Making Sense of Mathematics:

A Faster and Surer Pathway for Developmental Students?

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ABSTRACT

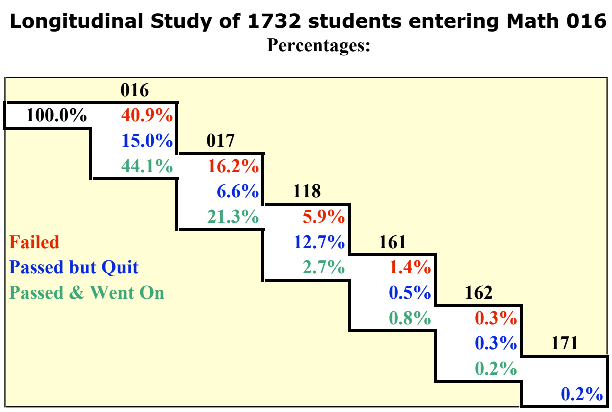
Of the students starting in Math 016 - Arithmetic, less than one quarter of one percent complete Math 171 - Calculus I. While one reason is surely the very length of the sequence, six semesters, another is the general reliance on memorization of atomized topics. This seminar will examine the possibility of a 5 credit-hour "prequel" to Math 165 - 166, decreasing the number of credit hours from arithmetic to differential calculus from 19 to 13 credits. A very tightly organized architecture of contents will be proposed with a focus on making sense of mathematics the way mathematicians do.

Outline

Contents

1 Motivation

1.1 Only 0.23% Of The Students Entering 016 Complete 171.



2 Proposed (5-0-0) "Alternative" To Math 016 - 017

2.1 Conceptual Learning.

Looking at Physics.

In order really to address the problem we are facing in Developmental Mathematics, we may turn to *David Hestenes*, a physicist whose use of Clifford Algebras as "the" language for physics, *Geometric Algebra*, was recognized in 2002 with the Oersted Medal[1].

Following are four excerpts from his Oersted lecture.

[1] The American Association of Physics Teachers' most prestigious award, it recognizes notable contributions to the teaching of physics. Recipients include many Nobel laureates.

Conceptual Learning

1. Conceptual learning is *a creative act*.
2. Conceptual learning is *systemic*.
3. Conceptual learning is *context dependent*.
4. The quality of learning is *critically dependent on the quality of conceptual tools* at the learner's command.
5. Expert learning *requires deliberate practice* with critical feedback.

The Necessity Of Conceptual Tools.

For students and scientists alike, what they know and learn about physics is *profoundly shaped by the conceptual tools at their command*.

Reconstructing Course Content.

Course content is taken [by many] as given, so the research problem is how to teach it most effectively. This approach [...] has produced valuable insights and useful results. However, it ignores *the possibility of improving pedagogy by reconstructing course content*.

Where To Start

To reform the mathematical language of physics,
you need to start all over at the most elementary level.

2.2 Starting From The Real World.

Mathematics As Physics.

What is important is the real world, that is physics, but it can be explained only in mathematical terms.

Denis Serre. Bulletin of the AMS, Vol 47 Number 1

A "Model Theoretic" Approach.

- *Model Theory*, a branch of mathematics, deals with the correspondence between real-world structures and their paper-world theories.
- The goal then is for the students to realize that by developing *symbolic systems*, they will be able to:
  - *Represent* different aspects of the real world,
  - Develop paper-world *procedures* to parallel real-world processes.

2.3 The Content Architecture.

The Governing Ideas.

The *Content Architecture*, presented below in 30 chapters suitable for a 15 weeks (5-0-0) course, "is not perfect" if only because the unfolding of the *story line* is not entirely smooth. However, it is based on the many *connections* between Arithmetic and Algebra by way of only a very few simple *ideas*:

- Different *Symbolic Systems* are needed to deal with different kinds of real-world situations.
- *Operations* are seen as *unary* rather than as *binary*, that is as *functions*. (As, historically, they used to.)
- *Equations* and *Inequations* appear in the context of "undoing" what a function does.
- *Laurent polynomials* generalize decimal numbers.

Symbolic System To Represent Bunches Of Items.

1. The real-world collection  can be represented by the *number-phrase* /// **Washingtons** where /// is a *numerator* and **Washingtons** is a *denominator*.

