aligned}$$

c. Then we can look at a *constant function* as just a *dilation* of either one of the functions $UNIT_+$ or $UNIT_-$.

For instance, the *constant function* C_{+4} whose input-output rule is

$$\begin{aligned} x \xrightarrow{\quad C_{+4} \quad} C_{+4}(x) &= (+4)x^0 \\ &= +4 \end{aligned}$$

is the same as the $(+4)$ *dilation* of the function $UNIT_+$:

$$\begin{aligned} x \xrightarrow{\quad (+4)UNIT_+ \quad} (+4)UNIT_+(x) &= (+4)\cdot\Big[(+1)x^0\Big] \\ &= (+4)x^0 \\ &= +4 \end{aligned}$$

but observe that it is also the same as the (-4) *dilation* of the function $UNIT_-$:

$$\begin{aligned} x \xrightarrow{\quad (-4)UNIT_- \quad} (-4)UNIT_-(x) &= (-4)\cdot\Big[(-1)x^0\Big] \\ &= (+4)x^0 \\ &= +4 \end{aligned}$$

d. From this point of view, we could get the graph of a *constant* function C_a from the graph of either *unit* function by a *vertical stretch* with *stretch factor a*. However, this would require that we define *negative* vertical stretching as well as *positive* vertical stretching and it will be simpler just to define *plain* **vertical stretching**, with the **stretch factor** an *unsigned* number, as follows: vertical stretching stretch factor

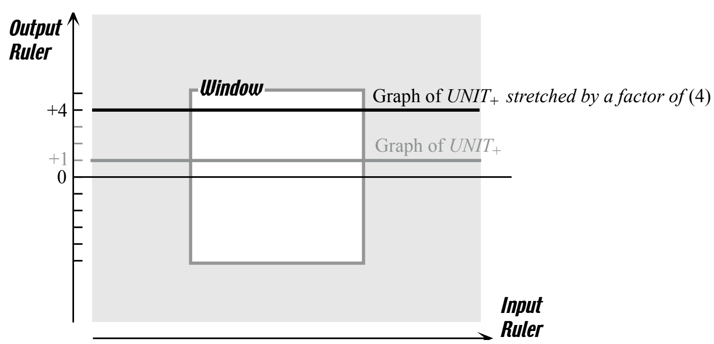
Then, to get the graph of C_a,

- when a is *positive*, look at the *stretched* graph of $UNIT_+$,
- when a is *negative*, look at the *stretched* graph of $UNIT_-$.

For instance, we get the *global quantitative graph* of the *constant* function C_{+4}, that is the function whose *input-output rule* is

$$x \xrightarrow{C_{+4}} C_{+4}(x) = (+4)x^0 = +4$$

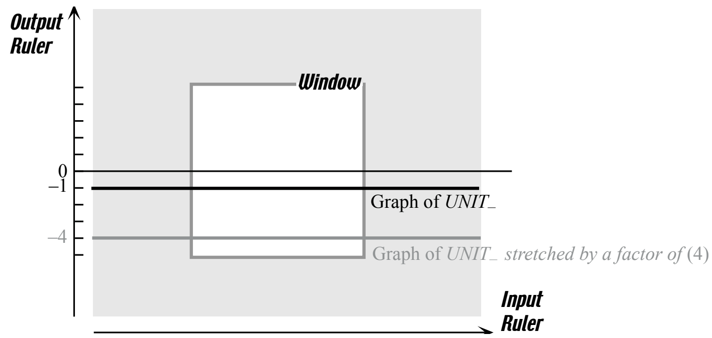
by a *vertical stretch* of the function $UNIT_+$ with stretch factor (4):



and we get the *global quantitative graph* of the *constant* function C_{-4}, that is the function whose *input-output rule* is

$$x \xrightarrow{C_{-4}} C_{-4}(x) = (-4)x^0 = -4$$

by a *vertical stretch* of the function $UNIT_-$ again with stretch factor (4):



The *advantage* of getting the graph this way is that it uses *dilation* and *stretching*, a tool that we will *have* to use with *all* other *polynomial functions*.

11.3.2 Operating On Constant Functions

1. Just like we looked upon constant functions as dilations of unit functions, we can dilate constant functions and this gives us constant functions. However, there is no reason here to limit ourselves to positive dilation factors.

a. When the dilation factor is *positive*, things are just the same as above.

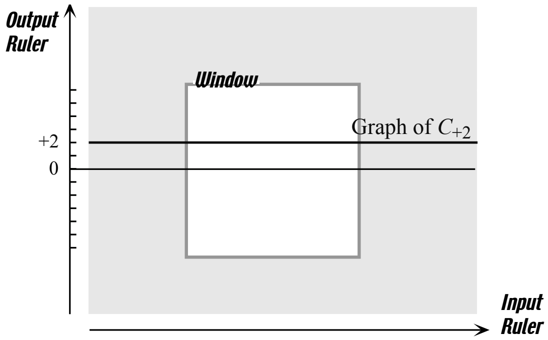
For instance, if we dilate the constant function C_{+2} by a factor of $(+3)$, we get the constant function $[(+2)C_{+2}]$:

$$\begin{aligned} x \xrightarrow{[(+3)C_{+2}]} [(+3)C_{+2}]\,(x) &= (+3)\cdot\Big[(+2)x^0\Big] \\ &= (+6)x^0 \\ &= +6 \end{aligned}$$

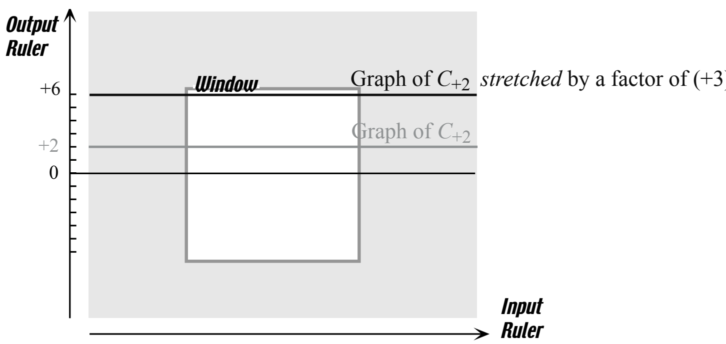
which is the same as the constant function C_{+6}

$$\begin{aligned} x \xrightarrow{C_{+6}} C_{+6}(x) &= (+6)x^0 \\ &= +6 \end{aligned}$$

From the graphic point of view, starting from the graphs of C_{+2}



we get the graph of $[(+2)C_{+2}]$ by a vertical (+3) stretch of the graph of C_{+2}



The resulting graph outputs +6 for *any* input and thus is the graph of C_{+6}.

b. When the dilation factor is *negative*, the graphic aspect is a bit more complicated.
For instance, if we dilate the constant function C_{+2} by a factor of (-3), we get the constant function $[(-3)C_{+2}]$:

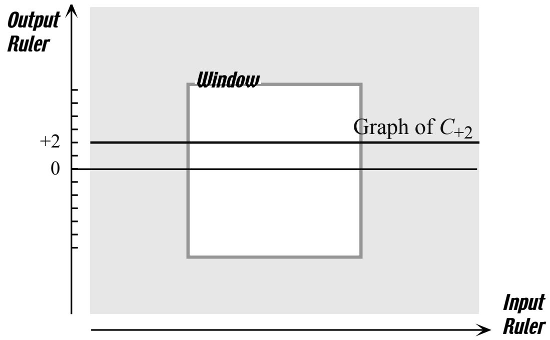
$$\begin{aligned} x \xrightarrow{[(-3)C_{+2}]} [(-3)C_{+2}]\,(x) &= (-3)\cdot\left[(+2)x^0\right] \\ &= (-6)x^0 \\ &= -6 \end{aligned}$$

which is the same as the constant function C_{-6}

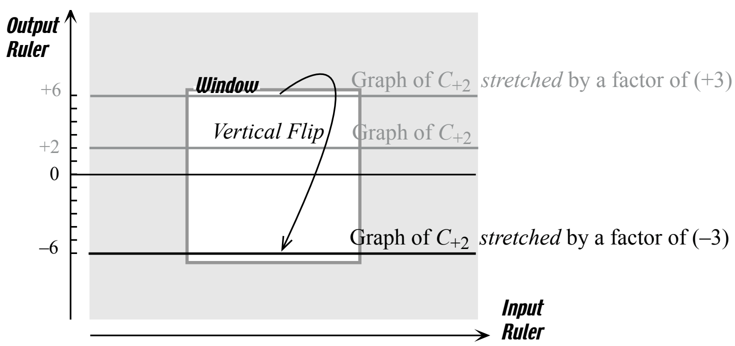
$$\begin{aligned} x \xrightarrow{C_{+6}} C_{+6}(x) &= (-6)x^0 \\ &= -6 \end{aligned}$$

From the graphic point of view, though, starting from the graphs of C_{+2}

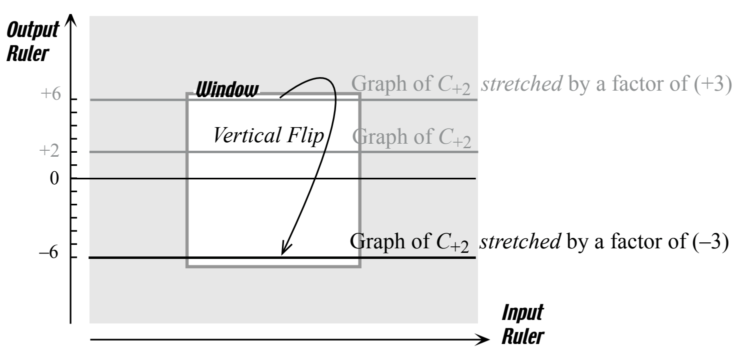
vertical flip



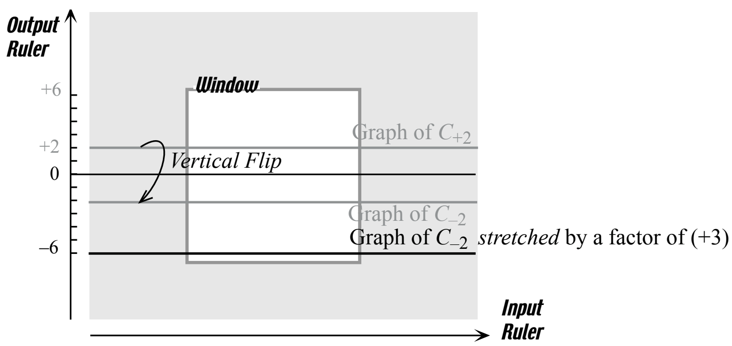
we get the graph of C_{+6} by a vertical (+3) stretch of the graph of C_{+2} followed by a **vertical flip**:



The resulting graph outputs +6 for *any* input and thus is the graph of C_{+6}.



