The Mathematical Memoirs of Sim Wobpa

With a Few Notes by A. Schremmer.

A first semester calculus is not all engineering students, it is not all physical science majors, it is not all computer science majors, it is not all social science majors, and it is definitely not all math majors. The Calculus I syllabus is designed for a general audience, "just plain folks".

(Goldstein, 1986)

(T)he difference between accuracy and honesty, between information and truth, between probity and, for example, justice, depends on how you phrased the memo.

(Enright, 1995)

I, Sim Wobpa, merely a student interested in mathematics, with only a background in polynomial algebra, wanted to enlist for the Great Calculus War but they told me that I was not ready and they signed me up for boot-camp. I bought the manual. Here is how it begins.

A firm foundation in algebra is necessary for success in college-level mathematics courses. *Algebra and Trigonometry*, Fourth Edition, is designed to help students develop their proficiency in algebra, and to strengthen their understanding of the underlying concepts.

Is that so? Is this one of these self evident truths or what? Anyway, this PREFACE is obviously not addressed to me who had to pay for a book I was not allowed to choose. I am told, though, that they are all clones of each other. The remaining twenty some pages are just there to convince the instructor to have this particular clone ordered by the bookstore. Why do I have to pay for advertising?

So, OK, I go to page 1. That ought to be for me. Hey, it is even *addressed* to me:

Real numbers are used in everyday life to describe quantities such as age, miles per gallon, container size, and population. To represent real numbers, you can use symbols such as

$$9,\ 0,\ \frac{4}{3},\ 0.666\ \ldots\ 28.21,\ \sqrt{2},\ \pi,\ \text{and}\ \sqrt[3]{-32}.$$

I guess it must make sense to instructors but it sure doesn't do much for me: What's this thing about "represent(ing) real numbers"? Is $\sqrt[3]{-32}$ a number or a symbol? And what about $\sqrt[2]{-32}$? Anyway, I never used anything like $\sqrt[3]{-32}$ in *my* life and they say that these are *real* numbers! Should I really have known all of this (80 pages)? And what makes them think that these booster shots will do it? Plus, we can't possibly need all of this right away and it is certainly not organized as a reference to be used as needed later on.

I noticed a proof though: "The length of the vertical side of the triangle is $|y_2-y_1|$, and the length of the horizontal side is $|x_2-x_1|$. By the Pythagorean Theorem, you can write … the Distance Formula". I guess I could. But then maybe not. I am having a lot of trouble with this stuff that has no beginning, no middle and no end. There is no logic to it. What is the point? I know, I should be patient but I can't help wondering if this course is really honest.

Finally we got to Chapter 1 and here is how it begins:

The Graph of an Equation.

In Section P.7 you used a coordinate system to represent graphically the relationship between two quantities. There, the graphical picture consisted of a collection of points in a coordinate plane (see Example 2 in Section P.7).

Frequently, a relationship between two quantities is expressed as an **equation** in two variables. For instance, $y = 7 - 3x$ is an equation in x and y. An ordered pair (a,b) is a **solution** or **solution point** of an equation in x and y if the equation is true when a is substituted for x and b is substituted for y. For instance, (1,4) is a solution of $y = 7 - 3x$ because $4 = 7 - 3(1)$ is a true statement.

In this section, you will review some basic procedures for sketching the graph of an equation in two variables. The **graph** of an equation is the set of all points that are solutions of the equation.

Help! This is too much. Again, why this distinction between x, y and a, b? What are these ordered pairs? What's this all about true statements? Why quantities and variables[1]? Did they forget that I am "unprepared"? What happened?

[*What happened is that, as once pointed out in the review of a Calculus text* (Dudley, 1988)*, authors must beware of reviewers. Just as, incidentally, non-tenured faculty must beware of the tenured faculty who will rate them and who in turn must beware of what could hamper their promotion to administrative positions.*]

One thing I do not understand is that, if the graph of an equation is the set of *all* points that are solutions of the equation, why can we get away as in Example 1 with plotting only *a few*? And, if we do have to take it for granted that it works, why don't they say so? What's the point of everything else then?

[*Sim Wobpa certainly has … a point. See Strangs' famous example. If only to be honest, the distinction between a finite bunch of points and a smooth curve ought to be made clear up front.*]

We continued with examples of how to "graph" various functions, with how to solve linear equations, with particular attention given to equations involving fractional expressions (presumably to show how sensitive they are to our needs) and a bit of "modeling"—I think the latter is what they used to call word problems in Basic Algebra. But then we came to quadratic equations and here I expected to see, finally, my first real proof, namely that of the celebrated quadratic formula that they showed us in Basic Algebra.

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[1] I looked them up in the index. Quantities do not appear at all and variables are on page 6 while I am still on page 2.

