MATH 161 EXAM II Discussions

[Run: 11/09/2016 at 18:6 Seed: 2081. Order of Checkable Items: Random.]

Xm **II-1.** Let f be the *affine* function specified by the **Boundary Value Conditions**:

$$\text{AND}\begin{cases} f(-1) = -7 \\ f(+4) = +3 \end{cases}$$

Find the global input-output rule that specifies f.

Discussion: This is a *global* question.

a. Since f is given to be an *affine function*, the global input-output rule of f is of the form:

$$x \xrightarrow{\quad f \quad} f(x) = ax + b$$

b. We use the information we have about the two inputs:

- Using the input -1, the global input-output rule gives

$$x|_{x\leftarrow -1} \xrightarrow{\quad f \quad} f(x)|_{x\leftarrow -1} = ax + b|_{x\leftarrow -1}$$
$$= a(-1) + b$$

  that is

$$-1 \xrightarrow{\quad f \quad} -a + b$$

  and since the first boundary condition was that

$$-1 \xrightarrow{\quad f \quad} -7$$

  because one input can have at most one output, we must have:

$$-a + b = -7$$

- Using the input $+4$, the global input-output rule gives

$$x|_{x\leftarrow +4} \xrightarrow{\quad f \quad} f(x)|_{x\leftarrow +4} = ax + b|_{x\leftarrow +4}$$
$$= a(+4) + b$$

  that is

$$+4 \xrightarrow{\quad f \quad} +4a + b$$

  and since the second boundary condition was that

$$+4 \xrightarrow{\quad f \quad} = +3$$

  because one input can have at most one output, we must have:

$$+4a + b = +3$$

c. We now solve the double equation problem where the unknowns are a and b as in Appendix 5.

$$\text{Both}\begin{cases}-a+b=-7\\+4a+b=+3\end{cases}$$

Ominussing the second equation from the first we get:

$$(-a+b)\ominus(+4a+b)=-7\ominus+3$$
$$(-a+b)\oplus(-4a-b)=-7\oplus-3$$
$$-a+b-4a-b=-7-3$$
$$-5a=-10$$
$$a=+2$$

and then, plugging $+2$ for a in either equation, we get $b=-5$.

So, the global input-output rule of f is:

$$x\xrightarrow{\quad f\quad}f(x)=+2x-5$$

Xm **II-2.** Let f be the function specified by the global input-output rule

$$x\xrightarrow{f}f(x)=-(x-1)(x-3)^2$$

Where is the output of f *negative*?

Discussion: To locate the inputs whose outputs will be negative amounts to solving the inequation

$$-(x-1)(x-3)^2<0$$

and we use the standard method to find the solution subset:

1. We solve the *associated equation* to find the boundary point(s) if any.

$$-(x-1)(x-3)^2=0$$

Since $f(x)=-(x-1)(x-3)^2$, $f(x)$ will be equal to 0 if and only if either factor is equal to 0, that is

$$\text{Either Or}\begin{cases}-(x-1)=0\\(x-3)^2=0\end{cases}$$

But then:

i. $-(x-1)=0$ gives $x=+1$ whose solution is $+1$

ii. $(x-3)^2=0$ gives $x=+3$ whose solution is $+3$

There are therefore two boundary points and three intervals which we must test:

2. We test the "outer" intervals:

- The interval $(-\infty,+1)$ using $-\infty$

- The interval $(+3, +\infty)$ using $+\infty$

and we test the "inner interval", $(+1, +3)$, using any input between the boundary points $+1$ and $+3$.

Xm **II-3.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{\quad f \quad} f(x) = +5x^2 + 7x - 6$$

Where, if at all, is the output of f *positive*?

Discussion: This is an inequation problem

$$+5x^2 + 7x - 6 > 0$$

and we use the standard method to find the solution subset:

i. We solve the *associated equation* to find the boundary point(s) if any.

$$+5x^2 + 7x - 6 = 0$$

We compute the discriminant of f: $(+7)^2 - 4(+5)(-6)$ and find that it is equal to $+169$. So, the boundary consists of the solutions $\frac{-7}{+10} + \frac{+13}{+10} = +\frac{3}{5}$ and $\frac{-7}{+10} - \frac{13}{+10} = -2$

ii. We check the boundary points against the given inequation to determine whether or not they are solutions and find that neither is a solution of the given inequation

iii. We test each interval and use the Pasch Theorem to determine whether or not the interval is part of the solution subset. Since the local input-output rule near $-\infty$ is

$$\text{Near } -\infty, x \xrightarrow{\quad f \quad} f(x) = +5x^2 + [...]$$

the output is positive everywhere except for the inputs that are in the interval between the two boundary points.