Observe that we could have done the flip first and then a (+3) stretch:



2. We will need to add constant functions.
For example, given the two constant functions, C_{+2} and C_{+5},

$$x \xrightarrow{C_{+2}} C_{+2}(x) = +2$$

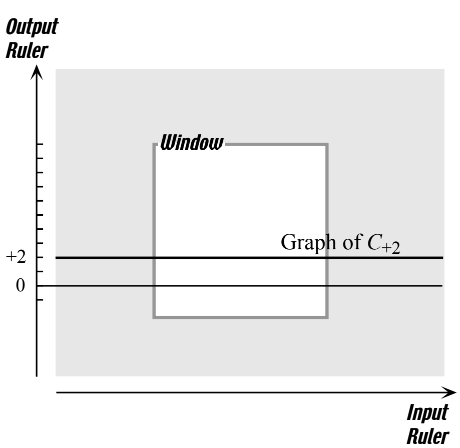
and

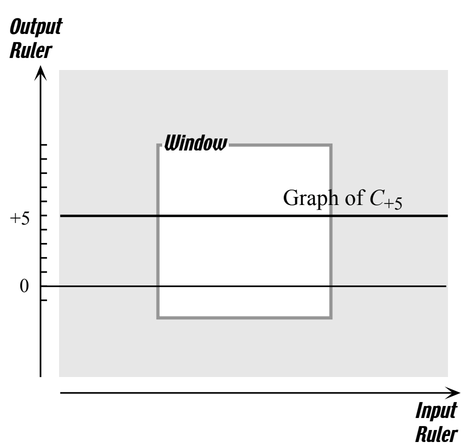
$$x \xrightarrow{C_{+5}} C_{+5}(x) = +5$$

we can add them as follows

$$\begin{aligned} x \xrightarrow{C_{+2} \oplus C_{+5}} \Big[C_{+2} \oplus C_{+5})\Big](x) &= C_{+2}(x) + C_{+5}(x) \\ &= [+2] + [+5] \\ &= +7 \end{aligned}$$

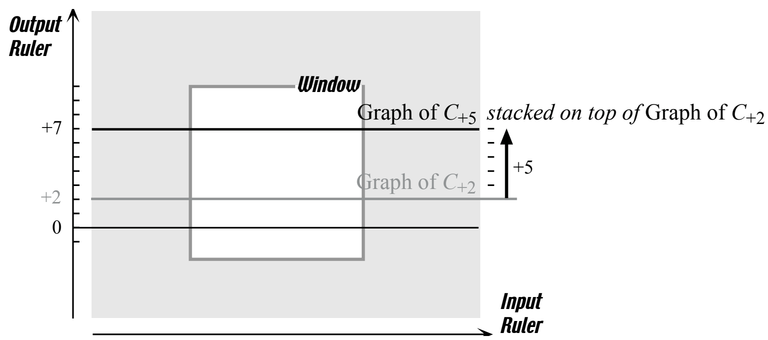
which is the constant function C_{+7}:
From the graphic point of view, starting from the graphs of C_{+2} and C_{+5}





stack

we get the graph of C_{+7} by **stacking** the graph of C_{+5} on top of the graph of C_{+2}



The resulting graph outputs +7 for *any* input and thus is the graph of C_{+7}.

11.4 Piecewise Constant Functions

11.5 Linear Functions

linear functions
linear coefficient

Functions whose *input-output rule* is of the form

$$x \xrightarrow{L_a} L_a(x) = ax^{+1}$$

where a stands for a given *signed* number, are called **linear functions**. L_a will be the *name* of the linear function *specified* by the signed number a. The given signed number a is called the **linear coefficient** of L_a.
For example, L_{-3} is the *linear function* whose input-output rule is

$$x \xrightarrow{L_{-3}} L_{-3}(x) = (-3)x^{+1}$$

that is the function whose *output* is equal to -3 multiplied by *one* copy of the input.

Note. Unless there is need for the above complete way of writing things, as there will be when, for instance, we have to deal with more than one linear function at a time, we shall usually write a simpler form such as

$$x \xrightarrow{L} L(x) = ax$$

In particular, the *exponent* $+1$ in the term ax^{+1} will often "go without saying".
For instance, instead of writing, as in the above example,

$$x \xrightarrow{L_{-3}} L_{-3}(x) = (-3)x^{+1}$$

we shall usually write

$$x \xrightarrow{L} L(x) = -3x$$

Then, when given an input, say $+4$, we write

$$\begin{aligned} +4 \xrightarrow{L} L(x)|_{\text{when x=+4}} &= -3x|_{\text{when x=+4}} \\ &= -3(+4) \\ &= -12 \end{aligned}$$

11.5.1 Graphing Linear Functions

As opposed to the case of *constant functions*, we cannot graph *linear functions* directly from the *input-output rule* and there is thus only one way to proceed.

1. First, we observe that *linear functions* are very similar to the *identity function*

$$\begin{aligned} x \xrightarrow{\quad ID \quad} ID(x) &= (+1)x^{+1} \\ &= +x \end{aligned}$$

and to the *opposite function*

$$\begin{aligned} x \xrightarrow{\quad OPP \quad} OPP(x) &= (-1)x^{+1} \\ &= -x \end{aligned}$$

namely the two "exceptional power functions" whose input-output rule involve *one* copy of the input. In fact, we can look at a *linear function* as just a *dilation* of either one of the functions ID or OPP.
For instance, the *linear function* L_{-3} whose input-output rule is

$$\begin{aligned} x \xrightarrow{\quad L_{-3} \quad} L_{-3}(x) &= (-3)x^{1} \\ &= -3x \end{aligned}$$

is the same as the function which is the (-3) *dilation* of the function ID:

$$\begin{aligned} x \xrightarrow{\quad (-3)ID \quad} (+3)ID(x) &= (-3) \cdot \Big[(+1)x^{0}\Big] \\ &= (-3)x^{1} \\ &= -3x \end{aligned}$$

but observe that it is also the same as the function which is the $(+3)$ *dilation* of the function OPP:

$$\begin{aligned} x \xrightarrow{\quad (+3)OPP \quad} (+3)OPP(x) &= (+3) \cdot \Big[(-1)x^{1}\Big] \\ &= (-3)x^{1} \\ &= -3x \end{aligned}$$

2. From this point of view, we could get the graph of a *linear* function L_a from either the graph of the function ID or the graph of the function OPP by a *vertical stretch* with *stretch factor* a. However, here as with *constant functions*, we shall use a *plain* vertical stretch with the *stretch factor* equal to SIZE(a), that is equal to the SIZE of the *dilation factor* and,

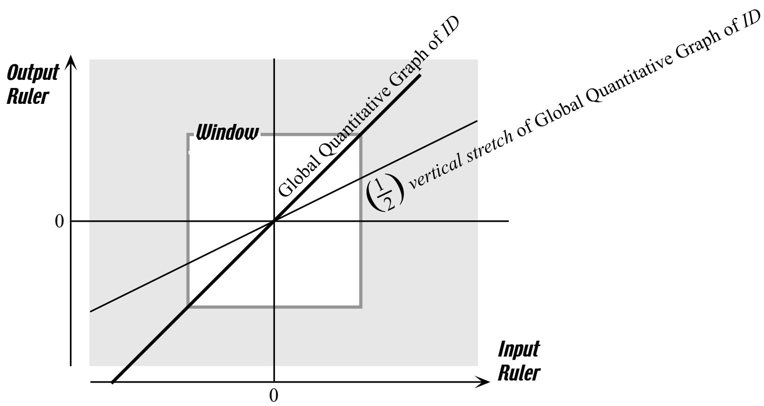
- when a is *positive*, stretch the function ID,
- when a is *negative*, stretch the function OPP.