2. Combinations of real-world collections can be represented by *combination-phrases*: 2 **Apples** + 3 **Bananas** where $+$ does not stand for addition but for "and" so that we should really write 2 **Apples** & 3 **Bananas**. Of course, what we have is a "*basket*", AKA *vector*.

   By the way, the "Arithmetic of Baskets" leads to *Linear Algebra* done in a completely painless way.

The Place Symbolic System.

3. With "large" collections, say four hundred and seven one-dollar bills, we must *exchange*: TEN one-dollar bills for a ten-dollar bill, TEN ten-dollar bills for a hundred-dollar bill, etc. We then get a combination of collections, each with less than TEN items and thus representable using only $1, 2, \ldots 8, 9$, and we can write 4 **Franklins** $\&$ 7 **Washingtons**.

   For computational purposes, we rewrite it as the *array*

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Roosevelts**
	4		7	

   which we can then encode with *base*-TEN *number-phrases* such as:

   0.407 **Clevelands**,

   40.7 **Hamiltons**,

   $4070.$ **Roosevelts**, etc

Metric – Exponential Symbolic System.

4. The place system, though, can only get us "so far" and to gain flexibility we need:

   - *Metric Prefixes* namely

KILO	HECTO	DEKA	—	DECI	CENTI	MILLI

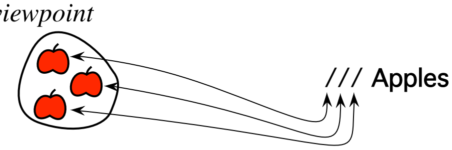
     to modify denominators. For instance, with **Hamiltons** as *base denominator*,
     - **Clevelands** become HECTO-**Hamiltons**
     - **Roosevelts** become CENTI-**Hamiltons**.

- *Exponential Suffixes* to modify numerators. For instance,
  - $6.31 \times 10^{+3}$ codes[2] .->At this point, $\times$ is only a *separator* and 10 is meaningless. "6.31 with . $\xrightarrow{\text{3 places}}$" i.e. 6310.
  - 6.31×10^{-2} codes "6.31 with . $\xleftarrow{\text{2 places}}$" i.e. 0.0631

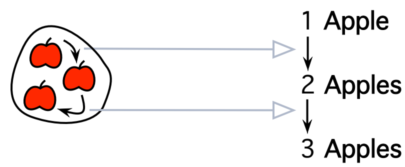
Using *arrays* shows how the two work together.

From The Real World To The Paper World.

By substituting for the *cardinal viewpoint*



the *ordinal viewpoint*



counting permits the development of *procedures*.

Procedures Based On Counting From . . . To . . . .

5. Comparison Of Collections: *Equalities* and *Inequalities*.
6. States And Actions: *Functions*.
7. Attaching A Collection To A Collection.

   Adding To: 5 Apples $\xrightarrow{\nearrow 2\text{ Apples}}$ 7 Apples

   Detaching A Collection From A Collection.

   Subtracting From: 5 Apples $\xrightarrow{\searrow 2\text{ Apples}}$ 3 Apples

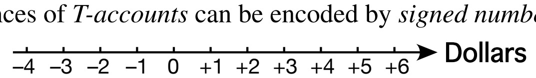
   5 Apples $\xrightarrow{\searrow 7\text{ Apples}}$?

Symbolic System To Represent Bunches of "Sided" Items.

The ancestor of Signed Counting is *Double Entry Accounting* which, to this day, remains a mathematical wonder.

Signed Arithmetic.

8. The balances of *T-accounts* can be encoded by *signed number-phrases* which can be imaged on *rulers*:



9. *Absolute Comparisons* versus *Size Comparisons*.
10. *Translations*: Moving back or forth from a given place:

    Milestone $+5$ $\xrightarrow{\leftarrow 7\text{ Miles}}$ Milestone -2

11. *Signed-Addition* and *Signed-Subtraction*.
    - $\oplus$ defined in terms of $>$, $<$, $+$ and $-$ in $\mathbf{N}$.
    - $\ominus$ shorthand for $\oplus$ *Opposite*.
      (Banks cannot *remove* an erroneous entry and can only *cancel* it by "adding the opposite" entry.)

[2] <

Multiplication.

"It Ain't No Repeated Addition". (Devlin's Angle, June 2008. Mathematical Association of America.)

> Pretending there is just one basic operation on numbers (be they whole numbers, fractions, or whatever) will surely lead to [students] assuming that numbers are simply an additive system and nothing more. Why not do it right from the start?