[*Sim Wobpa apparently hasn't been told that "proofs" are going out of style in college-level mathematics. In the almost 900 page book he is using, there are only eight other proofs. All in trigonometry! As he would say, what's the point?*]

They started with the method of factoring, like in Basic Algebra. But I remember that when I tried, back then, to get some practice by changing some of the numbers in an example, it wouldn't work any more. Some method, you never knew when it would work. But this time they did it also by completing the square and I guess that really is a *method* because they used a, b, c for the coefficients so it would have to work no matter what. Except that I thought you had to be careful with roots and they write $\sqrt{\frac{b^2 - 4ac}{4a^2}}$ without any qualm. Ah yes, on the next page they say that when b^2-4ac is negative the solution is not really a solution. I think this is a joke like when my boss told me he was going to give me a raise but it wouldn't be a real one because he would have to charge me the exact same amount for the air I breathed at work. Anyway, this completion of the square is a bit contrived. I don't know if I will ever be able to memorize it. Maybe I should not say this but a friend of mine told me that there is a really nice way to establish the quadratic formula but that I would have to wait a bit before she could show me. But then, they don't seem to really care about the proof and we had to spend most of our time on word problems except that they did not tell us all that we needed to give them the answer. We were supposed to assume it. I mean, can I assume anything I want? Finally, we did this weird thing they call complex imaginary numbers. At the end, they said it had to do with quadratic equations, to make up for the root of the negative thing. I don't know. And if they allow these imaginary numbers for x why don't they allow them also for a, b, c? Even my friend could not answer me that question. Then we went on to other types of equations, starting with

Polynomial Equations

> In this section you will extend the techniques for solving equations to nonlinear and nonquadratic equations. At this point in the text, you have only four basic methods for solving nonlinear equations—*factoring, extracting roots, completing the square,* and the Quadratic Formula. So the main goal of this section is to learn to *rewrite* nonlinear equations in a form to which you can apply one of these methods

In other words, to solve any equation, all you need is to rewrite it. That's pretty neat! My friend though says that's not true and that there are lots of equations nobody can solve. I don't believe it. She even mentioned two guys, something like Laurel and Hardy, who, she said, proved that nobody would ever be able to do it. But she said they are not in the index.

We finally made it to CHAPTER 2. FUNCTIONS AND THEIR GRAPHS but there are even more prerequisites. This time it's about graphing linear equations. I thought we were done with that. We started with

Lines in the Plane and Slope.

> The simplest mathematical model for relating two variables is the **linear equation** $y = mx + b$.

Wait a minute, are models just equations? And what about variables? This time I am going back to page 6 to figure out what a variable is:

One characteristic of algebra is the use of letters to represent numbers. The letters are **variables**.

I guess variables must be the same as letters! But then what's the difference? A bit further down we have:

DEFINITION OF AN ALGEBRAIC EXPRESSION.

A collection of letters (**variables**) and real numbers (**constants**) combined using the operations of additions, subtraction, multiplication, division, and exponentiation is an **algebraic expression**.

What's the fuss? Why not just say "letter" and be done with it?

And then what about these glorified word problems they call modeling? Is this what they call Data Analysis? I thought I had signed up for mathematics. All I wanted to know is to see how one establishes truth.

[*You could of course say some geometry is necessary for the sentence "the graph of an affine function is a (straight) line" to make sense. But, of course, that's not what the book does. As for "modeling", Sim Wobpa is more right than he could know since even "First Semester Calculus has no applications*" (Dudley, 1988).]

Blah, blah. I am getting sick of it. But my friend said that, just like arithmetic was about calculating with *known* numbers and algebra was about calculating with *unknown* numbers, calculus was going to be about calculating with *functions*, that is with the ways to get numbers out of other numbers. Well, I will have to see that.

We made it! We made it to functions but they are not what my friend said. Here is what *they* said:

DEFINITION OF A FUNCTION

A **function** f from a set A to a set B is a rule of correspondence that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

[*Note how slick: "A is…" but "B contains …" And not even a warning!*]

I am lost. I called my friend and she told me to forget about it, they were just showing off and she was going to explain it to me. First, she said, the differential calculus is just about finding the way things *change* but then, to see that something is changing, we have to look at it against something else that either does not change or changes differently like when you sit in an airplane you don't realize that you are moving unless you look out the window. So, she said, what we would do is to develop a *mathematical* framework in which we would look at how things change in terms of other things, that is as *functions* of other things.

