MATH **161** EXAM II Discussions

[Run: 03/30/2013 at 13:37 Seed: 6746. Order of Checkable Items: Random.]

II-1. Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = +x^3 - 3x^2 - 9x + 7$$

What is Slope-sign of f near -1?

Discussion: "What is Slope-sign of f near -1?" is a *local* question. The local input-output rule near -1 is:

$$-1 + h \xrightarrow{f} f(-1+h) = +(-1+h)^3 - 3(-1+h)^2 - 9(-1+h) + 7$$

$$= \Big[\quad\Big] + \Big[\quad\Big]h + \Big[\quad\Big]h^2 + \Big[\quad\Big]h^3$$

Slope-sign near -1 comes from the sign of the coefficient of h:

- The number of hs coming from $(-1+h)^3$ is $3(-1)^2 = +3$
- The number of hs coming from $-3(-1+h)^2$ is $3(2(-1)) = +6$
- The number of hs coming from $-9(-1+h)$ is -9

$$= \Big[\quad\Big] + \Big[+3+6-9\Big]h + \Big[\quad\Big]h^2 + \Big[\quad\Big]h^3$$

$$= \Big[\quad\Big] + \Big[0\Big]h + \Big[\quad\Big]h^2 + \Big[\quad\Big]h^3$$

So, here, what little slope there is comes from the coefficient of h^2:

- The number of h^2s coming from $+(-1+h)^3$ is $+3(-1) = -3$
- The number of h^2s coming from $-3(-1+h)^2$ is -3

$$= \Big[\quad\Big] + \Big[0\Big]h + \Big[-6\Big]h^2 + \Big[\quad\Big]h^3$$

Finally, from the graph of $h \longrightarrow -h^2$, we get that Slope-sign of f near -1 is $(\diagup, \diagdown)$

II-2. Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -3x^2 - 12x + 16$$

For which input(s), if any, is Concavity-sign of $f = (\cup, \cup)$?

Discussion: "For which inputs is Concavity-sign $= (\cup, \cup)$" is a *location* question and, in order to locate the subset where Concavity-sign $= (\cup, \cup)$, we need to:

i. *Locate* and *check* the *boundary* of the subset

ii. *Test* the *intervals* to get the *interior* of the subset

There are two ways to go about it depending on whether or not we trust our memory.

a. If we remember that for *quadratic* functions, the concavity is the same everywhere and is controlled by the coefficient of x^2, then since, here $a = -3$, the concavity is $(\cap, \cap)$ everywhere and therefore $(\cup, \cup)$ nowhere.

b. If we do not trust our memory, we convert the question "For which inputs is the concavity $(\cup, \cup)$?" to "For which inputs is the coefficient of h^2 *positive*?"

The local input-output rule near x_0 is:

$$x_0 + h \xrightarrow{f} f(x_0 + h) = -3(x_0 + h)^2 - 12(x_0 + h) + 16$$
$$= \Big[\quad\Big] + \Big[\quad\Big]h + \Big[\quad\Big]h^2$$

and since the only h^2 comes from $-3(x_0 + h)^2$, we have:

$$= -3(x_0 + h)^2 - 12(x_0 + h) + 16$$
$$= \Big[\quad\Big] + \Big[\quad\Big]h + \Big[-3\Big]h^2$$

and the concavity is $(\cup, \cup)$ nowhere.

II-3. Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = +x^3 - 3x - 2$$

Where is f *concave up*?

Discussion: "Where is f concave up?" translates to "Where is Concavity-sign $= (\cup, \cup)$?" which is a *location* question and, in order to locate the subset where Concavity-sign $= (\cup, \cup)$, we need to:

i. *Locate* and *check* the *boundary* of the subset

ii. *Test* the *intervals* to get the *interior* of the subset

There are two ways to go about it depending on whether or not we trust our memory.

a. If we remember that for *cubic* functions the *boundary* for concavity is $x_{0\text{-}concavity} = \frac{-b}{3a} = 0$ then we need only test the *intervals*.

Since the coefficient of x^3 is $a = +1$,

- The local graph near $+\infty$ shows that f is *concave up* (meaning $(\cup, \cup)$) when x is near $+\infty$.

– The local graph near $-\infty$ shows that f is *concave down* (meaning $(\cap, \cap)$) when x is near $-\infty$.

So, the concavity remains *concave up* as long as $x > 0$.

b. If we do not trust our memory, we convert the question "For which inputs is the concavity $(\cup, \cup)$?" to the question "For which inputs is the coefficient of h^2 *positive*?"