It's Multiplications.

12. As: • *Times Table*. ("Axiomatic" basis for *procedures*.)

    $3 \xrightarrow{2} 2 \times 3$ (In "times 2" table.)

    • *Dilation*.

    $-3 \xrightarrow{-2} -3 \otimes -2$ (-3 *flip-dilated* by a factor of 2)

    • *Additive Power*.

    4×7 shorthand for $\underbrace{4+4+4+4+4+4+4}_{\text{7 copies of 4}}$

    • *Measure*.

    $(7,4) \longrightarrow 7 \times 4$ (Area of a 7 by 4 rectangle.)

13. • *Co-Multiplication*: The real-world "Losing three bad apples that would have cost four cents each to dispose of is a twelve cents plus" is coded as

    $$-3\,\cancel{\mathsf{Apples}} \otimes -4\,\frac{\mathsf{Cents}}{\cancel{\mathsf{Apple}}} = +12\,\mathsf{Cents}$$

    where the *co-denominator* $\frac{\mathsf{Cents}}{\mathsf{Apple}}$ lives in the *dual space*.

Symbolic System To Represent Quantities of Stuff.

- The Hindu-Arabic Place System was originally constructed to represent larger and larger *collections of items*.
- With a mechanism for *subdivision*, it becomes the Decimal System with which we can represent *quantities of stuff*:
  - *The real real numbers are the decimal numbers.*

    (An Engineer.)
  - *No physical quantity has ever been measured with more than 15 or so digits of accuracy. Mathematicians, however, freely fantasize with infinite-precision real numbers. Nevertheless within pure mathematics the notion of a real number is extremely problematic.*

    (G. Chaitin. *How real are real numbers?* IBM Research.)

Decimal Arithmetic.

14. Smaller And Smaller:

    – "That of which TEN can be *exchanged* for . . . ",

    – [...] read as "something too small to matter *here*", that is as a precursor to Landau's $o[h^n]$,

    – *Approximations*.

    – $7 \times 10^{\pm 3}$ is now shorthand[3] for *7 multiplied/divided by 3 copies of* 10.

15. Images: *Points*, *Neighborhoods* and *Intervals*.

[3]Now 10 makes sense but $\times$ is still a *separator*.

16. *Division* and *Approximate Quotients*.
$\frac{5}{7} = 0 + [...] = 0.7 + [...] = 0.71 + [...] = 0.714 + [...] = \dots$

17. Fractions : Multiplication and Division.

18. Fractions: Addition and Subtraction.

Reverse Problems.

- Given a function, we usually want *output(s)* that satisfy a given requirement.
- A *problem* thus consists of a *data set* together with an *(in)equation*. To find its *solution subset*, we locate:
  1. The *boundary* by solving the *associated equation*,
  2. The *interior* by testing the regions.

Equations And Inequations.

19. Basic Problems (Counting Numbers): $n = n_0, n < n_0, \dots$

20. Basic Problems (Decimal Numbers): $x = x_0, x < x_0, \dots$

21. Translation & Dilation Reverse Problems: $x \longrightarrow x + b \geqq y_0, \dots \qquad x \longrightarrow a \cdot x < y_0, \dots$

22. Affine Reverse Problems: $x \longrightarrow a \cdot x + b = y_0, \dots$

23. Double Basic Problems, e.g. $\text{AND}\begin{cases} x < x_1 \\ x \geqq x_2 \end{cases} \quad \text{AND/OR}\begin{cases} x > x_1 \\ x \geqq x_2 \end{cases} \quad \text{EITHER/OR}\begin{cases} x \geqq x_1 \\ x < x_2 \end{cases}$

24. Double Affine Problems.

Laurent Polynomials.

- *Laurent polynomials* are essentially decimal numbers with x instead of 10 as basis. Anticipating a bit:

$$7x^{+2} + 5x^{+1} + 8x^0 + 6x^{-1} + 4x^{-2}\Big|_{x \leftarrow 10} = 758.64$$

  (Where, here too, $+$ really ought to be read as "and".)

- The algebra of Laurent polynomials is in fact "simpler" than the arithmetic of decimal numbers in that there is neither *carryover* nor *borrowing*.

Algebra of (Laurent) Polynomials.