For example,

a. We get the *global quantitative graph* of the *linear* function $L_{+\frac{1}{2}}$, that is the function whose *input-output rule* is

$$x \xrightarrow{L_{+\frac{1}{2}}} L_{+\frac{1}{2}}(x) = (+\frac{1}{2})x^1$$
$$= +\frac{1}{2}x$$

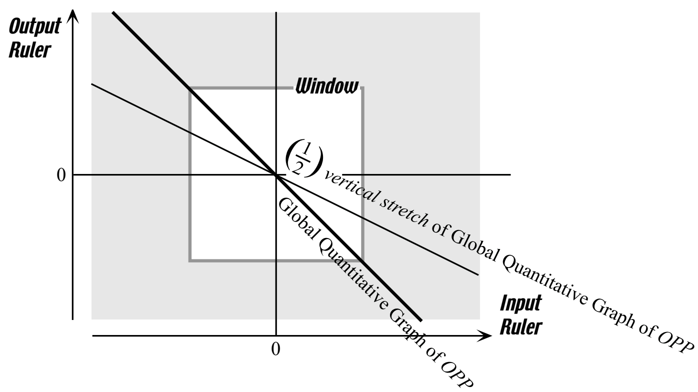
by a *vertical stretch* with stretch factor $(\frac{1}{2})$ of the function ID:



b. We get the *global quantitative graph* of the *linear* function $L_{-\frac{1}{2}}$, that is the function whose *input-output rule* is

$$x \xrightarrow{L_{-\frac{1}{2}}} L_{-\frac{1}{2}}(x) = (-\frac{1}{2})x^1$$
$$= -\frac{1}{2}x$$

by a *vertical stretch* again with stretch factor $(\frac{1}{2})$ but of the function OPP:



11.5.2 Operating On Linear Functions

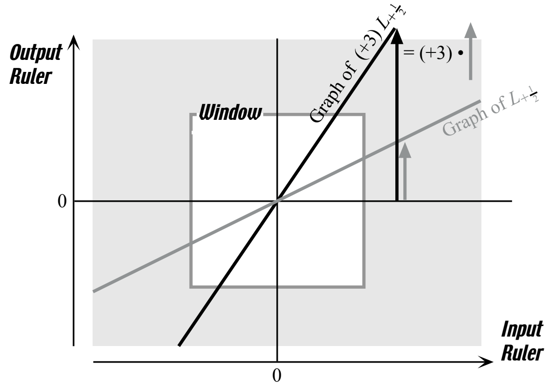
We can add and dilate *linear functions* just the way we dilated and added *constant functions* in Section 11.3. While essentially the same, stretching and stacking graphs will be just a little bit more difficult to see.

1. When we dilate by a *positive* factor,

For instance, given the linear function $L_{+\frac{1}{2}}$ we can dilate it by a factor of $(+3)$ as follows:

$$x \xrightarrow{\left[(+3)L_{+\frac{1}{2}}\right]} \left[(+3)L_{+\frac{1}{2}}\right](x) = (+3)\cdot\left(+\frac{1}{2}\right)x^1$$
$$= \left[(+3)\cdot\left(+\frac{1}{2}\right)\right]x^1$$
$$= \left(+\frac{3}{2}\right)x$$

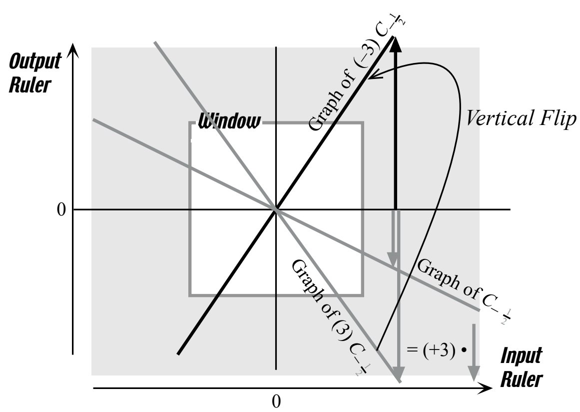
From the graphic point of view, we get the graph of $\left[(+3)L_{+\frac{1}{2}}\right]$ by a vertical $(+3)$ stretch of $L_{+\frac{1}{2}}$:



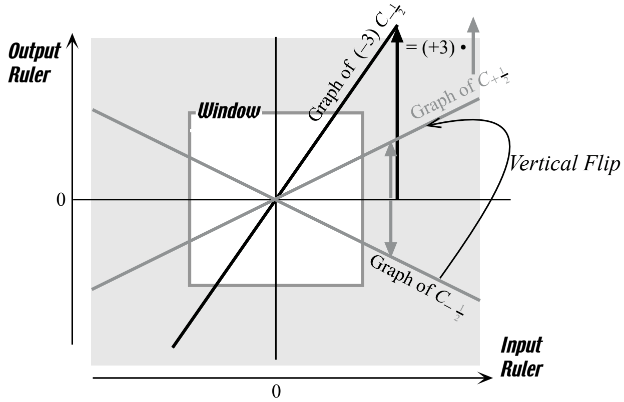
2. When we dilate by a *negative* factor, we must include a vertical flip. For instance, given the linear function $L_{-\frac{1}{2}}$ we can dilate it by a factor of (-3) as follows:

$$x \xrightarrow{\left[(-3)L_{-\frac{1}{2}}\right]} \left[(-3)L_{-\frac{1}{2}}\right](x) = (-3)\cdot\left(-\frac{1}{2}\right)x^1$$
$$= \left[(-3)\cdot\left(-\frac{1}{2}\right)\right]x^1$$
$$= \left(+\frac{3}{2}\right)x$$

From the graphic point of view, we get the graph of $\left[(-3)L_{-\frac{1}{2}}\right]$ by a vertical $(+3)$ stretch of $L_{-\frac{1}{2}}$ followed by a vertical flip:



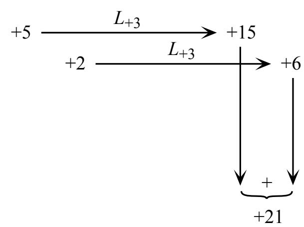
Observe that we could just as well have gotten the graph of $\left[(-3)L_{-\frac{1}{2}}\right]$ by a vertical flip of $L_{-\frac{1}{2}}$ followed by a (3) stretch:


Output
Ruler
Window
Graph of (−3) C−½
= (+3) •
Graph of C+½
Vertical Flip
0
Graph of C−½
Input
Ruler
0

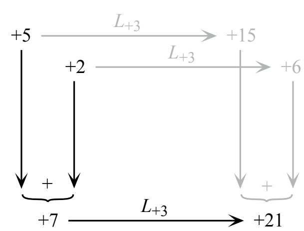
11.5.3 Linearity

Finally, we must mention an extremely important double feature which, among the functions in this text, *linear* functions will be the *only* ones to have.

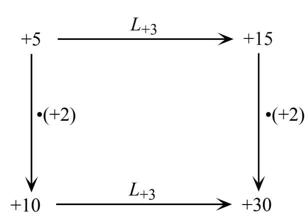
- Part of this feature is that if we input two numbers, say $+5$ and $+2$, into a linear function, say L_{+3}, and add the outputs:



  we would have gotten the same result by first adding the original numbers $+5$ and $+2$ and then inputting the result in the linear function:



- The other part of this feature is that, if we input a number, say $+5$, into a linear function, say L_{+3}, and multiply the output by some number, say $+2$, we would have gotten the same result by multiplying the original number $+5$ by the number $+2$ and inputting the result into the linear function L_{+3}:



The reader should check that *constant functions* do *not* have that feature.

linearity

Unsurprisingly, this double feature is called **linearity** but, surprisingly enough, there is a whole part of mathematics, called LINEAR MATHEMATICS, dealing with the far-reaching consequences of this apparently simple feature. In this text, though, we shall not make much use of it.

11.6 Affine Functions

affine functions
linear coefficient
constant term

Functions whose *input-output rule* is of the form

$$x \xrightarrow{A_{a,b}} A_{a,b}(x) = ax^{+1} + bx^0$$

where a and b stand for given *signed* numbers, are called **affine functions**. $A_{a,b}$ will be the name of the affine function *specified* by the two signed numbers a and b. The given signed number a is called the **linear coefficient** of $A_{a,b}$ and the given *signed* number b is called the **constant term** of $A_{a,b}$[1].
For instance, $A_{-3,+5}$ is the *affine function* whose input-output rule is

$$x \xrightarrow{A_{-3,+5}} A_{-3,+5}(x) = (-3)x^{+1} + (+5)x^0$$

that is of the function whose *output* is equal to -3 multiplied by *one* copy of x plus $+5$ multiplied by *zero* copy of x.

Note. Unless there is need for the above complete way of writing things, as there will be when, for instance, we have to deal with more than one affine function at a time, we shall write a simpler form such as

$$x \xrightarrow{A} A(x) = ax + b$$

In particular, the *exponent* $+1$ in the term ax^{+1} and the whole power x^0 in the term bx^0 will often "go without saying".
For instance, instead of writing, as in the above example,

$$x \xrightarrow{A_{-3,+5}} A_{-3,+5}(x) = (-3)x^{+1} + (+5)x^0$$

we shall usually write

$$x \xrightarrow{A} A(x) = -3x + 5$$

Then, when given an input, say $+4$, we write

$$\begin{aligned} +4 \xrightarrow{A} A(x)|_{\text{when x=+4}} &= -3x + 5|_{\text{when x=+4}} \\ &= -3(+4) + 5 \\ &= -7 \end{aligned}$$

[1]Educologists will rightfully observe that we are not being quite consistent: $ax^{+1} = ax$ is the *linear* term and $bx^0 = b$ is the *constant* term of which a and b are the *coefficients*.

linear combination

11.6.1 Graphing Affine Functions

In order to graph affine functions we need both operations.

