Notes from the Mathematical Underground

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As I have mentioned before, my goal in this column is to advocate (provoke?) an analysis and a discussion of the *mathematics* underlying the courses we teach, Basic Arithmetic, Calculus, Linear Algebra, … and of exactly *what* the students get out of it and of *why* they should get it. I think for my part that we take entirely too much for granted but this column is open to any discussion on the topic.

First, as teachers, we know all too little mathematics and what we know we often haven't really thought about. In fact, the late I. N. Herstein advocated the creations of a Ph.D. in mathematical *knowledge* alongside the conventional Ph.D. in mathematical *research*. Briefly, he argued that the research Ph.D. mostly resulted in infinite expertise in infinitesimal areas and that this was quite incompatible with the training of college students. As I recall, he said that the research Ph.D. produced people with no idea of how to teach anything outside their area of expertise, if that. (Hence the need for the "fat text".) On the other hand, the Ed.D. produces people who claim to know all about "how to teach" but have scant idea of what it is that they are teaching. (Hence … .) Predictably, his adjurations had no effect whatsoever. The only example I know of a thesis written along such lines is *A proposed Sophomore-Level Experimental Course in Geometric Algebra Based Primarily on the Work of Emil Artin.* (Judd, 1969) . But even it was more a paraphrase of (Artin, 1957) than the construction of a real course.

Also, mathematics is supposed to be *relevant*—hence the Data Analysis trend I suppose. But *what* data? Is any data relevant simply because it is data? When I was a child in elementary school, in France, we used to have, once a week, a half-hour lesson in *Education Civique*. I am afraid that we didn't take it very seriously but, while perusing some "*… with Applications*", I wondered why there was no application to "Civic Education". I have never seen "political" applications anywhere even if the administration of the *polis* ought to be of some relevance.

I would suggest that we come down from our ivory tower and ask that the students deal, at least once in a while, with "political" numbers and ponder the plausibility and/or the implications of items such as the following all taken last Spring, except for one, from *The Nation*. (Since I do not read *The National Review*, *The New Republic*, or even the *New York Times*, their readers are invited to send their own items to this column.) Given, for instance,

> Phil Gramm calls for the elimination of inheritance taxes. Pat Buchanan wants the exemption from inheritance tax raised from $600 000 to $5 million. Bob Dole would raise the exemption to only $3.5 million.—*January 1, 1996.*

it should not be too hard to find how much one's income tax would have to go up for the Federal Goverment to maintain its revenue at the present level. Similarly, what should be the rate of the much touted flat tax and how would it affect one's own taxes? By the way, oughtn't a really flat tax be *constant* as in "One Man, One Vote" rather than *proportional* (i.e. linear)? The following ought to yield a few good questions:

> A study produced by the Economic Policy Institute finds that in order for those booted from welfare to get jobs without displacing current workers, the wages of all low-income employees will have to fall about 12 percent. The E. P. I. estimates that competition from ex-welfare recipients could cost low-wage Americans about $36 billion a year in income—$8 billion more than the entire federal and state tab for welfare.— *January 1, 1996.*

Here is another relevant item. Given that

> Representative George Miller, in a letter signed by forty-four other Democrats to the ninety-one C.E.O.s of the nation's largest companies who had placed a two-page newspaper ad urging President Clinton and Congress to agree on a seven-year balanced budget, pointed out that, if corporations paid the same share of taxes they did in 1954, the deficit would be gone in one year, with no cuts required.— *January 22, 1996.*

and given the budget deficit, what is the income of these corporations? How does it compare with the aggregate income of all wage earners? How does it affect us? But the following items really point at a gold mine (!):

> By 1992 the super-rich had almost doubled their slice of the nation's wealth, from 22 percent in 1978 to 42 percent. Between 1983 and 1989 the top 20 percents of wealth holders received 99 percent of the total gains, and during that same period the top 1 percent got 62 percent of the new wealth generated during the eighties boom. Between 1989 and 1992 the super-rich 1 percent got 68 percent of all new wealth.— *Alexander Cockburn, January 8/15, 1996.*

> By 1989, the richest one-half of 1% increased their share of the nation's wealth from 24% in 1983 to 29% in 1989… The holding of those 500,000 families were worth $2.5 trillion in 1983. By 1989, they had risen to $5 trillion… The holding of those families grew by almost three times as much as the national debt grew during that same period. In fact, those 500,000 families could have paid off the entire national debt, not just its growth, and still have owned 10% more wealth than they did in 1983. — *Representative David Obey to the Center for National Policy as reported April 8, 1996.*

> Senator Bradley points out that in the twelve years between 1977 and 1989 the richest 1 percent of the population collected two thirds of the increase in personal income during those years. The middle class saw their income rise by 4 percent, while the rich saw theirs rise by 77 percent.— *Alan Ryan, New York Review of Books, March 21, 1996*.