25. Repeated Multiplications versus Repeated Divisions.

26. Laurent Monomials as "repeated dilations":
$a \cdot x^{\pm n}$ shorthand for "*a multiplied/divided by n copies of x*"

27. Addition and Subtraction.

28. Multiplication.

29. Binomial Expansions:
$(x_0 + u)^n$ for $n = 1$, $n = 2$, $n = 3$, Pascal Triangle.

30. Divisions in *descending* and *ascending* exponents.
    - Near ∞: $\frac{-12x^3+11x^2-17x+1}{-3x^2+5x-2} = +4x + 3 + 8x^{-1} + [...]$
    - Near x_0: $\frac{-12+23h-h^2-2h^3}{-3+2h} = +4 - 5h - 3h^2 + [...]$.

2.4 Learning from Text Instead of Lectures.

Lectures Are Inefficient

- Historically, note taking during lectures was the main way knowledge was transmitted but this no longer needs to be the case.
- Moreover, lectures do not facilitate understanding in that understanding requires:
  - *familiarization* which takes time while lectures press on,
  - *precise questions* which are difficult to formulate while trying to keep up with a lecture,
  - *explicit expression* which note taking rarely produces.

Explicative Texts Are Efficient.

While most textbooks available today are *prescriptive*, there is no reason why a textbook cannot *explain the why of things* by:

- Analyzing and developing *ideas* in detail,
- Preempting many likely *misunderstandings*,
- Enabling the students to formulate their own *questions*,
- Leaving class time for *discussions*.

Explicative Texts, Though, Must Be Transparent.

- An *idiosyncratic terminology* is necessary because:
  - Every single thing requires a *specific* name.
  - Names must be *explicit*, e.g. "0-output input" and "∞-output input" instead of "zero" and "pole".
- Nothing should go "without saying" so that explicit *default rules* should always be available as for when $x + 3x$ gives $3x$ because the absence of 1 was read as 0.
- Symbols should be as *context-free* as possible since context-bound meaning can be confusing as when $+$ as addition in **N** is used to define $+$ as addition in **Z**.
- There *can* be such a thing as *too much* explanation.

Explicative Texts Must Be Read (Pencil In Hand).

While explicative texts can be designed to meet the conceptual requirements of Developmental Students,

- A difficulty remains in that *learning from a text* is something entirely new for Developmental Students.
- A simple, no-cost solution would be for the explicative text to be used as *reading material* in a linked Eng 098.

An English 098 Instructor's View.

"*I did appreciate Schremmer's linguistic approach. The students who stayed to the end also appreciated his methods whether they passed or not. If we pursue another link, the English teacher should definitely read the math text with the students. Unfortunately, because I had my own reading to do, we did not read the math in English class as we should have done.*"

2.5 For Whom the "Alternative"?

No Prerequisite.

Nothing in the above contents requires from the students anything more than "common sense". All that is necessary for the students to take advantage of the *exponential character of conceptual learning* is for them to be able to persevere.

However, and whatever the reasons, some students cannot:

- Spare the necessary time,
- Conceive of learning as anything other than doing on the exam what they were shown how to do in class.

Attitude Assessment instead of Knowledge Test.

So, rather than on the basis of prior knowledge, placement into the "alternative" should be on the basis of the student's ability:

- To commit to a *sustained effort* in return for the instructor's availability and help,
- To accept that learning mathematics can be learning *how to make the case* that things are true or false.

2.6 Implementation.

"Free as in Free Speech and Free as in Free Beer".

A *freely modifiable* implementation of the above Content Architecture, including explicative text and ancillaries, i.e. daily homework, daily quiz, reviews and exams, is *freely available* from `http://www.freemathtexts.org/Standalones/RBA/Downloads.php`.

Other Approaches?

- The architecture proposed above is only one particular way to organize and present the contents.
- Within the "model theoretic" setting used here, other architectures are quite conceivable.
- As opposed to the "model theoretic" approach, a "formalist" approach would also be quite conceivable.

3 Conclusion

A Faster and Surer Pathway?

Whatever the significance of the percentages given at the outset, the proposed "alternative"

- Seems unlikely to harm Developmental Students,
- Would give more Developmental Students *immediate access* to college level courses.

Pathway To Where?

In particular, since the contents of the proposed "alternative" were reverse-engineered from the contents of Math 165, the two semester sequence of (4-0-0) courses, Math 165 - 166, Differential Calculus I and II, AKA the *Mathematics of Change*, would be a *natural continuation* of the proposed "alternative".

