Notes from the Mathematical Underground

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The "Calculus Initiative" turned out to be one more sad instance of the collective inability of college professors to understand what it means to *learn* mathematics. Consider, for example, (Zucker, 1996): To summarize, high school is OK. We, college professors, are OK. Students are not OK but redeemable if they would just apply themselves:

"The fundamental problem is that most of our current high school graduates don't know how to *learn* or even what it means to learn (a fortiori to understand) something. In effect they graduate high school feeling that learning must come down to them from their teachers. That may be suitable for the goals of high school, but it is unacceptable at the university level. *That the students must also learn on their own, outside the classroom, is the main feature that distinguishes college from high school.* (emphasis in the original)

The answer is "*so obvious that it is embarrassing* […]:

"It is possible to get college freshman to learn calculus fairly well, without resorting to utopian tricks such as enforced group projects. All we have to do is get the student to accept that learning is something that will take place mostly outside of class; that is *just insist that they grasp the underlying premise of college education*."

Specifically,

Students must be told *immediately* that they are about to face a big jump in level from high school. Most high school teaching is justifiably set to the needs of the least talented students in the class; the better students often become convinced by habit that this level is right for them too. They should be helped to recognize that the change is both appropriate and manageable.

My reason for taking issue with this particular article is because of where it appeared and because it quite embodies much of what is wrong with mathematical academia, including two-year colleges. The reasoning in the article is based on three, more or less explicitly stated, Uniqueness Theorems:

- Calculus is a corpus of knowledge one and indivisible, like the French Republic.
- There is one and only one way to learn it—the article does not disclose it—and one and only one way to "teach" it—by clearly presented, ex cathedra lectures. In fact, "*the instructor's job is primarily to provide a framework, with* some *of the particulars*, *to guide* [the students] *in doing* [their] *learning of the concepts and methods that comprise the material of the course*." The student is directed

"*to read the* [fat?] *textbook for comprehension. It gives the detailed account of the material of the course.*" Etc.

- There is one and only one kind of student. "*Of course, the way in which mathematics education in any particular college or university can be improved depends on the composition of its student body. The overall theme should be the same though.*"

This is overwhelming and impossible to rebut here. To illustrate what I consider to be the fundamental problem, and since it is universally acknowledged to be one of the tasks with which the students have the greatest difficulties, let me take graphing: In none of the texts I have consulted is there a mention that it is an *investigation*, driven by *logic*. Invariably, students are given a set of instructions to be followed, more of less blindly. For example, (Stewart, 1987), even though he invokes Polya in the preface, gives a three page "Checklist of Information for Sketching a Curve $y = f(x)$". Fortunately, it is reduced by (Anton, 1988), in the case of $f(x) = P(x)/Q(x)$, to the following "How to":

Step 1. Find the x-intercept of $P(x)$. At these values we have $f(x)=0$, so that the graph intersects the x-axis at these points.
Step 2. Find the x-intercept of $Q(x)$. At these values, $f(x)$ approaches $+\infty$ or $-\infty$, and the graph has a vertical asymptote.
Step 3. Compute $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. If either limit has a finite value L, then the line $y = L$ is a horizontal asymptote.
Step 4. The only places where $f(x)$ can change sign are at the points where the graph intersects the x-axis or has a vertical asymptote. Calculate a sample value of $f(x)$ in each of the open intervals determined by these points to see whether the graph is above or below the x-axis over the interval.
Step 5. From $f'(x)$ and $f''(x)$ determine the stationary points, inflection points, interval of increase, decrease, upward concavity, and downward concavity.
Step 6. If needed, plot a few well-chosen points and determine whether the graph crosses any of the horizontal asymptotes.

Why not just plot? But these authors are of the old school. So, let us turn to some "Revitalized Calculus". On page 81 of (Hughes-Hallett et al., 1994), we read

> Rational functions are those of the form $f(x) = p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials. Their graphs often have vertical asymptotes where the denominator is zero. If the denominator is nowhere zero, there are no vertical asymptotes. They may also have horizontal asymptotes which occur if $f(x)$ approaches a finite number as $x \to \infty$ or $x \to -\infty$.