1. First we observe that the *input-output rule* shows that *affine* functions can be looked upon as *combinations* of the two different types of "exceptional" power functions, that is of the power functions with *exponent* 0 and with *exponent* +1, namely:

- the two *unit* functions

$$x \xrightarrow{UNIT_+} UNIT_+(x) = +1$$

and

$$x \xrightarrow{UNIT_-} UNIT_-(x) = -1$$

- the *identity* function

$$x \xrightarrow{ID} ID(x) = +x$$

- and the *opposite* function

$$x \xrightarrow{OPP} OPP(x) = -x$$

For example, the *affine function*

$$x \xrightarrow{A} A(x) = -3x + 5$$

can be looked upon as a *combination* of a *dilation* of

$$x \xrightarrow{OPP} OPP(x) = -x$$

and a *dilation* of

$$x \xrightarrow{UNIT_+} UNIT_-(x) = +1$$

We shall call such a combination of dilations a **linear combination** and thus we shall say that affine functions are *linear combinations of the exceptional power functions.*

Observe, though, that there are four ways in which we can look at a *given* affine function as *linear combination* of *exceptional power functions.*

approximate linear function
band
median

2. So, just the way we introduced in Section 11.3 *stretching* as the graphical representation of *dilating* and *stacking* as the graphical representation of *adding*, we now need to introduce the graphical representation of *combination.*

As it happens, we can stack two functions of different type almost as easily as

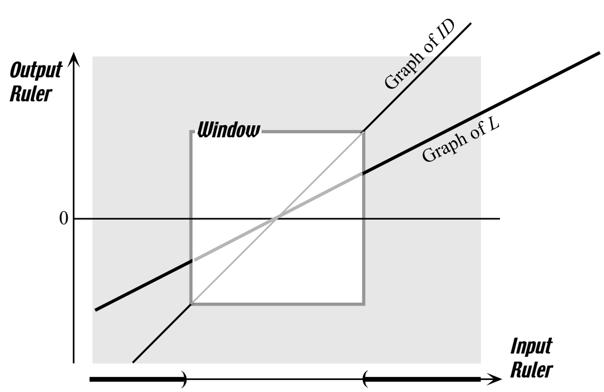
3. To get the graph of an *affine function*, we shall see it as a *linear combination of exceptional power functions*, graph these exceptional power functions individually, stretch them and stack the results.

11.6.2 Interpolating From A Local Graph Near Infinity

In what follows, we shall investigate the function $x \xrightarrow{A} A(x) = +\frac{1}{2}x + 17$.

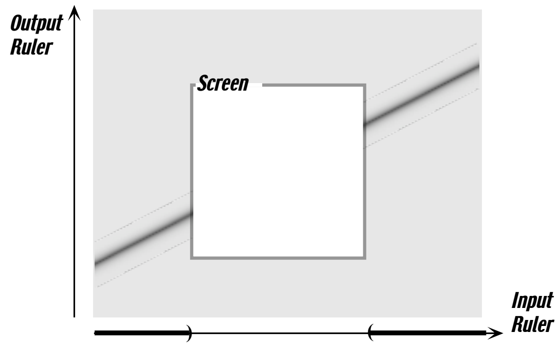
Our approach is based on the fact that, when x is near ∞, $+\frac{1}{2}x$ is larger in *size* than $+17$. So, we first look for the local graph of the *affine* function $x \xrightarrow{A} A(x) = +\frac{1}{2}x + 17$ when x is near ∞, that is for *large* inputs.

1. We begin by graphing the *linear function* $x \xrightarrow{L} L(x) = +\frac{1}{2}x$ as *dilation* of the *identity* function $x \xrightarrow{ID} ID(x) = +x$ with *dilation factor* equal to $\frac{1}{2}$.

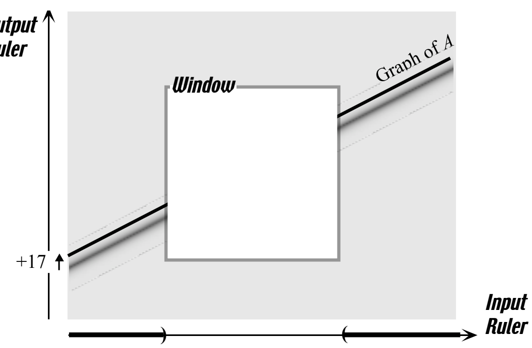


2. The graph of the **approximate linear function** $x \xrightarrow{L_{...}} L_{...}(x) = +\frac{1}{2}x + (...)$ is a **band** with the graph of L as **median**.

localize



3. Since, when x is large, $+17$ is small compared to x, the graph of $x \xrightarrow{A} A(x) = +\frac{1}{2}x + 17$ is somewhere in that band that acts therefore as a *local frame*.



In order to get a qualitative global graph, we must join the two pieces of the graph near ∞ in some way and the question is what information about the way we need to do so. For that, we shall need some *General Statements* about the graph inside the window and so we shall return to the issue after we have developed the latter in Section 11.6.4

In the meantime, we shall draw what we shall call the essential global graph, that is the simplest graph that fits the information we have.

11.6.3 Extrapolating A Local Graph Near A Finite Point

We now look at the output of an affine function for inputs that are restricted to being near a *given* input.

1. Given, for instance, the function $x \xrightarrow{f} A(x) = 3x + 17$, and given an input, say $x_0 = -5$, we **localize** the input-output rule at $x_0 = -5$ as

local input-output rule
fuzz

follows:

$$-5+h \xrightarrow{A} f_{-5}(h) = +3x+17|_{\text{when x=-5}}$$
$$= +3(-5+h)+17$$
$$= -15+3h+17$$

and, collecting terms of the same order of magnitude,

$$= +2+3h$$

We will call

$$-5+h \xrightarrow{A} f_{-5}(h) = +2+3h$$

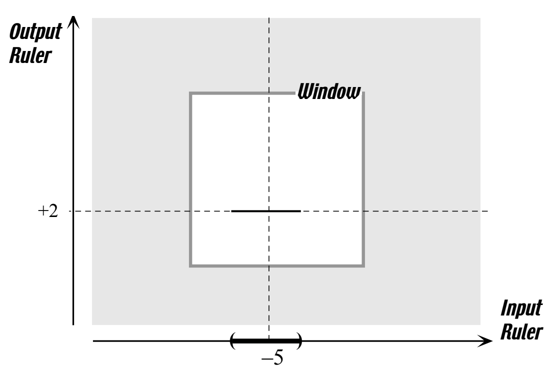
the **local input-output rule** when x is near x_0.
Note. The terms in the output of the local input-output rule are listed in order of *diminishing* order of magnitude.

2. We now look at the graphic representation of each of these terms by taking them into successive consideration:

a. We consider the effect of the term +2 in the output if it were alone:

$$-5+h \xrightarrow{A} +2$$

Graphically, this says that the output point for any input near −5 is *at* the point +2 on the output ruler so that the plot points are on the piece of horizontal straight line

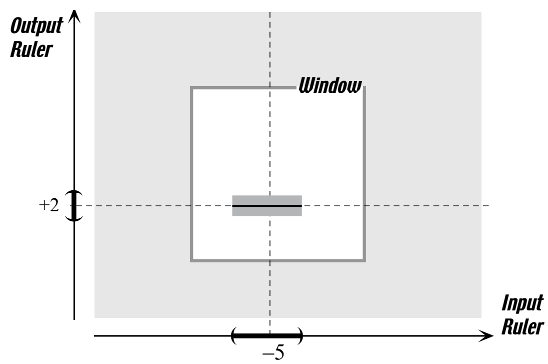


b. We consider the effect of adding some **fuzz** to +2 in the output:

$$-5+h \xrightarrow{A} +2+(...)$$

grey area
graphic fuzz
explicit

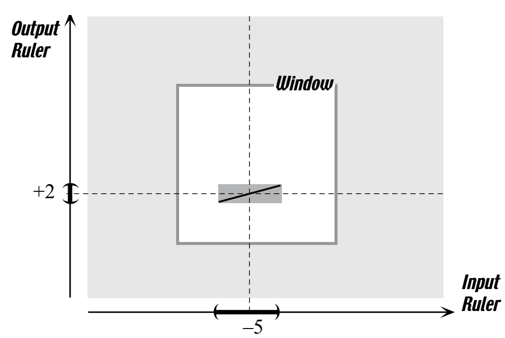
Graphically, this says that the *output points* for inputs that are near -5 will be *near* the point $+2$ on the output ruler, rather than exactly *at* the point $+2$, and that the *plot points* are now going to be, rather than exactly on the horizontal line, somewhere in the **grey area** which is the **graphic fuzz** that corresponds to the *fuzz term* (...) that we added in the output of the local input-output rule:



c. Finally, we **explicit** the order of magnitude of the fuzz by replacing (...) by $3h$:

$$-5 + h \xrightarrow{\ A\ } +2 + 3h$$

Graphically, this says that the plot points are not just anywhere in the grey area but on the piece of slanted line:



where the piece of slanted line is the graph of the power function

$$h \xrightarrow{\ p\ } p(h) = (+1)h^{+1}$$

translated near -5 *and added on top of* $+2$. because, since this is all qualitative, we ignore the size of the dilation factor, 3, but keep its sign, $+$.