To locate the *boundary*, we start from the local input-output rule near x_0 which is:

$$x_0 + h \xrightarrow{f} f(x_0+h) = +(x_0+h)^3 + 0(x_0+h)^2 - 9(x_0+h) + 7$$
$$= \Big[\quad\Big] + \Big[\quad\Big]h + \Big[\quad\Big]h^2 + \Big[\quad\Big]h^3$$

Concavity-sign near x_0 comes from the sign of the coefficient of h^2:

- The number of h^2s coming from $+(x_0+h)^3$ is $+3x_0$
- The number of h^2s coming from $0(x_0+h)^2$ is 0

$$= \Big[\quad\Big] + \Big[\quad\Big]h + \Big[3x_0\Big]h^2 + \Big[\quad\Big]h^3$$

Thus, the concavity is 0 when $3x_0 = 0$ and the boundary $x_{0\text{-}concavity} = 0$

We test the *intervals* as above.

II-4. Let f be the function specified by the global input-output rule

$$x \xrightarrow{\quad f \quad} f(x) = -6x - 5$$

and let g be the function specified by the global input-output rule

$$x \xrightarrow{\quad g \quad} g(x) = -8x - 9$$

find the input(s), if any, for which $g(x) \geqq f(x)$.

Discussion: "Find the input(s), if any, for which $g(x) = f(x)$ " is a *location* question so we need to

i. *Locate* and *check* the *boundary* of the subset

ii. *Test* the *intervals* to get the *interior* of the subset

i. To locate the *boundary*, we solve the (associated) *equation* $g(x) = f(x)$ that is we solve $-8x - 9 = -6x - 5$ which gives us $x_{boundary} = -2$

ii. To find the *interior*,

· Checking $+1000$ against the *inequation* $g(x) \geqq f(x)$ that is

$$-8x - 9 \geqq -6x - 5$$

gives $-8000 + [\ldots] \geqq -6000 + [\ldots]$ which is FALSE

· Checking -1000 against the *inequation* $g(x) \geqq f(x)$ that is
$$-8x - 9 \geqq -6x - 5$$
gives $+8000 + [\dots] \geqq +6000 + [\dots]$ which is TRUE

II-5. Let f be the function specified by the global input-output rule
$$x \xrightarrow{\ f\ } f(x) = +x^2 - 5x - 5$$
and let g be the function specified by the global input-output rule
$$x \xrightarrow{\ g\ } g(x) = -4x + 1$$
For how many input(s), if any, do the functions f and g return the same output?

Discussion: "For how many input(s), if any, do f and g return the same output" translates to "How many solutions does the equation $f(x) = g(x)$ have?". This is a global question but *not* a locating question. We have:
$$f(x) = g(x)$$
$$+x^2 - 5x - 5 = -4x + 1$$
$$+x^2 - x - 6 = 0$$
Computing the discriminant $b^2 - 4ac$ gives $+25$ which is *positive* so that the equation has *two* solutions.

II-6. Let f be the function specified by the global input-output rule
$$x \xrightarrow{\ f\ } f(x) = -\tfrac{3}{2}x^2 - 6x + 8$$
Locate $x_{0\text{-slope}}$

Discussion: "Locate $x_{0\text{-slope}}$" is a *location* question. There are two ways to go about it depending on whether or not we trust our memory.

a. If we remember that for *quadratic* functions $x_{0\text{-slope}} = \frac{-b}{2a}$ we get $x_{0\text{-slope}} = -2$

b. If we do not trust our memory, we convert the question "Locate $x_{0\text{-slope}}$" to "For which inputs is the coefficient of h equal to 0?" The local input-output rule near x_0 is:
$$x_0 + h \xrightarrow{\ f\ } f(x_0 + h) = -\frac{3}{2}(x_0 + h)^2 - 6(x_0 + h) + 8$$
$$= \Big[\quad\Big] + \Big[\quad\Big]h + \Big[\quad\Big]h^2$$
and since:

- the number of hs coming from $-\frac{3}{2}(x_0 + h)^2$ is $-3x_0$
- the number of hs coming from $-6(x_0 + h)$ is -6

$$= \Big[\quad\Big] + \Big[-3x_0 - 6\Big]h + \Big[\quad\Big]h^2$$
which gives $x_{0\text{-slope}} = -2$