Political arithmetic is clearly a hot potato and not the kind of thing one should embark on unprepared. Indeed, items such as the above ones do not really allow meaningful, pointed questions: Much more data would have to be collected and the original, unimpeachable sources would have to be located. Of course, even prepared to the hilt, one would be due, sooner or later, for some unpleasantness. But isn't it what academic freedom and tenure are all about? As for myself, if I would rather leave the legwork to people more Data Analysis inclined than I am, should anyone come up with usable material, I would want to be on the front line of the field testers.

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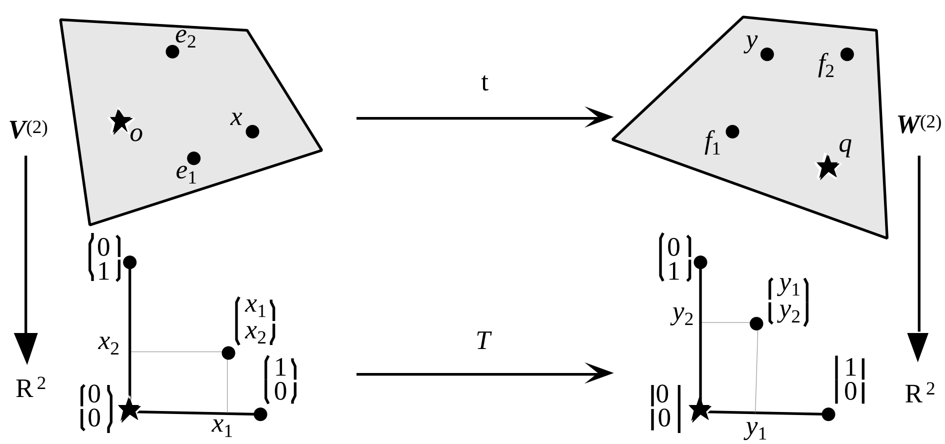
On several occasions already, I have raised the issue of contents architecture. I think that this is a most important issue and I would like to add here a further example of the kind of discussion I think should take place.

Linear Algebra is admittedly a very tough subject to organize *and* to make interesting. One architecture is historically well founded, deriving as it does from what used to be called "Higher Algebra": Start with systems of equations and end with linear transformations. This has of course the merit that, for students thoroughly familiar with systems of at least two or three equations, as was generally the case once upon a time, it can make sense to engage in a careful, detailed generalization via Gauss elimination in matrix form. This is the route taken, inter alia, by (Anton & Rorres, 1994) (and I must admit that I rather liked this opus of the illustrious author of "*In defense of the Fat Calculus Text*") as well as by (Strang, 1980), an author I respect enormously—the introduction alone is worth reading. However, I contend that the only reason this route appears natural is that it is the historical one. (Just as the order which, in arithmetic, makes us teach fractions ahead of decimals and integers.) Then, there is the unfortunate fact that none of *my* students, even the very sharp ones, is well-grounded in the solution of systems of equations.

Much more basically though, this fails to bring out the deep structure of linear algebra and its intimate relationship with the notions of transformation and space which, after all, are conceptually more fundamental than systems of equations. A really natural route would start with the coordinate-free notion of a geometric transformation, $\mathrm{t} : \boldsymbol{V}^{(m)} \to \boldsymbol{W}^{(n)}$, as distinguished, *once bases have been chosen* in $\boldsymbol{V}^{(m)}$ and $\boldsymbol{W}^{(n)}$, from its matrix representation $T : \mathrm{R}^m \to \mathrm{R}^n$:



Then, given b $\boldsymbol{W}^{(n)}$, and to quote Strang, "*(*the) *goal is a genuine understanding, deeper than elimination can give, of the equation* $\mathrm{t}(x) = b$"—as distinguished from the system of equations $\sum T_{ij} x_i = b_j$. What I would like to see is something like *A proposed Sophomore-Level … based on …*", say, (Halmös, 1958). No doubt, this route has structural problems of its own and this should be cause enough for an "exchange" to take place in this space.

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"Critical thinking" is rising to Dow-Jones heights, at least with second graders being asked to "fill in the missing numbers along the path

984 – 129 = ___ – 217 = ___ –

429 = ____ – 104 = ___ – 105 = ____ "

I thought that finding out that

984 – 129 = *1022* – 217 = 1284 –

429 = *959* – 104 = *960* – 105 = *855*

was only a mildly perverse exercise, even if the layout was a bit misleading, but the answer, at least as understood by the teacher, was

984 – 129 = *855* – 217 = *638* –

429 = *209* – 104 = *105* – 105 = *000*

Keep in mind though that both the teacher and the author(s) of the material (xyz, 199n) got the idea from *us*.

References

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xyz. (199n). Problem Solving and Critical Thinking, *Exploring Mathematics*, (Vol. 2, pp. 132): Scott, Foresman and Company.

ADDED IN PROOF. Political Arithmetic seems to have become popular of late. See for instance *America: Who Stole the Dream?* a series published in *The Philadelphia Inquirer* from September 8 to September