Radically Down Below: The Bottom Line

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> *Mathematics is neither accounting nor the theory of relativity. Mathematics is much more than the sum total of its applications no matter how important and diversified these may be. It is a way of thinking.*
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> Einar Hille, *Analysis, 1964*

Mathematics, somewhat like the Stock Market and, I think, very much for the same reasons, has gone through a rather incredible bull period. I recall my bewilderment when, fresh to these shores, I saw the pile, negligently piled up in a corner of the departmental office, of letters to the Head of the Mathematics Department at Penn begging him to suggest someone who might be willing to fill a desperately open position. This was in 1965 and it was not going to last and, some twenty-five years later, by then on the Hiring Committee at Community College of Philadelphia, I would receive over 300 applications a year and, except for maybe half a dozen, all with a Ph. D. in Mathematics.

Back in those days, mathematics was a requisite, if not a prerequisite. There was even Mathematics for Liberal Arts students. You *had* to "have" a year of calculus to apply to Medical, Law or Business School. *We* had it easy. Then came the lean years and the even leaner years and, I would argue, this is where we went completely wrong.

It is a bit like today's Democrats complaining that the Republicans legislate through the Supreme Court when they did the same when they had the chance. Similarly, everybody wanted us to screen their applicants. They didn't say it in so many words and just said that their students needed mathematics. Even though they did not bother to say why, we pretended to believe it and we took the job simply because it opened so many positions. Ph.D factories went in overdrive. Now, after having repeated for thirty years that mathematics was *useful*, that you *needed* mathematics to get ahead in life, we find that more and more of our customers are leaving us as they are finding out that, whatever they need, it ain't us. If Mathematics Departments have not yet begun to shrink it is only because we do not have to retire any more. We can die in the classroom, if not necessarily on our feet. Give it another few decades and Mathematics will have returned to its status of esoteric scholarship. There will be as many Abstract Algebra scholars as there are today of Ancient Greece. And students will learn again mathematics just because they feel like it, not because they are told they have to.

This did not have to be. We did not have to sell mathematics as something useful, the latest incarnation of which being Data Analysis and Modelling. After all, *we* like mathematics. So, why shouldn't most everyone? Mathematics is a wonderful world. It has both logic and beauty. It is the ideal environment in which to learn how to make convincing arguments. And everybody likes to compete and win. Or so I am told. So, why only on the Basketball court or the Baseball and Football fields? Could it be because, in these times of Absolute Relativism, "you may be right but I already made up my mind"? Could the First Amendment be really responsible for this? I don't think so.

In my continuing effort to demonstrate that the mathematical *contents* of what we teach are both ignored and useful … to their understanding, as well as interesting of and by themselves, let me use here the Fundamental Theorem of the Calculus, better known—at least by students—as the Fundamental Obfuscation Theorem. What is never said about it is that it is a Conservation Theorem. Could it be because that would make it obvious?

The Fundamental PI Theorem says that what enters but does not exit a house under surveillance remains in the house. Of course, one-dimensional spaces are hard to visualize even if, if we usually do not introduce two points as our first example of a circle, we often *do* use fields as first example of a vector space. Nevertheless, it *is* Stokes' Theorem and, if nothing immediately useful to the great unwashed masses, the fundamental truth about the Universe that it encapsulates—that nothing ever gets created or lost—is something that everybody ought to be deeply aware of and completely familiar with.

Since everybody is familiar with money, let me use it as pedagogical situation[1] and let us consider a bank account. At any given time, the account is in a state, the Balance, and we have a function of time telling us what gets transacted, i.e. deposited in or withdrawn out of the account. The sum of these transactions during any given period must obviously be equal to the difference betwen the Balances at the end and beginning of that period. You might call this the Fundamental Theorem of Accounting. A more complex avatar of this is that ΔPosition = $\sum$Net Income or, more explicitly, that Δ[Assets – Liabilities] = $\sum$[Incomes – Expenses][2].

ΔBalance does not, of course, pose any pedagogical problem. The difficulty is in measuring $\sum$Transactions in the continuous case. If the Differential Calculus can be said to be about solving the problem of interpolating-extrapolating plots into (global) *quantitative* graphs by *localizing* $f(x)$ into $f(x_0+h)$ and then assembling *local* graphs into *qualitative* (global) graphs[3] then, in the same manner, the Integral Calculus can now be said to be about the problem of interpolating-extrapolating (discrete) sums $\sum x_i$ [4].

Since any measure must be additive relative to intervals and sandwiched between the inscribed and the circumscribed rectangles **[1]**, **[2]**, **[3]**, approximating f by a piecewise constant function then gives a zeroth approximation of the integral and approximating by piecewise affine functions gives a first approximation. After that, we can go either one of two routes: we can either compute second degree approximations with Archimedes theorem, third degree approximations with Cavalieri's theorem, etc or we can stick with piecewise constant functions and

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[1] I am decidedly *not* modelling anything but I *am* a firm believer in the pedagogical virtues of Model Theory and Gödel's Completeness Theorem. More about this in some later column.

[2] Accountants, though, prefer to have a Liability called Retained Earnings = Previous Retained Earnings + $\sum$[Incomes – Expenses] so that Assets = Liabilities at any time. But I am not into "Applications".

[3] More about this in some later column.

[4] into $\int dx$ but while, at the time these statemens are first made, the students (think they) know what a graph is, other than AP students, they usually do not even pretend to know what an integral is.

diminish the length of the increments until they become *infinitesimal* which we can do either à la Euler or à la Robinson.

All of this raises the issue of how much rigor is necessary at this *pedagogical* stage. Those who, agreeing with e.g. McLane, would absolutely insist on absolute rigor at any given stage will of course be suitably appalled by the above. Still, if we will not deal in Voodoo Mathematics, there is room for a discussion here.

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VOODOO MATHEMATICS. A subject to be found somewhere between Mathematics and Mathematics Education in which mathematical statements are proven in painstaking details above and beyond the understanding of "just plain folks" who are then required to follow hermetic procedure by mimicking given examples in order to solve supposedly practical problems. Jack Vance **[4-6]** has shown that Voodoo is a quasi-recipe for disasters.

Here is an example from **[7]**. After the statement and proof of the theorem, we find two examples and 22 exercises in which to evaluate definite integrals and 5 exercises in which to find the area under the curve. Homework: 1) What's the relation between the two? 2) What are the two examples examples of?

References

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