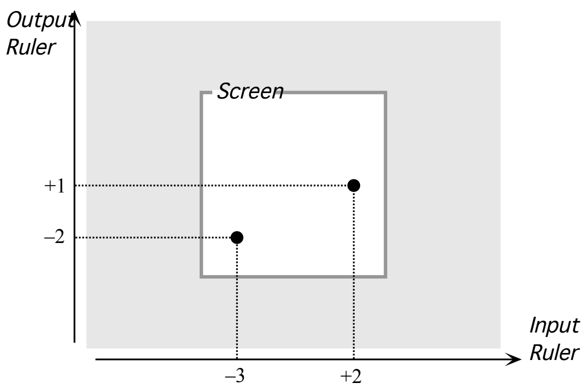
II-7. Let f be an *affine* function. Given the **Boundary Value Conditions**

$$\text{AND}\begin{cases} f(x)\ |_{x \leftarrow +2} = +1 \\ f(x)\ |_{x \leftarrow -3} = -2 \end{cases}$$

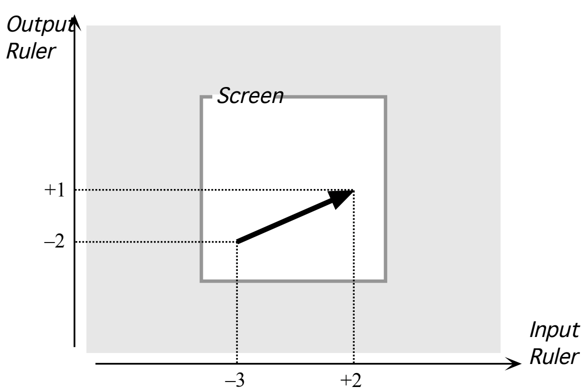
find the *slope* of the global graph of f.

Discussion: Finding the slope of the global graph makes sense only for an *affine function* because, for an affine function, the local slope is the same for all inputs while, for other kinds of functions, the local slope depends on the input so that other kinds of functions have no global slope.

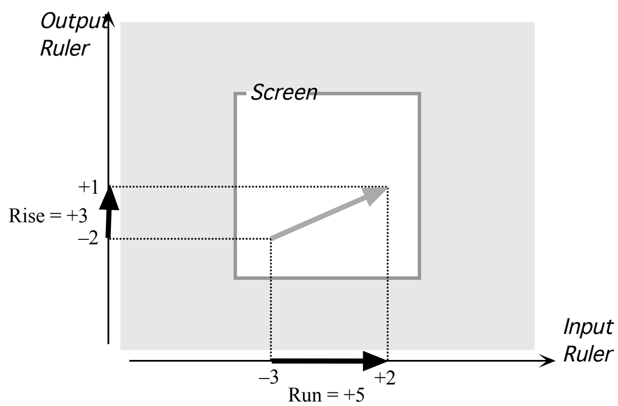
a. The given **Boundary Value Conditions** give us two input-output pairs, $(+2, +1$ and $(-3, -2$, therefore two plot points:



b. We pick one plot point for *start*, say $(-3, -2$. Then the other plot point $(+2, +1$ is *end*:



c. We then get the *run* and the *rise*:



and therefore the slope which is equal to $\frac{\text{Rise}}{\text{Run}}$

II-8. Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -x(x+1)^2$$

Where is the output of f *positive*?

Discussion: "Where is the output of f *positive*?" is a *location* question so we need to:

i. *Locate* and *check* the *boundary* of the subset

ii. *Test* the *intervals* to get the *interior* of the subset

Since the function is factored, we see that the *boundaries*, that is the 0-output inputs are 0 and -1.

Since $f(x)$ must be *positive*, the boundaries 0 and -1 are *not* in the solution subset.

We then *test the intervals*:

- Checking -1000

$$\begin{aligned} -1000 \xrightarrow{f} f(-1000) &= -(-1000)(-1000 + [\ldots])^2 \\ &= -(-1000)(-1000)^2 + [\ldots] \\ &= +\ \textit{large} \end{aligned}$$

- Checking $-\frac{1}{2}$

$$\begin{aligned} -\frac{1}{2} \xrightarrow{f} f(-\frac{1}{2}) &= -(-\frac{1}{2})(-\frac{1}{2} + 1)^2 \\ &= -(-\frac{1}{2})(-\frac{1}{2})^2 \\ &= +\ \textit{bounded} \end{aligned}$$

- Checking +1000

$$+1000 \xrightarrow{f} f(+1000) = -(+1000)(+1000 + [\ldots])^2$$
$$= -(+1000)(+1000)^2 + [\ldots]$$
$$= -\ large$$

So, we get:

II-9. Let f be the function specified by the global input-output rule

$$x \xrightarrow{\quad f \quad} f(x) = -12x + 4$$

Find the 0-height input(s) of f, if any.