Xm **II-4.** Let f be an *affine* function. Given that

$$\text{AND}\begin{cases} f(x)\ |_{x\leftarrow +7} = -1 \\ f(x)\ |_{x\leftarrow +2} = +3 \end{cases}$$

find the *slope* of the global graph of f.

Discussion: We can proceed in either one of two ways:

i. We can get the *global input output rule* and then from our knowledge of the fact that the coefficient of h in the jet near x_0 is a, it follows that a is going to be the *global slope*.

ii. From our knowledge of the fact that the *global graph* of an affine function is a straight line, we need only compute $\frac{rise}{run}$ from the point $(+7, -1)$ to the point $(+2, +3)$.

Xm **II-5.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -5(x-2)(x-8)$$

Locate $x_{\text{0-slope}}$.

Discussion: f is a *quadratic function* so we know that $x_{\text{0-slope}} = \frac{-b}{2a}$. And since $-5(x-2)(x-8) = -5x^2 + 50x - 80$, we get that $x_{\text{0-slope}} = \frac{-50}{-10} = +5$

Xm **II-6.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{\quad f \quad} f(x) = -(x-3)^2 - 7x - 21$$

Find Slope-sign of f near ∞.

Discussion:

a. The local input-output rule near ∞ is

$$x \text{ near } \infty \xrightarrow{\quad f \quad} f(x) = -(x + [...])^2 - 7x - 21$$
$$= -x^2 + [...]$$

b. So, Slope-sign of f near $\infty = (\searrow, \nearrow)$

Xm **II-7.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{\quad f \quad} f(x) = -12x - 4$$

Find the 0-height input(s) of f, if any.

Discussion: The 0-height inputs are those inputs whose output is 0. In other words, the 0-height inputs are the solutions of the equation

$$f(x) = 0$$

that is, here, of the equation

$$-12x - 4 = 0$$
$$-12x - 4 + 4 = 0 + 4$$
$$-12x = +4$$
$$x = -\frac{1}{3}$$

Xm **II-8.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = x(x-5)(x+3)$$

Where is the output of f equal to 0?

Discussion: We *solve* the equation $f(x) = 0$ that is the equation $x(x - 5)(x + 3 = 0$. This is a *cubic* equation and, normally, we don't know how to solve a *cubic* equation. Here, though, the cubic is *factored* and so solving amounts to the same as solving:

$$\text{OR}\begin{cases} x = 0 \\ x - 5 = 0 \\ x + 3 = 0 \end{cases}$$

So, the inputs for which $f(x) = 0$ are 0, $+5$, -3

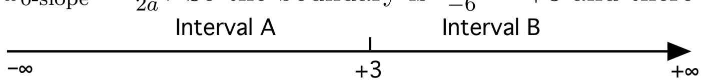
Xm **II-9.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{\ \ f\ \ } f(x) = -3x^2 + 18x - 5$$

Near which input(s), if any, is the slope of f positive?

Discussion: This is a global problem which we solve in two steps:

i. Locate the *boundaries*, that is the input(s), if any, near which the slope is 0. From our knowledge of quadratic functions, we know that $x_{0\text{-slope}} = \frac{-b}{2a}$. So the boundary is $\frac{-18}{-6} = +3$ and there are two intervals:



ii. Test the intervals: Near ∞, $f(x)) = -3x^2 + [...]$ and so

- Near $-\infty$, Sign slope $f = \diagup$
- Near $+\infty$, Sign slope $f = \diagdown$

So, the slope of f is positive in Interval A.

Xm **II-10.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{\ f\ } f(x) = +x^3 - 3x^2 - 9x - 7$$

What is Slope-sign of f near $+3$?

Discussion: The slope-sign near a given input *usually* comes from the sign of the coefficient of h in the *jet* near the given input.

