Notes From The Mathematical Underground

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In this Second Part of the last of *my* Notes From The Mathematical Underground[1] I will use matrix multiplication as another illustration of how the *way* we define things can affect how students *understand* and *use* them.

As I have insisted repeatedly, we should never use "lone" numbers but only expressions involving a **numerator** and a **denominator**, as in 3♥, with the result that, rather than *additions*, expressions such as 3♥ + 5♣ are really **combinations** in which "+" means "AND".

Starting then from the fact that a *combination* is nothing more than a **list**, we will follow Sila (Student Interested in Linear Algebra) as, using the **combination language**, she develops[2] a **list language** in a familiar situation that involves **shopping lists** (*of* objects) together with **price lists** (*for* these objects)[3]. To distinguish, we will write shopping lists *horizontally* and price lists *vertically*[4].

Say Sila buys $\left(5 \textit{ apples},\ 8 \textit{ bananas}\right)$ at $\begin{pmatrix} 45 \textit{ cents per apple} \\ 12 \textit{ cents per banana} \end{pmatrix}$ in which *cents per apple* and *cents per banana* are **denominators** as well as *apple* and *banana*. Sila has of course no difficulty computing the **value** of her shopping list under the price list. She sets up in *list* language

$$\left(5 \textit{ apples},\ 8 \textit{ bananas}\right) @ \begin{pmatrix} 45 \textit{ cents per apple} \\ 12 \textit{ cents per banana} \end{pmatrix}$$

which she reads in *combination* language as

5 *apples* **at** 45 *cents* ***per*** *apple* AND 8 *bananas* **at** 12 *cents* ***per*** *banana*

Now she replaces the symbol "**at**" by the multiplication sign and the symbol "***per***" by a fraction bar:

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[1] Again, I would hope that someone will continue to take a less than conventional look at what we *claim* we are doing versus what we are really *achieving* as demonstrated by, say, the **Third International Math and Science Study** (TIMSS). Those not in the know might look up **http://nces.ed.gov/timss/**

[2] The actual development is of course much longer than space permits here.

[3] Thus, from the very start, we involve the **dual** space.

[4] As it corresponds to $f(x)$, writing vectors as *columns* and covectors as *rows* is more common. But, as we shall see, writing $(x)f$ corresponds to *natural* English and also has the advantage of delaying the introduction of **transposition**.

$$5\ apples \bullet 45\ \frac{cents}{apple}\ \text{AND}\ 8\ bananas \bullet 12\ \frac{cents}{banana}$$

After "canceling" the denominators *apple* and *banana*, she now has a *common denominator*, *cents*, and can replace the *combination* symbol "AND" by the *addition* symbol "+" and compute

$$5\bullet 45\ cents + 8\bullet 12\ cents = 225\ cents + 96\ cents = 321\ cents$$

Thus, in (slightly modified) *list* language,

$$\left(5\ apples,\ 8\ bananas\right) @ \begin{pmatrix} 45\ \frac{cents}{apple} \\ 12\ \frac{cents}{banana} \end{pmatrix} = 321\ cents$$

Then, Sila wonders about a **reverse problem**: Other than $\left(5\ apples,\ 8\ bananas\right)$, is there a shopping list, (x *apples*, y *bananas*), that would be worth 321 *cents* under the same price list $\begin{pmatrix} 45\ \frac{cents}{apple} \\ 12\ \frac{cents}{banana} \end{pmatrix}$? This brings her to the equation $45x + 12y = 321$. Dually, and similarly, any price list $\begin{pmatrix} s\ \frac{cents}{apple} \\ t\ \frac{cents}{banana} \end{pmatrix}$ under which the shopping list $\left(5\ apples,\ 8\ bananas\right)$ would still be worth 321 *cents* must be a solution of the equation $5s + 8t = 321$. Etc, etc.

Sila now wants to **exchange**[5] her *fruits* for *vegetables* at the rate of 1 *apple* for 3 *yams* and 7 *zucchinis* and 1 *banana* for 2 *yams* and 5 *zucchinis*. She rewrites this as the **matrix**

$$\begin{pmatrix} 3\ \frac{yams}{apple} & 7\ \frac{zucchinis}{apple} \\ 2\ \frac{yams}{banana} & 5\ \frac{zucchinis}{banana} \end{pmatrix}$$

Now she computes that, for her $\left(5\ apples,\ 8\ bananas\right)$, she will get

$$\left(5\ apples,\ 8\ bananas\right) @ \begin{pmatrix} 3\ \frac{yams}{apple} & 7\ \frac{zucchinis}{apple} \\ 2\ \frac{yams}{banana} & 5\ \frac{zucchinis}{banana} \end{pmatrix} =$$

$$= \left(5\ apples \bullet 3\frac{yams}{apple} + 8\ bananas \bullet 2\frac{yams}{banana},\ 5\ apples \bullet 7\frac{zucchinis}{apple} + 8\ bananas \bullet 5\frac{zucchinis}{banana}\right)$$

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5 Note the role played here, just as in arithmetic, by the notion of *exchange*.