Discussion: "Find the 0-height input(s) of f, if any." is a *location* question. Since the height is the output, we must solve the equation $f(x) = 0$, that is the equation $-12x + 4 = 0$.

The solution of $-12x + 4 = 0$ is $+\frac{1}{3}$ so $x_{\text{0-height}} = +\frac{1}{3}$

II-10. Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -3x^2 - 6x + 1$$

What is(are) the lowest bounded *output(s)*, if any?.

Discussion: In the case of a *quadratic* function, the lowest/highest bounded output is the output for $x_{\text{0-slope}}$. Since a is negative, the output for ∞ is *negative* and the output for $x_{\text{0-slope}}$ is the *highest* bounded output.

II-11. Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = x(x^2 + x + 1)$$

Where is the output of f equal to 0?

Discussion: "Where is the output equal to 0?" is a locating question which translates to the question "Solve the equation $x(x^2 + x + 1) = 0$" which amounts in turn to solving

$$\text{EITHER} \begin{cases} x = 0 \\ x^2 + x + 1 = 0 \end{cases}$$

The first equation has the solution 0. To solve the second equation, we compute the *discriminant*, $b^2 - 4ac$, which here turns out to be equal to -3 so that the second equation has no solution.

II-12. Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -3x^3 + x^2 - 5x - 11$$

Where is the slope of f equal to 0?.

Discussion: "Where is the slope of f equal to 0?" is a *locating* question.

The local input-output rule near x_0 is:

$$x_0 + h \xrightarrow{f} f(x_0 + h) = -3(x_0 + h)^3 + (x_0 + h)^2 - 5(x_0 + h) - 11$$
$$= \Big[\quad\Big] + \Big[\quad\Big]h + \Big[\quad\Big]h^2 + \Big[\quad\Big]h^3$$

and since:

- the number of hs coming from $-3(x_0 + h)^3$ is $-9x_0^2$
- the number of hs coming from $+(x_0 + h)$ is $+2x_0$
- the number of hs coming from $-6(x_0 + h)$ is -6

$$= \Big[\quad\Big] + \Big[-9x_0^2 + 2x_0 - 6\Big]h + \Big[\quad\Big]h^2 + \Big[\quad\Big]h^3$$

and we must solve the *quadratic* equation $-9x^2+2x-6 = 0$. We compute the discriminant, b^2-4ac, which here computes to $(+2)^2-4(-9)(-6) < 0$. So, the slope can never be equal to 0.

II-13. Let the function f be specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -x^2 + 4x + 12$$

For which input(s), if any, is the output of f equal to 0?

Discussion: "For which input(s), if any, is the output of f equal to 0?" is a global question. Since the function is *quadratic*, we compute the discriminant, $b^2 - 4ac$, which here computes to $(+4)^2 - 4(-1)(+12) = +64$. So, there are two solutions which are -2 and $+6$.

II-14. Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = +3(x - 5)^2 + 2x + 13$$

Find Slope-sign of f near ∞.

Discussion: "Find Slope-sign of f near ∞" is a local question.

$$x \text{ near } \infty \xrightarrow{f} f(x) = +3(x + [\ldots])^2 + [\ldots]$$
$$= +3x^2 + [\ldots]$$

So the local concavity near ∞ is $(\cup, \cup)$ and therefore the slope-sign is $(\diagup, \diagdown)$. (Remember to look at the local graph near ∞ while facing ∞)

II-15. Let f be the *affine* function specified by the **Boundary Value Conditions**:

$$\text{AND}\begin{cases} f(+3) = +1 \\ f(-3) = +3 \end{cases}$$

Find the global input-output rule that specifies f.