1992 Report Of The Institutional Research Office.

- "Of those attempting the first course in their respective calculus sequence, 654 did so between Fall 1988 and Fall 1989[4]. While 67 (*12.5%*) of the 538 traditional track students finished that sequence, 56 (*48.3%*) of the 116 differential calculus sequence did likewise, revealing an association between the latter sequence and completion ($\chi^2(1) = 82.14), p < .001$)."
- "If the additional requirement of passing Math 172 or Engineering 172 is added, 27 (5.0%) of the tradtional sequence students completed all courses, while 6 (5.2 %) of the 116 differential calculus students did likewise."

At The Very Least?

1. According to the 2001 Report, 10% of the students attempting 016 completed 017.

   Assume *half* that percentage for the students attempting the "alternative": *5%*.
2. According to the 1992 Report, 48% of the student who attempted Math 165 completed Math 166.

   Assume *half* that percentage for the students coming from the "alternative": *24%*.
3. Then, of the students attempting the "alternative", *1.2%* would complete Math 166.
4. This would be a *five-fold improvement* over the current situation.

Moreover.

- Other courses might be accessible from the proposed "alternative",
- Other courses might be adjusted to take advantage of the proposed "alternative",
- Other "alternatives" might be designed that would be better suited for other courses and/or sequences.

4 Appendix

Contents of Math 165

About three quarters of the contents of Math 165 were re-implemented a few years ago for Math 161[5]:

- The emphasis is on the analysis of individual "algebraic functions".
  - The *local analysis* is by way of *local polynomial approximations* near ∞ and near bounded inputs.
  - The *global analysis* uses the local analysis near a generic x_0 but beyond that is of course more or less ad hoc.
- *Derivatives* are not defined by way of limits but, following *Lagrange*, as the functions outputting $n! \times$ the coefficient of h^n in the polynomial approximation of f near x_0.

[4]Fall 1988 was chosen as one endpoint because that was the first semester in which the differential calculus sequence was offered; Fall 1989 was chosen as the other endpoint because it allowed students two complete calendar years to complete their chosen sequence.

[5]Freely available from `http://www.freemathtexts.org/Standalones/RAF/Downloads.php`

Part I Of Math 161.

- The Toolbox.

  1. Relations And Functions: The *Fundamental Graphic Problem*.
  2. Towards Local Analysis: *Arithmetic* of *large*, *small* and *bounded* numbers.
  3. Graphic Local Analysis. *Local Features*: Height, Slope, and Concavity. Extremal input-outputs.
  4. From Local Graphs to Global Graphs: *The Essential Question*.

- The Gauges: *Power Functions*

  5. Regular Positive-Exponent Power Functions. (Exp. > 1.)
  6. Negative-Exponent Power Functions.
  7. Exceptional Power Functions. (Exponents 0 & 1.)

Part II Of Math 161.

- Plain Polynomial Functions.

  8. *Constant Functions* and *Linear Functions*.
  9. *Affine Functions*: Local Analysis near ∞, x_0.
  10. Global Analysis: The *height* must be 0 somewhere. The *slope* is given by a constant function.
  11. *Quadratic Functions*: Local Analysis near ∞, x_0.
  12. Global Analysis: The *concavity* is given by a constant function. The *slope* is given by an affine function, so must be 0 somewhere. The number of 0-height inputs depends on Sign $f(x)|_{x \text{ near } x_{0\text{-slope}}}$ compared to Sign $f(x)|_{x \text{ near } \infty}$.
  13. *Cubic Functions*: Local Analysis near ∞, x_0.
  14. Global Analysis. The *concavity* is given by an affine function so must be 0 somewhere. The *slope* is given by a quadratic function so the number of 0-slope inputs depends on Discriminant f'.

Part III Of Math 161

- Rational Functions.

  15. Algebra Review:
      - Binomial Expansions ("Alternative" 29)
      - Divisions in Descending and Ascending Exponents ("Alternative" 30)
  16. Local Analysis near ∞ (Requires division in *descending* exponents).
  17. Local Analysis near x_0 (Requires division in *ascending* exponents).
  18. Global Analysis. Contrary to (plain) polynomial functions, rational functions have a wide variety of *bounded graphs*.

Differential Calculus.
The fourth hour in Math 165 then provides ample time to:

- Develop the *recursive aspect* of the derivatives,
- Focus more formally on *continuity* and *smoothness*,
- Prove all the *usual theorems*.