Let's Face It!

No one can be as fatuous as a faculty. Take me for instance. I kind of liked my latest column. I mean my first column. I expected to get a lot of mail. Say at least two, maybe three irate letters. Of course, not a one! Well, I will just take it for granted that you agree with me that mathematics is useless.

Another myth I wanted to debunk is that mathematics is hard. It depends of course on what mathematics we have in mind. It also depends on what mathematics we already know. It mostly depends on how we know the mathematics we think we know. But hard we make it in any case!

Take limits for instance. No. Wrong column. Take just about any topic in elementary mathematics, say how numbers are usually introduced and dealt with. I find the usual treatment far more than just plain abysmal.

Let's start with whole numbers. Here, I want to stay away from semantic considerations other than to say that we use numbers only because it is more convenient to write 6 than to write . (The bite (byte?) has been taken by Apple Computer Inc. I cannot help it.) To define 6 as 1+1+1+1+1+1 just makes no sense.

But, for now, I want to concentrate on the syntactic aspect. What is wrong there is that to divorce whole numbers from decimal numbers is not only totally uncalled for but in fact makes everything harder. The point is just that we have only so many symbols at our disposal. Let's say that we have the usual TEN symbols (but with small children I normally start with THREE). But since we cannot represent collections consisting of more than 9 entities of a kind, the only solution is to change any TEN entities of one kind for one entity of another kind.

To illustrate this, let's use the following sequence of well known entities (money is the only decimal system in this country.): Clevelands, Franklins, Hamiltons, Washingtons, Dimes, Cents, Mills, for which the rate of change is TEN for ONE. Notice that the distinction between whole and decimal numbers just vanished.

It is obvious that 7.254 Hamiltons is the same as 725.4 Dimes if we say that the unit refers to the digit that is ... pointed at!

Say now that Jane has 5 Hamiltons, 6 Washingtons, 4 Dimes and that Dick has 7 Washingtons, 8 Dimes. Together, they have twelve dimes so that they must change TEN of them for one Washington. They now have fourteen Washingtons and must change TEN of them for one Hamilton. Altogether, they now have: 6 Hamiltons, 4 Washingtons, 2 Dimes. Could anyone out there explain what got "carried over"?

If Jane has 3 Hamiltons and 5 Dimes and she must pay 7 Dimes she would have to change one Washington for TEN dimes but as she does not have any Washington, she must first change one Hamilton for TEN Washingtons. She can then change one of the Washingtons for TEN dimes so that she now has fifteen dimes from which to pay the seven dimes that she owes. She is left with 2 Hamiltons, 9 Washingtons, 8 Dimes. Could anyone out there explain where the "borrowing" occured?

Of course the above should be experienced. But I claim further that this idea of change subtends a large amount of mathematics which we teach as disconnected and disjoint topics. It certainly organizes and unifies arithmetic and, therefore, elementary algebra

which, by the way, I claim, should be learned together with arithmetic. But I will leave this for another column so as to give you time to first think about it.
Notice that, to introduce the metric system, I only have to translate the names of the entities to Kilo$, Hecto$, Deka$, $, Deci$, Centi$, Milli$. Could anyone out there explain what happened to these conversion factors we ram down the students' throats?
We can even get in the same way the exponential notation or even plain English. Thus, for instance, 47.2 Kmiles is synonymous with 47.2 • 10+3 miles and 47.2 Thousand miles.
But, you are saying, what does this have to do with whether or not mathematics is hard? Observe that all we have to memorize here is the sequence of the units' names and, if you want, that the unit refers to the digit being pointed at. For instance, to introduce scientific notation, just define it as the form in which the unit refers to the leftmost non-zero digit. The rest follows logically and should be left to the student.