11.6.4 General Statements

unspecified point

1. We localize

$$x \xrightarrow{\ A\ } A(x) = ax + b$$

near a yet **unspecified point** x_0:

$$\begin{aligned} x_0 + h \xrightarrow{\ A\ } f_{x_0}(h) &= ax + b|_{\text{when x}=x_0+h} \\ &= a(x_0 + h) + b \\ &= ax_0 + ah + b \end{aligned}$$

and, collecting terms of the same *order of magnitude*, we get

$$= [ax_0 + b] + [a]h$$

2. Observe that the *order of magnitude* of the term $[ax_0 + b]$ is 0 because $[ax_0 + b]$ is "of the order of $10^0 = 1$", which is to say that it is "in the ones", while the order of magnitude of the term $[a]h$ is -1 because $[a]h$ is "of the order of $10^{-1} = \frac{1}{10} = 0.1$" which is to say that it is "in the tenths". This is the reason why we write $[a]h$ in second position after $[ax_0 + b]$.

3. This gives us the following two *general statements*:

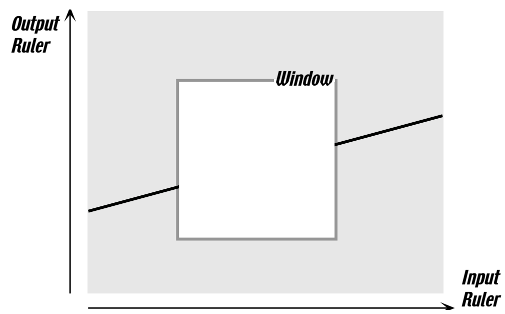
- *The* height *of the local graph, which is given by the coefficient of* h^0*, that is* $[ax_0 + b]$*,* depends *on* x_0 *and therefore will be* different *"everywhere", that is for all inputs.*
- *The* slope *of the local graph, which is given by the coefficient of* h^{+1}*, that is* $[a]$*, is* independent *of* x_0 *and therefore will be the* same *everywhere, that is for all inputs.*

11.6.5 Qualitative Global Graph

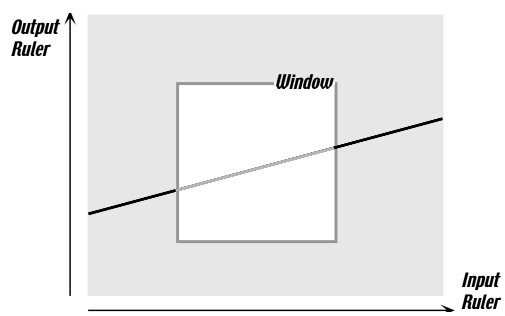
On the basis of the above *general statements*, we can now obtain a qualitative global graph in either one of two ways:

1. We can start from the local graph near infinity of Section 11.6.2:

finite graph

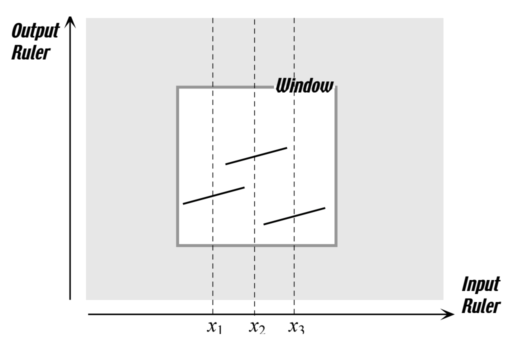


the two pieces of which we join with a *straight* line, the **finite graph**.

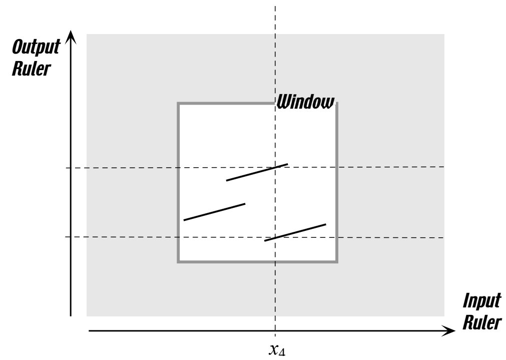


2. We can start from local graphs near finite points as follows:

a. We construct local graphs near, say, three different points, x_4, x_2, x_3. On the above basis, they would look something like this:

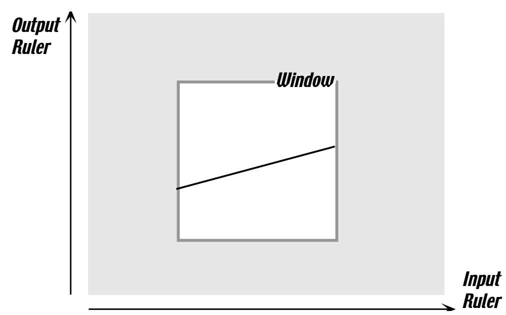


b. However, this is not possible because that would mean that inputs such as x_4 would have *two* outputs:



whereas the *input-output rule* will pair the input x_4 with *only one* output, namely $+3x_4 + 17$.

c. As a result, the *local* graphs *must* all line up and so the *finite* graph *must* be a straight line:



3. Of course, the finite graph must line up with the local graph near ∞ and, either way we get the *general statement* :

The global graph of an affine function is a straight line.

11.6.6 Global Problems

- Where is the output equal to/smaller than/larger than a given number? *algebraically*/in SIZE.
- Where is the output smallest/largest)? *Algebraically*/In SIZE? Absolutely/In a given interval?
- BOUNDARY VALUE PROBLEM: What is the *input-output rule* of an affine function given its output for two different inputs.
- INITIAL VALUE PROBLEM: What is the *input-output rule* of an affine function given its *slope* and given its output for a given input.

11.7 Piecewise Affine Functions

11.8 Quadratic Functions

11.9 Cubic Functions

11.10 Quartic Functions And Above

Transition inputs

Chapter 12

Rational Functions

Transition inputs (inputs that are just outside the window. needed for the cases where slope and/or concavity are lost and need to be recovered by division.)

Important for image interpretation because they are the first functions to have steep gradients near finite inputs. See Meyer?

12.1 Local Investigations

12.1.1 Local investigation of the rational function

$$x \xrightarrow{f} f(x) = \frac{x}{x^2 - 1}$$

when x is near 0.

$$x \text{ near } 0 \xrightarrow{f} f(x)|_{\text{when } x=0+h=h} = \left.\frac{x}{x^2-1}\right|_{\text{when } x=h} = \frac{h}{h^2-1}$$

which we must rewrite as

$$= \frac{h}{-1+h^2}$$

because h^2 is smaller in size than -1

At this point we have a choice:

- We can compute blindly
- We can think ahead about what is going to happen and take advantage of it so as to do as little work as possible.

We look at each choice separately:

- When we compute blindly, we must divide:

$$-1 + 0h + h^2 \;\overline{\big)\qquad +h \qquad}$$

Observe that we wrote in $0h$ to make sure things will remain lined up later on.

1. We divide the first into the first:

$$\begin{array}{r} -h \\ -1 + 0h + h^2 \;\overline{\big)\qquad +h \qquad} \end{array}$$

This gives us

$$x \text{ near } 0 \xrightarrow{f} f(x)|_{\text{when } x=h} = \frac{h}{-1+h^2} = -h + (...)$$

but, since $-h$ is one of the *exceptional* power functions, it does not have all three features:

  – If we want the sign of the *output* or the sign of the *slope*, the power function $-h$ will give it to us.
  – If we want the sign of the *concavity*, it is inside (...) and we must push the division to dig it out.

2. We multiply

$$\begin{array}{r|llll} & -h & & \\ \hline -1+0h+h^2 \;) & +h & & \\ & +h & +0h^2 & -h^3 \end{array}$$

3. We subtract

$$\begin{array}{r|llll} & -h & & \\ \hline -1+0h+h^2 \;) & +h & & \\ & +h & +0h^2 & -h^3 \\ \hline & & & -h^3 \end{array}$$

4. We divide the first into the first

$$\begin{array}{r|llll} & -h & & +h^3 \\ \hline -1+0h+h^2 \;) & +h & & \\ & +h & +0h^2 & -h^3 \\ \hline & & & -h^3 \end{array}$$

This gives us

$$x \text{ near } 0 \xrightarrow{\;f\;} f(x)|_{\text{when } x=h} = \frac{h}{-1+h^2}$$
$$= -h + h^3 + (...)$$

and the power function $+h^3$ gives us the sign of the *concavity*.

- When we think ahead, we see that dividing $-1+(...)$ (instead of $-1+h^2$) into $+h$ will give us $-h+(...)$ which will give us the sign of the *output* and the sign of the *slope* but not the sign of the *concavity*.
  If we need the sign of the *concavity* then we must divide as above.

12.1.2 Local investigation of the rational function

$$x \xrightarrow{\ f\ } f(x) = \frac{x^3 - 8}{x - 1}$$

when x is near $+2$.

$$x \text{ near } +2 \xrightarrow{\ f\ } f(x)|_{\text{when } x=+2+h} = \left.\frac{x^3-8}{x-1}\right|_{\text{when } x=+2+h}$$
$$= \frac{(+2+h)^3 - 8}{+2+h-1}$$
$$= \frac{(+2+h)^3 - 8}{+1+h}$$

At this point we have a choice:

- We can compute blindly
- We can think ahead about what is going to happen and take advantage of it so as to do as little work as possible.

