Notes from the Mathematical Underground

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> "There is not much we can do about that, after all"
> "But don't you see? We act in stupid and shortsighted ways and then we behave as if we didn't have any responsibility for those actions. Somehow that justifies our continuing to behave in the same shortsighted ways. Instead of trying to change, we hope it works better this time."
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> Vonda N. McIntyre, *Starfarers.* Ace Books 1989

Given that the mathematical content of the best selling textbooks is rapidly approaching 0, and the popular consent that this entails, I should perhaps not be too surprised if, to the best of my knowledge, these Notes have had no more than two readers. Yes, there may be some hurt author's pride here. But don't you see? …

To take an example, consider Precalculus: For the remaining few of us who do not think that Precalculus is yet another name for Basic Algebra, it's sole raison d'être is that the Bolzano-Cauchy-Weirstrass embodiment of the Calculus is unlearnable without an a priori, fair measure of "mathematical maturity". But even *we* wind up "previewing" in the context of Basic Algebra the very questions the Calculus was invented to discuss and, since we are profoundly "answer oriented" (but that in itself is another story), pretending to answer them. In any case, apart from being fundamentally dishonest, what this must surely do is to kill in our students any need and/or desire to learn the Calculus: Why should they bother when they already have the answers? The pious *theory* is that the given answers are only temporary ones, intended to be improved upon in Calculus. The sad *reality* though is that the overwhelming majority of the students successful in Precalculus never even attempts Calculus. You will say: "There is not much we can do about that, after all … it's their choice. Instead of trying to change, we hope it works better this time … with technology and applications." But, when we refuse to do away with a Precalculus whose avowed goal is to deflower if not rape the Calculus, when we refuse to redesign it in a way that makes sense to students right out of Basic Algebra, aren't we reneging on our promise to bring *mathematics* to the great unwashed masses ?

As it happens, already back in 1797, a time when mathematicians "*were more interested in the formal use of infinite processes than in their rigorous proofs* [and whose] *results are very striking*" (Copson), Lagrange had designed an approach, explicitly for pedagogical reasons, to free the differential calculus from "*any consideration of infinitesimals, vanishing quantities, limits and fluxions and reduce it to the* algebraic *study of finite quantities*" (Lagrange, 1797). In other words, Lagrange's approach to differential calculus requires only Basic Algebra. Moreover, as I showed in the Spring 98 Notes, because proofs in this context make basic algebraic sense, it gives "just plain folks" a sense of *local coherence.*

Here I would like to discuss in some detail its other immense advantage, namely the "story line" it allows and the sense of *global coherence* it brings. As one of the authors of the "Harvard Calculus" once put it—but now appears to have entirely forgotten, the conventional approach to Calculus "*fails to put the proper emphasis on what the subject is all about, namely functions of a real variable*" (Gleason, 1967)]. And, a function being given in even the simplest manner, namely by a rule giving the output in terms of the input, the problem that immediately confronts us is how to *picture* it. But, if we invariably begin by saying that a plot can never, ever determine a function—we may even give Strang's famous example of cos n, $1 \leq n \leq 1000$ at this point, we then proceed to do just that and may even require the students to get graphing calculators to cover up the dastard deed.

And that is precisely “it”: What information about a function f do we need to turn a plot into a *global quantitative* graph. Since *local qualitative* graphs are easy to get, the issue boils down to how to assemble them into a *global qualitative* graph to serve as a guide in joining plot points smoothly. Specifically, how do we: 1) *interpolate* local graphs into a *proximate* graph (i.e. the part of Graph f that would show say on a computer screen) and 2) *extrapolate* them to get Graph$_{\infty}f$ (i.e. the part of Graph f that would fall outside said screen) and, in particular, how do we get Graph f_{∞} (i.e. the part of Graph f for which x is near ∞)?

It is fairly natural to want to proceed "inside-out", that is first to construct a proximate graph from which then to infer Graph f_{∞}. However, aside from the rapidly increasing difficulty of obtaining the *critical* points (among which the *change* points are to be found), the simplicity of this rather Ptolemaic viewpoint is somewhat illusory if only because most proximate change points are *non-essential* in the sense that they can be smoothed out. Take, for instance, the graph in Figure 1 of a polynomial function of degree 4 with three turning points:

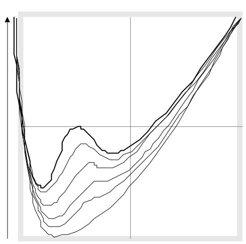

Figure 1

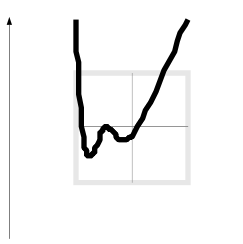

Figure 2

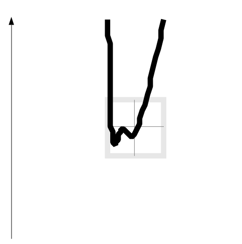

Figure 3

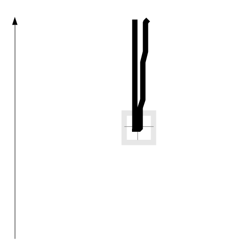

Figure 4

The middle turning point can be merged with either the right-hand turning point, as in Figure 1, or with the left-hand one.