I told her the plane thing was not really convincing because anybody would know the plane was moving. Then never mind the plane she said, what about taxes? How much tax do you pay? Like, everybody pays the same tax? No matter what? I said no, taxes were always *in relation to* something else like income, real estate, sales…. Besides, she said, to say that $25,000 in income taxes is a lot of taxes is still meaningless because it would depend on how much the income was. She said that I should think of income tax as a *function* of income. I said I guessed that's what I was doing when I was looking at the tables in the 1040. So, she said, instead of the above, I should use something like:

DEFINITION. By **function** we mean any process, device, procedure, agency, converter, translator, etc. that **outputs** *at most one* number for each and every number we **input**.

I asked her what was this thing about "at most one", why not just say "one"? She said a function absolutely could *not* return more than one output for any given input because a function cannot give us a choice. Like how much tax to pay on a given income. But a function need not return any output at all.

Then I reminded her she had promised to tell me about the Quadratic Formula and the solutions that were not really solutions. She said that there was more to it than solving the equation $ax^2 + bx + c = 0$ and that what we really have is a function that outputs $f(x) = ax^2 + bx + c$ when we input x and that solving the equation is just finding those inputs whose output is 0 but that it would be better to take a larger view, namely to look for the graph of f. She was right, it turned out to be a much larger view and it did take me quite some time to get it but I think that it makes sense and was worth it.

First she asked me to accept that the *local* graphs of the power functions near 0 and near infinity were as the *plots* suggested but, she said, she would never ask me to accept anything like that ever again. OK. After thinking about small and large powers of 10, I could believe in these graphs. Though she said never to forget that I had taken them on faith. She is right. After all, she is not a teacher, she could be wrong. Then she made me investigate constant and affine functions *locally*, based on the local graphs of the power functions, and she then made me extrapolate local graphs to *qualitative global graphs* that is graphs that summarize all their features and on the basis of which I would then be able to turn *plots* into *quantitative graphs*. Then, in exactly the same spirit, she made me start the *local* investigation of quadratic functions. I know this sounds like gobbledygook so, this time, I will go into more detail.

We have $f(x) = ax^2 + bx + c$, that is a function that is the sum of the first three power functions. But the trouble is that, *globally*, we have no ground on which to approximate it by some simpler function. But when x remains near x_0, that is when $x = x_0+h$ where h is small, we have

$$
\begin{aligned}
f(x_0+h) &= a(x_0+h)^2 + b(x_0+h) + c \\
&= a(x_0^2 + 2ax_0h + h2) + b(x_0+h) + c \\
&= [ax_0^2 + bx_0 + c] + [2ax_0 + b]h + ah^2
\end{aligned}
$$

that is, *locally*, f is the sum of three functions which are now of diminishing *numerical* importance. However, all contribute *visible qualitative* information: a *constant* function giving the *height* of the graph, a *linear* function giving the *slant* of the graph and a *squaring* function giving the *bending* of the graph.

This gave us the following *global* results:

i. The *bending* is the same everywhere since it does not depend on x_0 and is controlled by a.

ii. The *slant* depends on x_0 and, since it is an affine function of x_0, it will be 0 at $x_{0\text{-slant}} = \frac{-b}{2a}$ which, a priori, may or may not be a *turning* point but which here it has to be because, otherwise, the (sign of the) bending would change which, by **i.**, it cannot.

iii. The *height* depends on x_0. Can it be 0? We compute $f(x_{0\text{-slant}}) = \frac{-\text{Discriminant } f}{4a}$ where Discriminant $f = b^2 + 4ac$. Since $f(x_{0\text{-slant}})$ is *extremal*, we then reason that, depending on the sign of Discriminant f versus the sign of a, there will be zero, one or two 0-*height* points.

When there is just one 0-height point, it must be $x_{0\text{-slant}}$. To see what happens in the other cases, we note that here, since we did not leave anything out in $f(x_0+h)$, h need not really be small for the equality to hold exactly. So we use u instead and we get:

$$f(x_{0\text{-slant}}+u) = \frac{-\text{Discriminant } f}{4a} + au^2$$

and we have $f(x_{0\text{-slant}}+u) = 0$ whenever $u^2 = \frac{\text{Discriminant } f}{4a^2}$ so that $x_{0\text{-height}} = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$.

Not bad! You know, it seems *logical* to me. In fact, rather than to memorize the quadratic formula, I found I liked it better solving quadratic equations that way. And on top of it, it follows immediately that the local graph near any point, finite or infinite, can be extrapolated in essentially only one way. My friend says that's what mathematics is all about: thinking logically. She said that thinking logically pays but I don't know about that. Not in boot-camp.

Note. The identity of Sim Wobpa is clearly of no interest, he is just a casualty on the battlefields of the Calculus wars. As for the manual, it is enough to say that the market indicates it has our overwhelming support. There is no clue as to who Sim Wobpa's clearly misguided friend might have been.

References

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