$= (5 \bullet 3\ yams + 8 \bullet 2\ yams,\ 5 \bullet 7\ zucchinis + 8 \bullet 5\ zucchinis)$

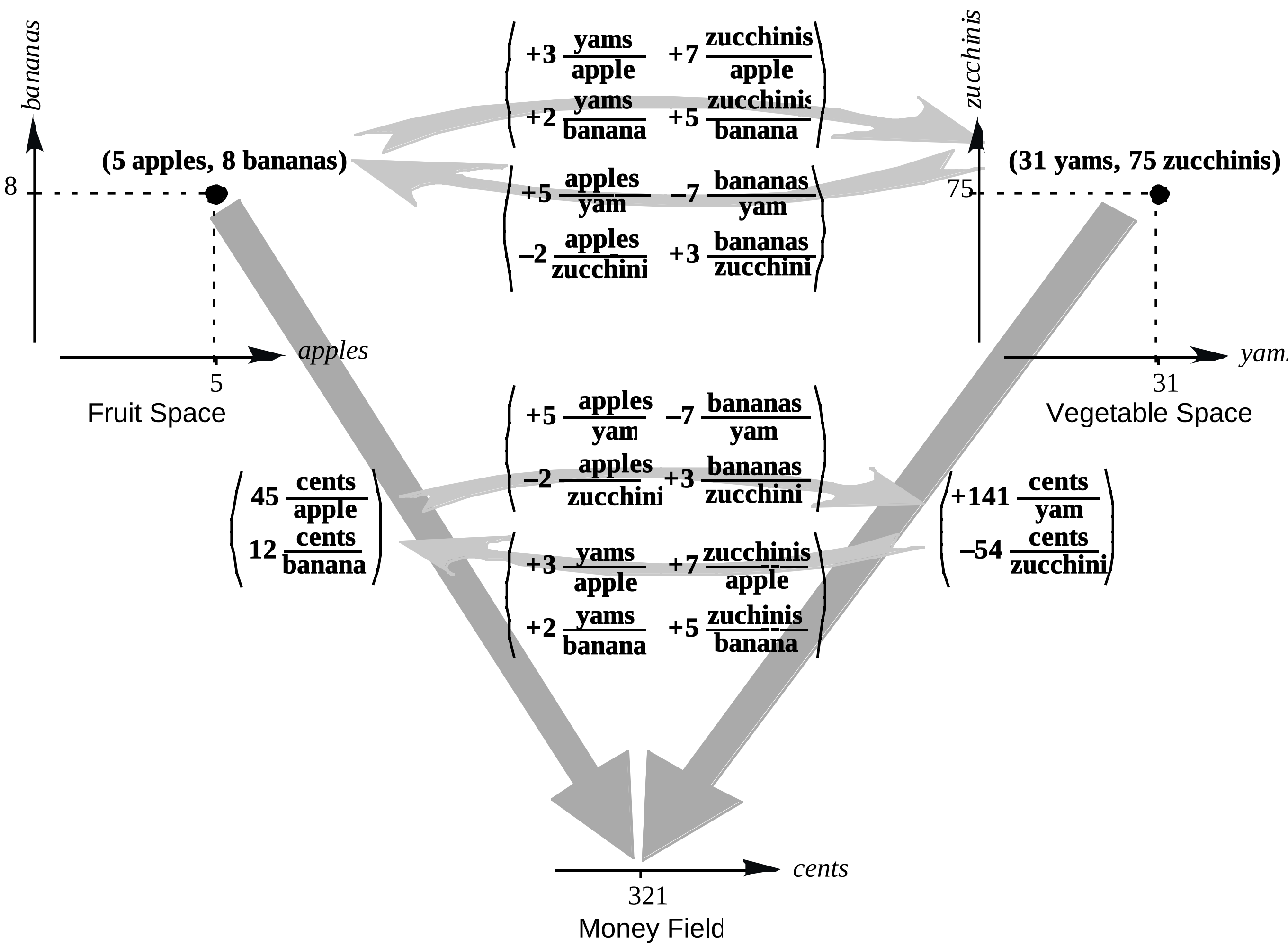
$= ([5 \bullet 3 + 8 \bullet 2]\ yams,\ [5 \bullet 7 + 8 \bullet 5]\ zucchinis)$

$= (31\ yams,\ 75\ zucchinis)$

Sila finds that the **identity matrix** in the *fruit* space is $\begin{pmatrix} 1\ \frac{apples}{apple} & 0\ \frac{bananas}{apple} \\ 0\ \frac{apples}{banana} & 1\ \frac{bananas}{banana} \end{pmatrix}$

and then that the **inverse matrix** is $\begin{pmatrix} +5\ \frac{apples}{yam} & -7\ \frac{bananas}{yam} \\ -2\ \frac{apples}{zucchini} & +3\ \frac{bananas}{zucchini} \end{pmatrix}$.

And, to make a long story short, looking at the dual situation, she eventually gets the very nicely symmetrical diagram



Thus, here just as in arithmetic, it is because we omit *denominators* that students cannot see the reason "behind" all that we "show and tell" them.

Comments, criticisms and rebuttals are very welcome and should be sent to:

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