In either case, though, we need to know the addition formula:

$$(x_0 + h)^3 = x_0^3 + 3x_0^2 h + 3x_0 h^2 + h^3$$

- When we compute blindly, we write:

$$\frac{(+2+h)^3 - 8}{+1+h} = \frac{+8 + 3(+2)^2 h + 3(+2)h^2 + h^3 - 8}{+1+h}$$
$$= \frac{+3(+2)^2 h + 3(+2)h^2 + h^3}{+1+h}$$
$$= \frac{+12h + 6h^2 + h^3}{+1+h}$$

and we must divide:

$$+1+h \;\overline{\big)\quad +12h \quad +6h^2 \quad +h^3}$$

1. We divide the first into the first:

$$\begin{array}{r} +12h \\ +1+h \;\overline{\big)\quad +12h \quad +6h^2 \quad +h^3} \end{array}$$

This gives us

$$x \text{ near } +2 \xrightarrow{\ f\ } f(x)|_{\text{when } x=+2+h} = \frac{+12h + 6h^2 + h^3}{+1+h}$$
$$= +12h + (...)$$

but, since this is one of the *exceptional* power functions, it does not have all three features:

- If we want the sign of the *output* or the sign of the *slope*, the power function $+12h$ will give it to us.
- If we want the sign of the *concavity*, it is inside $(...)$ and we must push the division to dig it out.

2. We multiply

$$\begin{array}{r|llll} & +12h & & \\ \hline +1+h & +12h & +6h^2 & +h^3 \\ & +12h & +12h^2 & \end{array}$$

3. We subtract

$$\begin{array}{r|llll} & +12h & & \\ \hline +1+h & +12h & +6h^2 & +h^3 \\ & +12h & +12h^2 & \\ \hline & & -6h^2 & +h^3 \end{array}$$

4. We divide the first into the first

$$\begin{array}{r|llll} & +12h & -6h^2 & \\ \hline +1+h & +12h & +6h^2 & +h^3 \\ & +12h & +12h^2 & \\ \hline & & -6h^2 & +h^3 \end{array}$$

This gives us

$$x \text{ near } +2 \xrightarrow{\ f\ } f(x)|_{\text{when } x=+2+h} = \frac{+12h + 6h^2 + h^3}{+1+h} = +12h - 6h^2 + (...)$$

and the power function $-6h^2$ gives us the sign of the *concavity*.

- When we think ahead about what $(+2+h)^3$ is going to be equal to, we see that it is going to start with $(+2)^3 = +8$ so that, because of the -8 at the end, the numerator will really start with the next term in the *expansion* of $(+2+h)^3$, that is with $+12h$ so that we will have:

$$\frac{(+2+h)^3 - 8}{+1+h} = \frac{+12h + (...)}{+1 + (...)} = +12h + (...)$$

Here again, since this is one of the *exceptional* power functions, it does not have all three features:

- If we want the sign of the *output* or the sign of the *slope*, the power function $+12h$ will give it to us.

– If we want the sign of the *concavity*, it is inside (...) and we must push the division to dig it out. Since the next term in the expansion of $(+2+h)^3$ is $+6h^2$, we write

$$\frac{(+2+h)^3-8}{+1+h} = \frac{+12h+6h^2+(...)}{+1+h}$$

where we included the $+h$ to balance our including the $+6h^2$. Note that there is no (...) after that because the $+h$ is all there was in (...). So, we must divide

$$+1+h \,\big)\overline{\quad +12h \quad +6h^2 \quad + (...)}$$

instead of

$$+1+h \,\big)\overline{\quad +12h \quad +6h^2 \quad +h^3}$$

as we did above.

So, in this case, the economy is not a big one.

12.1.3 Local investigation of the rational function

$$x \xrightarrow{f} f(x) = \frac{x-1}{x^3-8}$$

when x is near $+2$.

$$x \text{ near } +2 \xrightarrow{f} f(x)|_{\text{when } x=+2+h} = \left.\frac{x-1}{x^3-8}\right|_{\text{when } x=+2+h}$$
$$= \frac{+2+h-1}{(+2+h)^3-8}$$
$$= \frac{+1+h}{(+2+h)^3-8}$$

At this point we have a choice:

- We can compute blindly
- We can think ahead about what is going to happen and take advantage of it so as to do as little work as possible.

In either case, though, we need to know the addition formula:

$$(x_0+h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3$$

- When we compute blindly, we write:

$$\frac{+1+h}{(+2+h)^3-8} = \frac{+1+h}{+8+3(+2)^2h+3(+2)h^2+h^3-8}$$
$$= \frac{+1+h}{+3(+2)^2h+3(+2)h^2+h^3}$$
$$= \frac{+1+h}{+12h+6h^2+h^3}$$

and we must divide:

$$+12h + 6h^2 + h^3 \;\big)\; +1 \qquad +h$$

1. We divide the first into the first:

$$\begin{array}{r} +\frac{1}{12}h^{-1} \\ +12h + 6h^2 + h^3 \;\big)\; +1 \qquad +h \end{array}$$

2. We don't need to go any further since the first term we got, $+\frac{1}{12}h^{-1}$ has *slope*.

- When we think ahead about what $(+2+h)^3$ is going to be equal to, we see that it is going to start with $(+2)^3 = +8$ so that, because of the -8 at the end, the denominator will really start with the next term in the *expansion* of $(+2+h)^3$, that is with $+12h$ so that we will have:

$$\frac{+1+h}{(+2+h)^3-8} = \frac{+1 + (...)}{+12h + (...)}$$
$$= \tfrac{1}{12}h^{-1} + (...)$$

which should be compared with the division above.

So we have

$$x \text{ near } +2 \xrightarrow{\quad f \quad} f(x)|_{\text{when } x=+2+h} = \frac{+1+h}{(+2+h)^3-8}$$
$$= \frac{1}{12}h^{-1} + (...)$$

and, since this is a negative power function, it has all three features.

12.2 Global Investigations

12.2.1 Investigate the rational function Rat_2 whose input-output rule is

$$x \xrightarrow{Rat_2} Rat_2(x) = \frac{x-1}{x^3}$$

1. We look at inputs that are near ∞, that is at *large* inputs (most of them are since the screen is only a very, very small part of the universe).

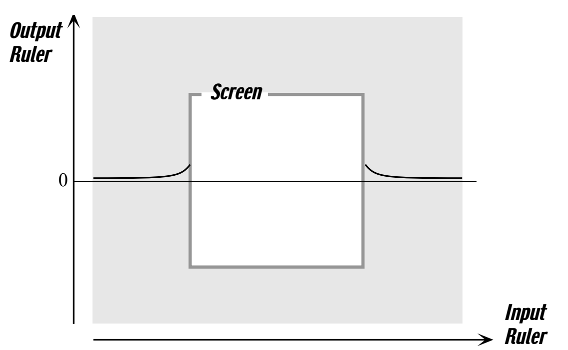
a. From the *input-output rule* point of view, we compute:

$$\begin{aligned} x \text{ near } \infty \xrightarrow{\quad Rat_2 \quad} Rat_2(x)|_{\text{when } x \text{ is near } \infty} &= \left.\frac{x-1}{x^3}\right|_{\text{when } x \text{ is near } \infty} \\ &= \frac{x + (...)}{x^3} \\ &= \frac{1}{x^2} + (...) \\ &= (+1) \cdot x^{-2} + (...) \end{aligned}$$

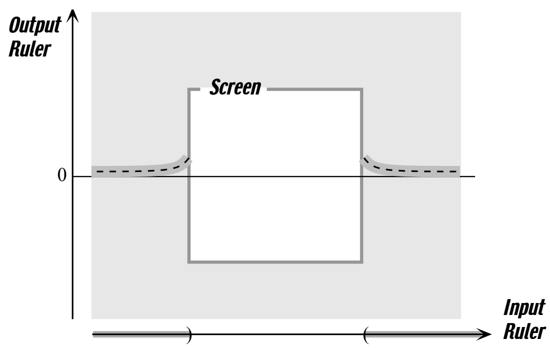
In other words, when x is near ∞, the output of the rational function Rat_2 is approximately the same as the output of the power function whose input-output rule is

$$x \xrightarrow{Power} Power(x) = (+1) \cdot x^{-2}$$

b. From the *graphic* point of view, the local graph of the power function when x is near ∞ is

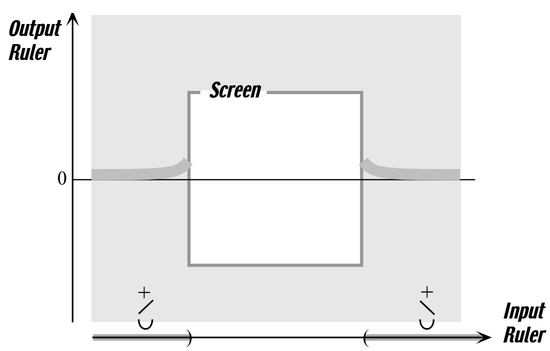


Then, the fact that $Rat(x)|_{\text{when } x \text{ is near } \infty} = (+1) \cdot x^{-2} + (...)$ means that, when x is near ∞, the local graph of Rat_2 is in a strip whose median is the graph of the power function:



(The width of the strip corresponds to the (...) in the input-output rule.)

c. Looking at the *features* of the graph when x is near ∞



we see that:

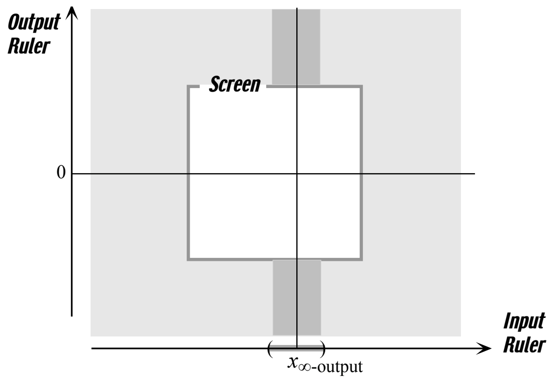
- the output does not switch sign.
- the slope switches sign,
- the concavity does not switch sign,

2. We now ask if, when x is *finite*, the output $Rat_2(x)$ must also be *finite*.

a. From the *graphic* point of view, we are asking if the graph of Rat_2 for finite input is completely within the screen.