This is followed by the example $y = \dfrac{x^2 - 5}{x - 1}$, in which the behavior at infinity is obtained by a table of powers of 10 but the behavior near 1 is not investigated. After that, an admittedly brief search does not turn up any graphing. Similarly, (Ostebee & Zorn, 1995) just gives a definition and the example $r(x) = \dfrac{x^2}{x^2 - 1}$.

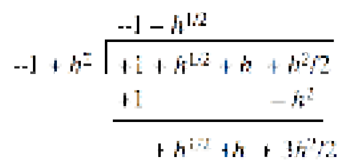
Well, I suppose that the investigation of rational functions is already going the way of Greek and Latin. To show why I deplore this disappearance, let me give an example of what I ask from my students after they have gone through a "toolbox" in which they see how all the local properties of a function can be read off the appropriate terms of its polynomial approximation at that point. Investigating a function then consists in finding out the extent to which global features can be derived from local ones. We start with polynomial and rational functions, but I will take an example which occurs later on in the course.

Investigate the function $f(x) = \dfrac{\sqrt{x} + e^x}{x^2 - 1}$.

The students are supposed to go through two phases: The investigative phase in which they can cooperate and the writing of a report which they have to do alone. *I only get to see their report*. Here is *my* report. I use, in lieu of little ohs, the symbol (…)—pronounced "a little bit", as in $\frac{1}{3} = 0.33 + (\ldots)$.

Since f involves a root, we must first determine the *allowable* inputs. Since $\sqrt{x}$ is not defined for negative inputs and since this is the only restriction, all inputs from 0 on are allowable.

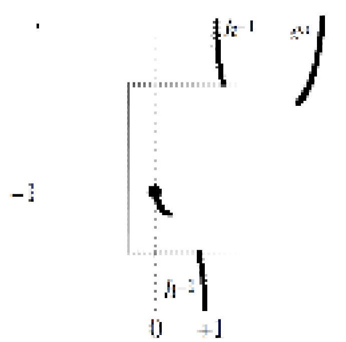
At 0, we have $f(0) = \dfrac{0 + e^0}{0^2 - 1} = \dfrac{+1}{-1} = -1$. It is worthwhile to find the local graph near 0.



We have $f(h) = \dfrac{h^{1/2} + 1 + h + h^2/2 + (\ldots)}{-1 + h^2}$

$= \dfrac{1 + h^{1/2} + h + h^2/2 + (\ldots)}{-1 + h^2} = \dfrac{+1 + (\ldots)}{-1 + (\ldots)} = -1 + (\ldots)$

but, to get variation and concavity, we must divide *before* we approximate. We find $f(h) = -1 - h^{1/2} + (\ldots)$. So, right of 0, the graph starts straight down like the graph of $-\sqrt{x}$.

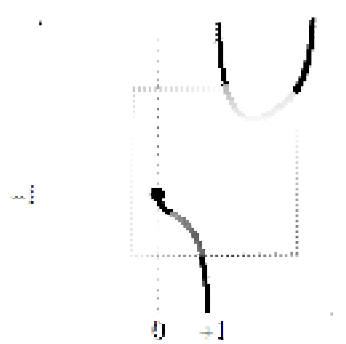


We continue by looking at the graph "outside the window". For large *positive* inputs: $f(x)_{+\infty} = \dfrac{e^x + (\ldots)}{x^2 + (\ldots)} = e^x + (\ldots)$

We now must see if the graph could escape from the window through the top or the bottom. In other words, we must see if f has poles. $x^2 - 1 = 0$ gives -1 and $+1$ as possible poles but -1 was ruled out by $\sqrt{x}$. We look at the graph near +1:

$f(1+h) = \dfrac{\sqrt{1+h} + e^{1+h}}{(+1+h)^2 - 1} = \dfrac{1 + h/2 + (\ldots) + e[1 + h + (\ldots)]}{1 + 2h + (\ldots) - 1}$

$= \dfrac{e+1 + (e+1/2)h + (\ldots)}{2h + (\ldots)} = \dfrac{e+1 + (\ldots)}{2h + (\ldots)} = \dfrac{e+1}{2}\, h^{-1} + (\ldots)$



Altogether then, there must be an *even* number of (finite) zero points, an *odd* number of (finite) turning points and an *odd* number of (finite) inflection points. To find out if there might be fluctuations*, we should compute $f'(x)$ and solve $f'(x) = 0$ which may or may not be feasible.