Discussion:

a. The information that the function f is *affine* gives us that the global input-output rule of f is of the form

$$x \xrightarrow{\ f\ } f(x) = ax + b$$

so that to find the global input-output rule reduces to finding a and b.

b. The **Boundary Value Conditions** give us:

$$\text{AND}\begin{cases} +3 \xrightarrow{\ f\ } f(+3) = a(+3) + b = +1 \\ -3 \xrightarrow{\ f\ } f(-3) = a(-3) + b = +3 \end{cases}$$

c. Reducing the system of two equations

$$\text{AND}\begin{cases} +3a + b = +1 \\ -3a + b = +3 \end{cases}$$

to

$$\text{AND}\begin{cases} a = -\frac{1}{3} \\ b = +2 \end{cases}$$

gives the input-output rule:

$$x \xrightarrow{\ f\ } f(x) = -\frac{1}{3}x + 2$$

II-16. Let f be the function specified by the global input-output rule

$$x \xrightarrow{\ f\ } f(x) = +\frac{2}{3}x + 2$$

find its global graph.

Discussion: There are two approaches depending on how much we know about affine functions

i. We can find two input-output pairs by picking two inputs and using the global input-output rule to find the outputs, and since the function is *affine*, its global graph is the straight line that goes through the two plot points.

ii. Since the given function f is affine, it has a global slope which is the coefficient of x in the global input-output rule.

Then, we find one input-output pair by picking one input and using the global input-output rule to find the output, and since the function is *affine*, its global graph is the straight line that goes through the one plot point with the given global slope.

II-17. Let f be the function specified by the global input-output rule

$$x \xrightarrow{\quad f \quad} f(x) = -3x^2 + 18x - 55$$

Near which input(s), if any, is the slope of f *positive*?

Discussion: "Near which input(s), if any, is the slope of f *positive*?" is a *location* question so we need to

i. *Locate* and *check* the *boundary* of the subset

ii. *Test* the *intervals* to get the *interior* of the subset

There are two ways to go about it depending on whether or not we trust our memory.

a. If we remember that for *quadratic* functions, $x_{\text{slope-change}} = x_{\text{0-slope}} = \frac{-b}{2a}$, then the boundary is $+3$.

We "test" near $\pm$ by looking at the local graph near ∞, that is the graph of $-3x^2$. Since the concavity near ∞ is $(\cap, \cap)$, the slope near ∞ is $(\diagdown, \diagup)$. So the slope of f is positive for all inputs smaller than $+3$

b. If we do not trust our memory, we convert the question "For which inputs is the slope positive?" to "For which inputs is the coefficient of h *positive*?"

II-18. Let f be the function specified by the global input-output rule

$$x \xrightarrow{\quad f \quad} f(x) = +(x-3)(x+2)$$

Where, if at all, is the output of f *positive*?

Discussion: "Where, if at all, is the output of f *positive*?" is a *location* question which translates here to locating the solution subset of the inequation

$$+(x-3)(x+2) > 0$$

So:

i. We *locate* and *check* the *boundary* of the solution subset. Since the *associated equation* is in factored form, we see imediately that the boundaries are $+3$ and -2. We check the boundary points against the given inequation and find that neither is a solution of the inequation.

ii. We *test* the *intervals* to get the *interior* of the solution subset :

- To test near ∞, we use $x\, large$:
$$+(x-3)(x+2) = +(x+[...])(x+[...]) = +x^2+[...] > 0$$
- To test between $+3$ and -2, we use 0:
$$+(0-3)(0+2) = -6 < 0$$

so that the output is positive everywhere except in an interval (namely except in $[-2,+3]$.)

II-19. Let f be the function specified by the global input-output rule
$$x \xrightarrow{\quad f \quad} f(x) = +3(-x+4)^2 - 2x + 13$$
Find Height-sign of f near ∞.

Discussion: "Find Height-sign of f near ∞" is a local question so we use the local input-output rule near ∞:
$$\begin{aligned} x \xrightarrow{\quad f \quad} f(x) &= +3(-x+[...])^2 - 2x + 13 \\ &= +3x^2 - 2x + 13 \\ &= +3x^2 + [...] \end{aligned}$$
So, Height-sign of f near $\infty = (+,+)$

II-20. Given the function f whose global input-output rule is
$$x \xrightarrow{\ f\ } f(x) = -(x-5)^2(x+3)$$
what is Concavity-sign f near $+5$?

Discussion: "What is Concavity-sign f near $+5$" is a local question so we use the local input-output rule near $+5$:
$$\begin{aligned} +5+h \xrightarrow{\quad f \quad} f(+5+h) &= -(+5+h-5)^2(+5+h+3) \\ &= -h^2(+8+h) \\ &= -8h^2 + [...] \end{aligned}$$
So, Concavity-sign f near $+5$ is $(\cap, \cap)$