Another way to ask the same question is the following:

Is there a *finite* input, call it $x_{\infty\text{-output}}$, such that the local frame is



(if the answer to the one is "yes", the answer to the other is "no" and if the answer to the one is "no", then the answer to the other is "yes".)

b. From the input-output rule point of view, we are asking if there is a *finite* input, call it $x_{\infty\text{-output}}$, whose output is ∞? But since we do not know how to compute with ∞ itself, we ask:
In ENGLISH: Is there a *finite* input, call it $x_{\infty\text{-output}}$, such that when x is near $x_{\infty\text{-output}}$, then $Rat_2(x)$ is near ∞?
In ALGEBRA: Is there a *finite* input, call it $x_{\infty\text{-output}}$, such that when $x = x_{\infty\text{-output}} + h$, $Rat_2(x)$ is *large*?
In other words, we want

$$Rat_2(x)|_{\text{when } x=x_{\infty\text{-output}}+h}$$

to be large
i. So, first we localize near a temporarily unspecified x_0

$$x = x_0 + h \xrightarrow{Rat_2} Rat_2(x)|_{\text{when } x=x_0+h} = \left.\frac{x-1}{x^3}\right|_{\text{when } x=x_0+h}$$
$$= \frac{[x_0+h]-1}{[x_0+h]^3}$$
$$= \frac{[x_0-1]+h}{x_0^3+3x_0^2h+3x_0h^2+h^3}$$
$$= \frac{[x_0-1]+(...)}{x_0^3+(...)}$$

Note. Observe that since

$$Rat_2(x)|_{\text{when } x=x_0} = \frac{x_0-1}{x_0^3}$$

the last line of the previous computation could have been obtained easily.

ii. Now since the output $Rat_2(x)|_{\text{when } x=x_0}$ is a fraction, there are two ways that it could be large:

- if its numerator, $[x_0-1]+(...)$, were large. But $(...)$ is small and $[x_0-1]$ cannot be large since x_0 is *finite*. So the numerator cannot be large.
- if its denominator, $x_0^3+(...)$, were small. But, while $(...)$ is indeed small, x_0^3 is not. So for the denominator to be small, it must reduce to $(...)$ and so x_0^3 must be equal to 0, that is $x_0 = 0$

Altogether then, the only neighborhood in which the output can be large is when x is near 0

3. We investigate the behavior of the function Rat_2 when x is near $x_{\infty\text{-output}}$, that is when x is near 0.

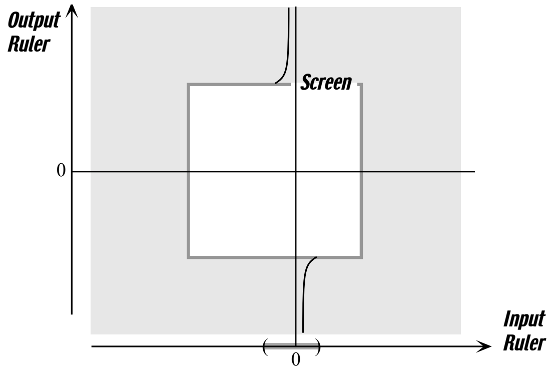
a. From the *input-output rule* point of view, we compute

$$
\begin{aligned}
x = 0 + h = h \xrightarrow{Rat_2} Rat_2(x)|_{\text{when } x=h} &= \left.\frac{x-1}{x^3}\right|_{\text{when } x=h} \\
&= \frac{-1+h}{h^3} \\
&= \frac{-1+(...)}{h^3} \\
&= \frac{-1}{h^3}+(...) \\
&= -h^{-3}+(...)
\end{aligned}
$$

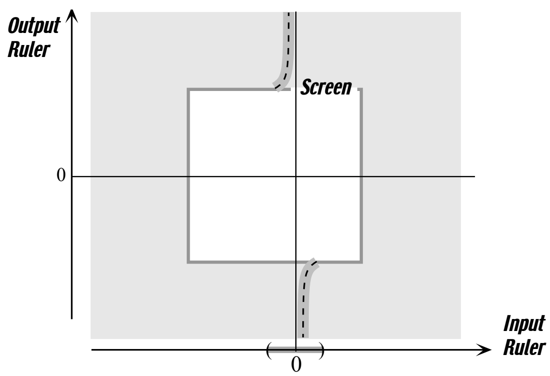
In other words, when x is near 0, the output of the rational function Rat_2 is approximately the same as the output of the power function whose input-output rule is

$$x \xrightarrow{Power} Power(x) = (-1) \cdot x^{-3}$$

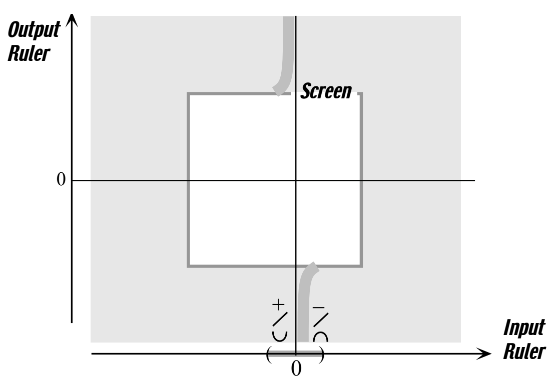
b. From the *graphic* point of view, the local graph near 0 of the power function is



Then, the fact that $Rat_2(x)|_{\text{when } x=h} = -h^{-3} + (...)$ means that, when x is near 0, the local graph of Rat_2 is in a strip whose median is the graph of the power function



c. Looking at the features of the graph when x is near 0

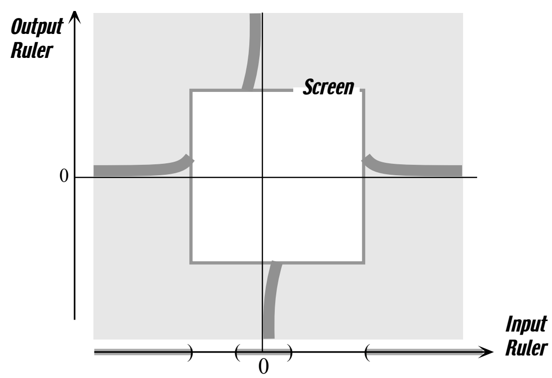


we see that:

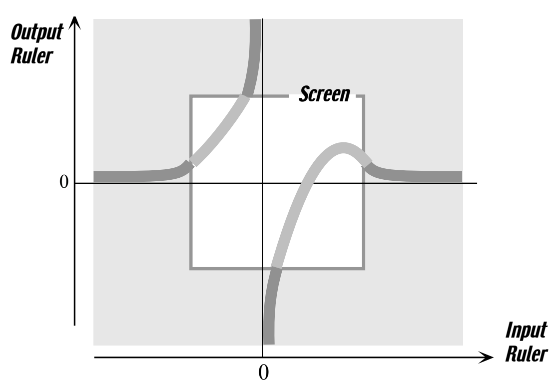
essential graph

- the output switches sign.
- the slope does not switch sign,
- the concavity switches sign,

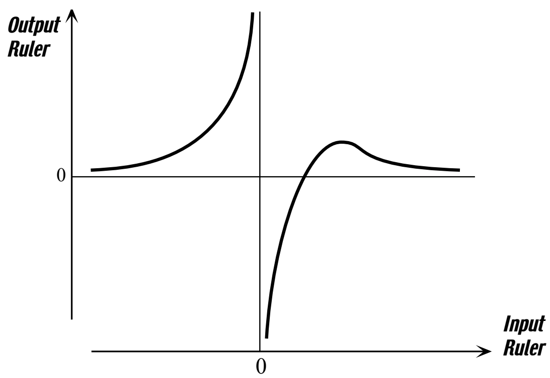
4. Altogether, we have that the graph outside the screen is in the shaded strips



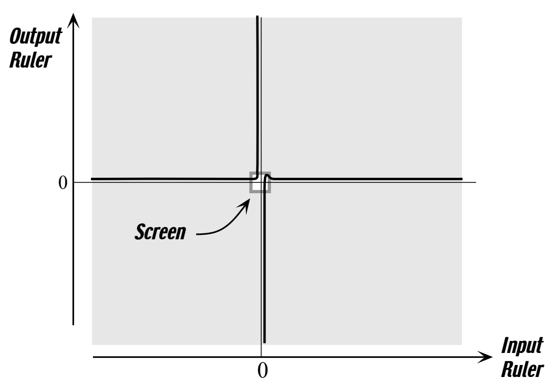
When we join smoothly these local graphs, we obtain the **essential graph**, that is the simplest possible qualitative global graph for the function Rat_2:



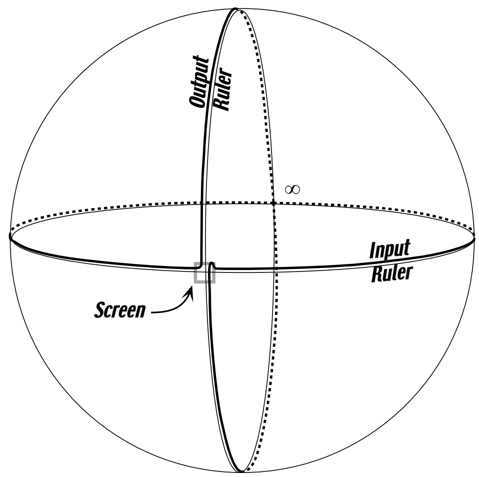
In most textbooks, this would be simply called "the" graph and shown as



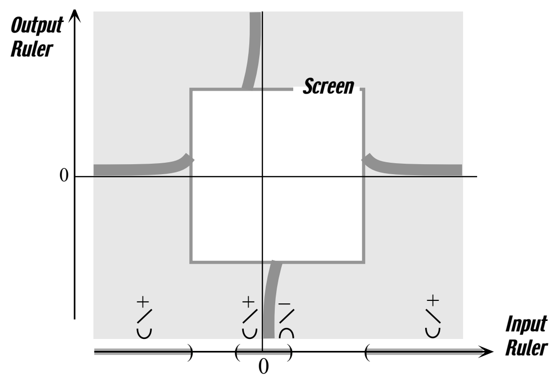
which is quite misleading since, as an anthropocentric concession, even the screen above was already disproportionately large and a graph more evocative of the size of the screen relative to the universe, would be:



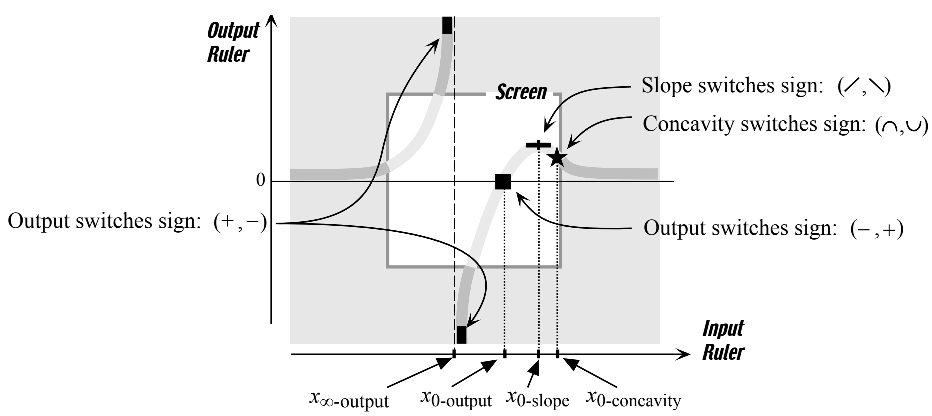
In fact, an even more realistic picture would be:



5. Looking at the features that we already found



we can get a certain amount of additional, mostly *qualitative*, information:



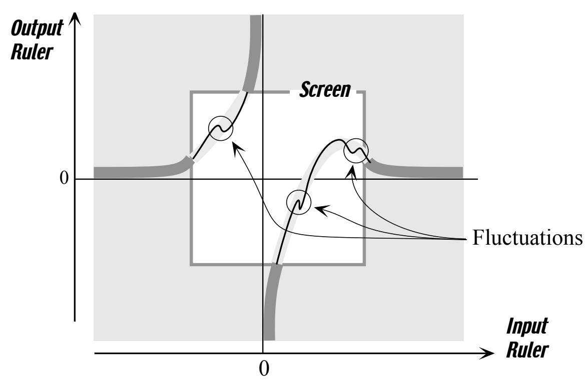
essential features
fluctuations
locate
input-slope rule
input-concavity rule

In other words, we can see that there has to be *at least*:

- one *finite* input near which the concavity switches sign, namely the 0-concavity input $x_{\text{0-concavity}}$
- one *finite* input near which the slope switches sign, namely the 0-slope input $x_{\text{0-slope}}$
- two *finite* inputs near which the output switches sign, namely the 0-output input $x_{\text{0-output}}$ and the ∞-output input x_{∞}.

We said "at least" because these features, which we shall call the **essential features** of the function Rat_2, are those that are forced by what happens *beyond* the screen just like, for instance, tides on earth are caused by the moon.

But, on the basis of our investigation, we cannot tell if the actual graph might not be, for instance,



In other words, the graph may have **fluctuations**, that is min-max pairs. But these would not be caused by what happens beyond the screen and they would just be like, for instance, waves on the sea.

6. We would now like to **locate** the input intervals, if any, for which one of the three features has a *given* sign. This is quite independent from most of the previous investigation since this will involve only the input-output rule and solving an *inequation* which, as usual (See Section 1.3), means: **(i)** looking for those inputs where the output is either 0 or ∞ because, for a rational function, these are the only inputs where the output *could* switch sign. **(ii)** testing each of the intervals to find the inputs where the output actually *does* switch sign.

Unfortunately, here we will only be able to locate the inputs whose *output* has a given sign because we are not in a position to *locate* $x_{\text{0-slope}}$ and $x_{\text{0-concavity}}$ since we do not have an **input-slope rule** and an **input-concavity rule**. (These will be derived in Differential Calculus.)

For example, in what follows, we will want to locate the input intervals, if any, for which the output is *positive*.

a. We look for the inputs where the output *could* switch sign, that is the inputs where the output is either 0 or ∞.
i. We already located $x_{\infty\text{-output}}$, namely we found above that $x_{\infty\text{-output}} = 0$.
ii. We locate $x_{0\text{-output}}$ exactly in the same manner: As already computed above,

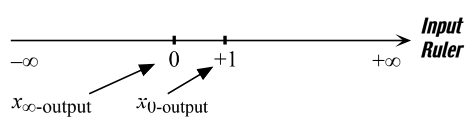
$$x = x_0 + h \xrightarrow{Rat_2} Rat_2(x)|_{\text{when } x=x_0+h} = \frac{[x_0 - 1] + (...)}{x_0^3 + (...)}$$

Now, since the output $Rat_2(x)|_{\text{when } x=x_0}$ is a fraction, there are two ways that it could be small:

- if its numerator, $[x_0 - 1] + (...)$, were small.
  But, while $(...)$ is indeed small, $[x_0 - 1]$ is not. So, for the numerator to be small, it must reduce to $(...)$ and so $[x_0 - 1]$ must be equal to 0, that is $x_0 = +1$
- if its denominator, x_0^3, were large.
  But x_0^3 cannot be large since x_0 is *finite*. So the denominator cannot be large.

Altogether, the only possibility is $x_{0\text{-output}} = +1$

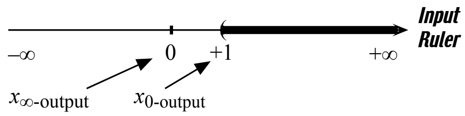
b. We now test each one of the intervals determined by $x_{\infty\text{-output}} = 0$ and $x_{0\text{-output}} = +1$



i. To test the interval from $+1$ to $+\infty$, we can use any number larger than $+1$. We use a *large* positive number:

$$\begin{aligned} +\text{large} \xrightarrow{Rat_2} Rat_2(x)|_{\text{when } x=+\text{large}} &= \left.\frac{x-1}{x^3}\right|_{\text{when } x=+\text{large}} \\ &= \frac{+\text{large} + (...)}{(+\text{large})^3} \\ &= \frac{+1}{(+\text{large})^2} + (...) \\ &= +\text{small} + (...) \end{aligned}$$

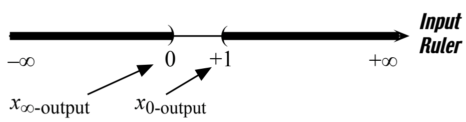
So the output will be positive in the whole interval from $+1$ to $+\infty$. Since this *is* what is being required, we *highlight* the interval:



ii. To test the interval from $-\infty$ to 0, we can use any number smaller than 0. We use a *large* negative number:

$$-\text{large} \xrightarrow{Rat_2} Rat_2(x)|_{\text{when } x=-\text{large}} = \left.\frac{x-1}{x^3}\right|_{\text{when } x=-\text{large}}$$
$$= \frac{-\text{large} + (...)}{(-\text{large})^3}$$
$$= \frac{+1}{(-\text{large})^2} + (...)$$
$$= +\text{small} + (...)$$

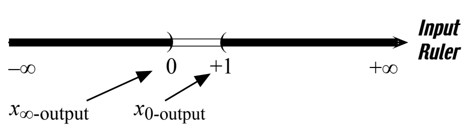
So the output will be positive in the whole interval from +1 to $+\infty$. Since this *is* what is being required, we *highlight* the interval:



iii. To test the interval from 0 to +1, we can use any number between 0 and +1. Say we use +0.5:

$$+0.5 \xrightarrow{Rat_2} Rat_2(x)|_{\text{when } x=+0.5} = \left.\frac{x-1}{x^3}\right|_{\text{when } x=+0.5}$$
$$= \frac{+0.5-1}{(+0.5)^3}$$
$$= \frac{-0.5}{+0.125}$$
$$= -0.25$$

So the output will be negative in the whole interval from 0 to +1. Since this is *not* what is being required, we *blank-out* the interval:



Chapter 13

Introduction to Initial Value Problems

text

Chapter 14

Exponential Functions

text

Chapter 15

Logarithm Functions

text

Chapter 16

Circular Functions

Chapter 17

Inverse Circular Functions

Chapter 18

Hyperbolic Functions

text

Chapter 19

Inverse Hyperbolic Functions

text

Index