* Pairs of opposite (finite) turning points.

Again, this is *my* report and I do not expect the students necessarily to write like that. Moreover, I agree with Zucker that, initially, the students have no idea of what I want. I keep telling them that they have to make a case as if they were lawyers in front of a jury and that I will act the attorney on the other side, challenging what they assert. I demand that they write what they are going to do and why. In English. Otherwise I will not read them. I do end up getting readable reports which I then take apart. Of course, I have to read quite a number of them before they shape up. But eventually they do. And, that *is* doing mathematics.

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Reporting on $f(x) = \dfrac{e^x}{x^2 - 1}$, a student once included a spreadsheet (Excel) plot that disagreed with her graph (Figure 1). I was intrigued and indeed found that Excel plots are quite unstable as they change *qualitatively* depending on very small incremental changes. Looking at [–25,+30], Excel shows an exponential: $f(x)$ equals 0 until x = 25 and then moves rapidly to 20,000,000. The poles don't show at all. Looking at [–1.5,+1.8] by 0.05 increments, I got the plot in Figure 2 which, since +1 is an odd pole (multiplicity 1), I found a bit … odd. But then I tried 0.06 and, to my vast relief, got a plot I could believe in (Figure 3) even though it too gave no hint of $f(x)$ going to infinity as x goes to $+\infty$.

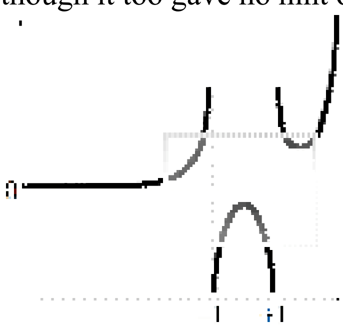

Figure 1

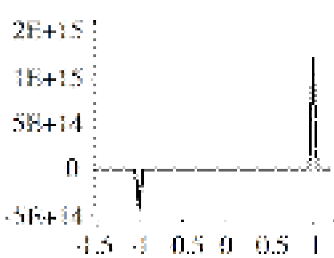

Figure 2

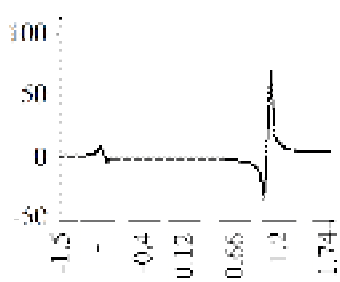

Figure 3

Lesson #1: No such thing as a *global quantitative* graph. Lesson #2: So much for Excel. Lesson #3: Graphing *qualitatively* with little ohs is nice.

While on the subject of high tech, and not to appear prejudiced, let me give an example of what I think certainly justifies its use. Back round the turn of the century, Poincaré showed that a lot of information could be obtained about the solution of a differential equation by looking at its phase portrait, a parametric family of curves. The trouble of course was that, apart from a few trivial equations, phase portraits were all but impossible to get in finite amounts of time. With the advent of specialized applications though, the situation has now changed to a point where the standard course in ODE, in which the goal is only to *obtain* solutions in closed form or series solutions, can be replaced by a course in

Dynamical Systems in which one learns to *investigate* the behavior of the solution. See for instance (Artigue & Gautheron, 1983; Hubbard & West, 1991). Thus, here is a case where technology has allowed a complete rethinking of a course, changing it from empty formalism to useful meaning. Unfortunately, I don't know any other.

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