On second thought then, we take a more Keplerian viewpoint, that is an "outside-in" approach in which we attempt to infer the proximate graph from Graph$_{\infty}f$. An *essential* feature then is one that is visible from infinity. For instance, Figure 1 might be the result of zooming in from Figure 4 in which the only visible finite turning point is the counterpart of ∞ being a turning point and with the other two appearing as a *fluctuation*.

Since both the way we expand a function f and the way its local graphs relate to its global graph depend on the nature of f, the differential calculus reduces to the investigation of the interplay between *local* and *global* analyses in the successive cases of progressively more complicated types of functions. (It is of course no accident that this progression parallels that of the number systems—positive-powers of ten, whole numbers; negative-powers of ten, decimal numbers; rational numbers; irrational numbers; transcendental numbers—and their decimal expansions offer a constant source of inspiration.) We shall find that: **i.** All functions (that just plain folks are ever likely to encounter) are *almost polynomial almost everywhere* (i.e. their expansion, except possibly near infinity and a few finite points, is always a polynomial plus a small remainder) and **ii.** It is in fact the behaviour near these *exceptional* points that *essentially* determines the behaviour everywhere else.

Starting with x^n which we will take as gauge functions, when $n > 1$, all that matters is that x^n takes large values near ∞ and small values near 0 (0 is a zero and ∞ is a pole) and that their order of magnitudes is defined by a comparison theorem (Spring 98 Notes). But x^0, because it lacks variation and concavity as well as because it cannot take large values near ∞ and small values near 0, is completely *pathological*. On the other hand, x^1 is pathological only because it lacks concavity. (Which is what makes affine functions at once very useful and very unrealistic.)

The next step, as usual, is to look at linear combinations. When the degree of the polynomial function P_n is *low* ($n \leq 3$), we find that we can extrapolate the global graph from the local graph near just *one* finite point. This can be any old point in the case of constant and affine functions but, in the case of quadratic and cubic functions, it has to be the point where the function's behaviour is the dual of its behaviour near ∞: the turning point for quadratic functions and the inflection point for cubic functions. The problem is how to generalize this to *higher* degrees: Will the global graph of $P_n(x)$ be controlled by its change points of rank $n–1$ (i.e. transversal solutions of $P_n^{(n-1)}(x) = 0$)? Or perhaps by those of rank 1? Or rank 2? What remains true is that infinity controls the *essential* proximate behaviour and, essentially, polynomial functions are nothing but power functions with a few fluctuations thrown in.

Negative-power functions simply dualize the positive-power functions: 0 is a pole and ∞ is a zero. That the duality is explicited by $x \to 1/x$ "explains" why x^0 should equal 1 since, in arithmetic, it is 1 that separates "small" from "large".

At this point, it would be natural to study Laurent polynomials (linear combinations of *integral* power functions) but this would be … unconventional and we move on to rational functions which we find to be approximately polynomial everywhere except, because positive-power functions are not closed for division, near their poles and, when d° Numerator ≤ d° Denominator, near infinity. In both these cases they behave like negative-power functions and the essential behaviour is now controlled by the finite poles as well as by infinity. *Finite* points can thus control the global behaviour but only if their output explodes off screen. Which is why acting as we do in Precalculus as if zeros (i.e. critical points of rank 0) were control points is, to put it as gently as possible, very … "misleading". Note that *even* poles are turning points (Figure 5) and *odd* poles are inflection points (Figure 6). Also, since they are only *affine* approximations of $\text{Graph}_\infty f$ and thus lack concavity, *asymptotes* cannot control the essential graph (Figures 7 and 8) so that their importance is quite overrated.

