# Notes From The Mathematical Underground

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*The opinions expressed are those of the author, and should not be construed as representing the position of AMATYC, its officers, or anyone else.*

This is Part I of the last of my Notes From The Mathematical Underground[1] which I would like to devote to another major subject of enquiry in mathematics education, one as important as *content analysis* and just as ignored, namely *language analysis*. Both are, in fact, deeply intertwined in the learning process. I will show in two examples[2], here exponential notation and, in Part II to appear in the Fall, matrix multiplication, how a carefully chosen if very informal phrasing can not only make sense enough that its memorization involves a minimum of stress but will also enable students to develop and formulate *spontaneously* many of the associated *operational* "rules".

Start in ARITHMETIC by letting the students *read* $3(5)^{+2}$ as "3 *multiplied* by 2 copies of 5". Stress that it is the "+" in front of the 2 that indicates that the 3 is to be *multiplied* by the 2 copies of 5. Consequently, students should *write* 3(5)(5).

Follow up immediately with letting the students *read* $3(5)^{-2}$ as "3 *divided* by 2 copies of 5" where it is the "–" in front of the 2 that indicates that the 3 is to be *divided* by the 2 copies of 5. Consequently, students should *write* $\frac{3}{(5)(5)}$ which, it should be pointed out, also reads as "3 *divided* by 2 copies of 5" but where it is the *bar* that indicates that the 3 is to be *divided* by the 2 copies of 5.

Naturally, just as is the same as 1 , $(5)^{+2}$ is the same as $1(5)^{+2}$ and of course $(5)^{-2}$ is the same as $1(5)^{-2} = \frac{1}{(5)^{+2}}$ but now at least the students can see where the 1 in the numerator comes from.

Only after that should students be allowed to *compute* 3(5)(5) to 75, $\frac{3}{(5)(5)}$ to $\frac{3}{25}$ or 0.12, and $(5)^{-2}$ to $\frac{1}{25}$ or 0.04: It is only when they see *reading*, *writing* and *computing* as three conceptually distinct actions that the students can

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[1] I would hope though that, even in these severely conservative times, some sort of Mathematical Underground will survive.

[2] In neither of which "+" will stand for addition!

*meaningfully* write $3(5)^{+2} = 75$, $3(5)^{-2} = \frac{3}{25} = 0.12$ or $(5)^{-2} = \frac{1}{25} = 0.04$. Anything less than that is pure voodoo.

Observe that none of the above involves more than the *natural* numbers as, *here*, the symbols "+" and "–" are mere *code* for "*multiplied by*" and "*divided by*". On the other hand, the coefficient and the original from which the copies are made can be any kind of numbers as long as they can be multiplied in some way.

Now the so-called "laws of exponents" go entirely *without saying* which is to say that they *make sense*. For instance, by *counting* (multiplicative) copies, students have no difficulty computing $3(5)^{+4} \cdot 7(5)^{+2} = 3(5)(5)(5)(5) \bullet 7(5)(5) = 21(5)(5)(5)(5)(5)(5) = 21(5)^{+6}$. Counting copies merely parallels, and is just as natural[3] as, counting s to get 4 + 2 = + = = 6 . It is when students grow tired of counting copies and casually state what we usually inflict them as "law" that something significant has occurred. And, since division involves similarly subtraction of the number of copies, students again spontaneously extend the "first law" to *signed* exponents.

In exactly the same way, $4\left[7(5)^{+2}\right]^{+3}$ means $4\left[7(5)^{+2}\right]\left[7(5)^{+2}\right]\left[7(5)^{+2}\right]$, i.e. 4•(7)(5)(5)•(7)(5)(5)•(7)(5)(5) etc. And again, students will get tired of doing this and, before long, will figure that what they have is 4 multiplied by 3 copies of 7 and 3 times 2 copies of 5 , in other words $4(7)^{+3}(5)^{+2\cdot 3}$.

The next step is to let the students read $\sqrt[3]{125}$ as "that number 3 copies *of which* multiply to 125". This takes some time as it is not a turn of phrase that comes naturally to most students. *But*, after they have become familiar with it, we *can* ask the question $\sqrt[3]{247} = (247)^{+?}$ and since the *only* thing they know about $\sqrt[3]{247}$ is that 3 copies of it multiply to 247, it is natural for them to say that 3 copies of $(247)^{+?}$ should also multiply to 247. In other words, we must have $\left[(247)^{+?}\right]^{+3} = 247^{+1}$, that is $(247)^{+?\cdot 3} = 247^{+1}$ so that the inescapable conclusion is that ?•3 = 1 and therefore ? = $\frac{1}{3}$.

Even though logic is not always *immediately* convincing, once students agree to let $x^{+3}$ (which must be the same as $x^{+\frac{3}{1}}$) stand for the phrase "the number we *get* when we multiply 3 copies *of x*", it is not too hard for them to accept that $x^{+\frac{1}{3}}$

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[3] Needless to say, working on the number of copies is just what logarithms are all about.

should stand for the **dual phrase** "the number we must multiply 3 copies *of* to *get* *x*". Then of course, $x^{+\frac{2}{3}}$ means "the number we get when we multiply 2 copies of the number we must multiply 3 copies *of* to get *x"*.

While students initially find this kind of reading quite difficult, the duality is of the essence and, of course, it is exactly the same as that involved in the notion of fraction: The only meaningful answer to "What is a quarter?" is naturally that "It takes four *of them* to make up a dollar" so that three quarters are "3 of which it takes 4 to make up a dollar".

To tell ALGEBRA students to "raise both sides to the third power" to solve $\sqrt[3]{3x+7} = 2x - 1$ is just voodoo to them. On the other hand, students *used* to read the left hand side as "that number 3 copies of which multiply to 3*x* + 7" will eventually, and of themselves, write $\sqrt[3]{3x+7} \cdot \sqrt[3]{3x+7} \cdot \sqrt[3]{3x+7}$ = 3*x* + 7 and therefore 3x + 7 = $(2x-1) \cdot (2x-1) \cdot (2x-1)$ and, consequently, 3x + 7 = $(2x-1)^{+3}$.

In PRECALCULUS/CALCULUS, to "show" that, near 0, the power function $x^m$ is *below* $x^n$ when $m > n > 0$, let, say, $x = 1/10$ and simply point out that the more *copies* of 1/10 are used, the smaller the product will be. Etc, etc.

Thus, definitions too have their logic and rather than glossing over that logic and "simply" let the students memorize them, it pays both in the short and the long run to analyze what makes them work.

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