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Another way in which we make things much harder than they are is by divorcing whole numbers from signed numbers. Of course this has the triple advantage of confusing the students endlessly, of making us therefore endlessly necessary and of allowing us endlessly to lament our students' ineptitude.
Consider the way you think of negative (whole) numbers. Here is a typical example: You have 5 dollars and you lost 3 dollars. How much do you now have? Looks trivial doesn'it? How about you have 3 dollars and you lost 5 dollars? This one will let you to do a lot of hand waving.
The problem is that the two summands are of a different nature: the first one is an initial state (you have x dollars) on which the second summand acts as an operator (you lost y dollars) which results in a final state (you now have z dollars). This sure is not the easy way to present an internal binary operation.
What you really should do is to compose operators: you have an undisclosed number of dollars as an initial state, you win (or lose) x dollars, you then win (or lose) y dollars which gets you to the final state. The question then is: what single winning (or losing) operator would get you to that same final state?
One advantage is that now the intuitiveness of the example does not depend on what the respective values of x and y are. But a more powerful advantage is that addition is now an internal operation, that is one in which the nature of the two entries is the same.
In fact, we should write +(–5$) +(+3$) +(–4$). This way, we can read the outer + as a unary operator: enter a 5$ loss (withdrawal, etc), enter a 3$ gain (deposit, etc), enter a 4$ loss. One reason is that we can then read –(+3$) as remove a 3$ deposit. It is fairly easy to make the students realize that since they cannot remove an entry (it is in fact illegal for an accountant to do so), the only way to straighten out the balance is by entering the opposite of the entry that was to be removed (in accounting terms it is called an adjusting entry).
A second advantage is that multiplication is neatly distinguished from addition.
But first, let's face how you define multiplication of whole numbers. As an additive power, right? You say that 2 • 3 is just another way to say 3 + 3, right? And it generalizes nicely too: 2 • 1 third is just 1 third + 1 third. Great. How about 2 thirds • 3 fifths? (By the way, how about 2 feet • 3 feet? Oh well, that's advanced mathematics and

anyway the bell just rang. Of course, the next class will start with: "The rule for multiplying fractions is ..." . But did you notice that, if you think of fractions as operators, then it is quite easy to figure out what the equivalent fraction is? Hint: just find a nice state to start with. Like 60.)
Thus we must again distinguish multiplication as unary operator on states such as "in doubling one's investment" from multiplication as binary operation on unary operators as in "doubling followed by tripling equals sextupling". We should read (–5)(+3)(–4) as: change the sign of the state to the opposite and quintuple it, then keep the sign of the state as it is and triple it, then change the sign of the state and quadruple it. This amounts to: change the sign of the (initial) state and multiply by 60.
An even more powerful advantage is that we can start from the outset with signed decimal numbers.
That does it. This goes against a pedagogical experience of some two thousand years. Well, at least two hundred years. How does he want students who already have a hard time with whole numbers to be able to understand signed decimal numbers from the outset? This guy writes complete nonsense! And he has tenure? What a disgrace to the profession!
Did anyone out there try it? The Greeks, Peano and Dedekind notwithstanding.
Recently, we had an opening in the mathematics department and, to obtain a short list from which to decide whom to interview, we asked the candidates to answer a questionnaire. Two of the questions were:

As part of one's course load at CCP, one is likely to teach remedial arithmetic. One area where many students have difficulties is subtraction of signed numbers. What do you think the nature of these difficulties are? How might you approach teaching this?

How can our having learned higher mathematics, e.g. abstract algebra, help us teach elementary courses, e.g. basic algebra?

I leave the answers to your imagination.

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How come we have so little faith in the strength and elegance of mathematics that we can only resort to the usual sugar coating dismal mean little tricks? Might this have anything to do with the following which was quoted in a recent article [1]:

"In Russia, my colleagues and I liked to complain about a deteriorating level of mathematical preparedness of high school graduates, but I had to come to this country to see what I could not imagine in my wildest dreams. ... When I talk before conventions of high school principals and show them standard math textbooks used all over the USSR ... they are angry with me. They tell me that it is a scientifically established fact that no more than [the] 5% of 'mathematically gifted' kids can study such stuff. Well, everybody in Russia who is not medically certified [learning disabled] masters these texts."

Of course, since it's a scientifically established fact, what is the author arguing with those principals about? What am I writing about?

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1. H. Clemens. Is There A Role For a Mathematician in Math Education? Notices of the AMS. 36(5) (1989) 542-544.

Exercises. 1. Why can't apples and bananas be added? Compare 3 + 2 with 3x + 2y.
2. If an apple changes for 23 ¢ and a banana for 12 ¢, for how many cents does 3 + 2 change for?
3. Compare with 3 quarts + 2 fifths. You may or may not change them for ¢, eggs or even twentieths.
4. Why is the bottom number in a fraction called the denominator and the top number the numerator?