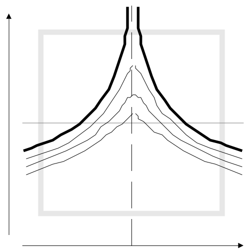

Figure 5

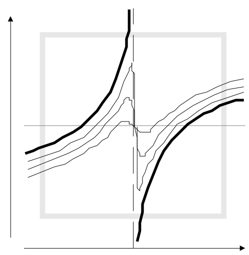

Figure 6

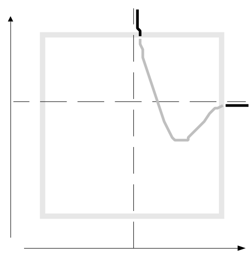

Figure 7

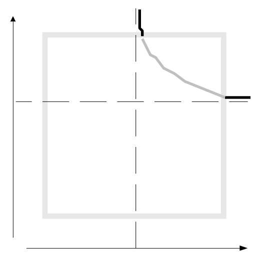

Figure 8

With fractional-power functions, defined by $[f(x)]^{q} = x^{p}$, the Ptolemaic passage from local to global becomes more subtle. When p=1, q=2 for instance, $\text{ROOT}_2(x_0+h) = \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}\, h - \frac{1}{8x_0\sqrt{x_0}}\, h^2$ gives that, except near 0 and ∞, ROOT is approximately quadratic everywhere and all local graphs look essentially the same. But then, while the global graph in Figure 10 is certainly compatible with the local information sampled in Figure 9, it is certainly not obvious that it should be the only possible one. And then there remains the question of the behaviour near +∞.

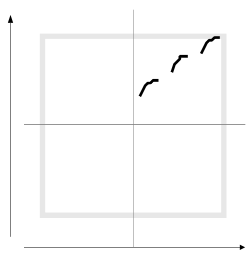

Figure 9

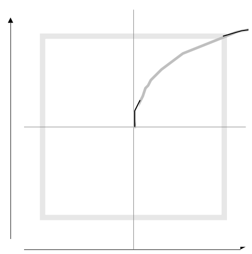

Figure 10

In the Kepler view however, $[f(x)]^{2} = x$ gives $\text{ROOT}(0^+) = 0^+$, $\text{ROOT}(+\infty) = +\infty$ and $f'(x) = \frac{1}{2f(x)}$ gives $\text{ROOT}'(0^+) = +\infty$, $\text{ROOT}'(+\infty) = 0^+$ and $f''(x) = \frac{-f'(x)}{2[f(x)]^2}$ gives $\text{ROOT}''(0^+) = -\infty$, $\text{ROOT}''(+\infty) = 0^-$ and thus the essential graph in Figure 10. (In fact, since $f'(x) = 0$ only near +∞, ROOT can have no fluctuation.) Note that the duality between 0 and ∞ continues to hold.

An additional advantage is that we can handle the exponential[1] in just the same manner as above. We find that EXP is approximately polynomial everywhere except near infinity and, once we show that EXP(x) can never be 0, that all local graphs look essentially the same. Again, the usual graph is compatible with the local information and we may or may not want to establish that it is the only possible one. To prove that EXP behaves near +∞ like a super positive-power function (in that it beats all positive-power functions to +∞) and near –∞ like a super negative-power functions (in that it

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[1] Note that by defining EXP as the inverse of the integral of the reciprocal of the identity function and therefore relagating it to Calculus Two, as is usually done, it is precisely those students who can least afford it whom we prevent from studying the exponential.

beats all negative-power functions to 0^+) is more difficult but still quite doable. See (Lang, 1976). The duality between x^{+n} and x^{-n} thus extends to that of EXP(x) and EXP($-x$). Although the duality between 0 and ∞ is somewhat broken, we find again 1 as "midpoint" between 0 and ∞. We study LN similarly as solution of $xf'(x) = 1$ and composites such as $f(x) = \frac{\sqrt{x} + e^x}{x^2 - 1}$ can be investigated quite simply (Spring 97 Notes).

COS and SIN exhibit yet another type of behavior as, near infinity, they can take any value between –1 and +1 repeatedly. But, because they inherit the symmetries of the circle, we can reduce their investigation to the interval $[0,\pi/4]$ in which t is small enough that we can use the polynomial approximations $\text{COS}_n(t)$ and $\text{SIN}_n(t)$.

To conclude: First, there are of course many simple topics, e.g. partial fractions, that I didn't even mention here but, perhaps by now not unsurprisingly, even they fit the above framework quite naturally. Second, how should the story … end? Traditionally, this is about where we would be about to move into integral calculus but it is tempting to return instead to the real world with *damped* oscillations and thus end the course with a brief introduction to second order differential equations with constant coefficients! Note that this is getting to be the fashionable thing to do … in Calculus Two (but only to provide an alternate approach to EXP). Finally, and as I have mentioned before, this approach is the ideal preparation for a course in Dynamical Systems that, for most students, would be much more appropriate follow up than Calculus Two.

References

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