The local input-output rule near $+3$ is

$+3 + h \xrightarrow{\ f\ } f(+3 + h) = +(+3 + h)^3 - 3(+3 + h)^2 - 9(+3 + h) - 7$

and we work on the jet:

$$= \Big[\quad\Big] \oplus \Big[\quad\Big] h \oplus \Big[\quad\Big] h^2 \oplus \Big[\quad\Big] h^3$$

Since

- The h-term coming from $+(+3+h)^3$ is $+1 \times 3 \cdot (+3)^2 h = +27h$
- The h-term coming from $-3(+3+h)^2$ is $-3 \times 2 \cdot (+3)h = -18h$
- The h-term coming from $-9(+3+h)$ is $-9h$

we have

$$= \Big[\quad\Big] \oplus \Big[+27-18-9\Big]h \oplus \Big[\quad\Big]h^2 \oplus \Big[\quad\Big]h^3$$
$$= \Big[\quad\Big] \oplus \Big[0\Big]h \oplus \Big[\quad\Big]h^2 \oplus \Big[\quad\Big]h^3$$

So, the input $+3$ happens to be *critical for the slope* and we must therefore get the coefficient of h^2. Since

- The h^2-term coming from $+(+3+h)^3$ is $+1 \times 3 \cdot (+3)h^2 = +9h^2$
- The h^2-term coming from $-3(+3+h)^2$ is $-3 \times h^2 = -3h^2$

we now have

$$= \Big[\quad\Big] \oplus \Big[0\Big]h \oplus \Big[+9-3\Big]h^2 \oplus \Big[\quad\Big]h^3$$
$$= \Big[\quad\Big] \oplus \Big[0\Big]h \oplus \Big[+6\Big]h^2 \oplus \Big[\quad\Big]h^3$$

and the Slope-sign near $+3$ is therefore $\langle \diagdown, \diagup \rangle$.

Xm **II-11.** Given the function f whose global input-output rule is

$$x \xrightarrow{f} f(x) = +4(x-2)^3$$

what is Concavity-sign f near $+2$?

Discussion: The concavity-sign near a given input *usually* comes from the sign of the coefficient of h^2 in the *jet* near the given input. The local input-output rule near $+2$ is

$$\begin{aligned} +2+h \xrightarrow{f} f(+3+h) &= +4\left[(+2+h)-2\right]^3 \\ &= +4\left[+2+h-2\right]^3 \\ &= +4^3 \end{aligned}$$

So, the jet near $+2$ is

$$= \Big[0\Big] \oplus \Big[0\Big]h \oplus \Big[0\Big]h^2 \oplus \Big[+4\Big]h^3 + [...]$$

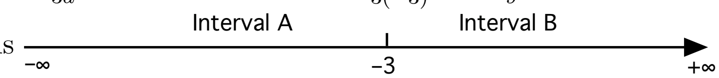
and the Concavity-sign near $+2$ is therefore $\langle \cap, \cup \rangle$.

Xm **II-12.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -3x^3 - 27x^2 - 2x + 3$$

Where is f *concave up*?

Discussion: This is a *location problem* so:

i. Locate the *boundary*. For the concavity sign of a *polynomial function*, the boundary is $x_{0\text{-concavity}}$. From our knowledge of cubic functions, we know that $x_{0\text{-concavity}} = \frac{-b}{3a}$. So, the boundary is $\frac{-(-27)}{3(-3)} = \frac{+27}{-9} = -3$ and there are two intervals 

ii. Test the intervals. Near ∞, $f(x) = -3x^3 + [...]$ and so

- Near $-\infty$, Sign Concavity $f = \cup$
- Near $+\infty$, Sign Concavity $f = \cap$

So, f is concave up in Interval A.

Xm **II-13.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = +3x^2 - 12x - 24$$

For which input(s), if any, is Concavity-sign of $f = (\cap, \cap)$?

Discussion: This is a *location problem* so:

i. Locate the *boundary*. For the concavity sign of a *polynomial function*, the boundary is $x_{0\text{-concavity}}$. From our knowledge of quadratic functions, we know that there is no $x_{0\text{-concavity}}$ because the concavity near every input is equal to a. So, there is no boundary and there is only one interval.

ii. Test the interval. Near ∞, $f(x) = +3x^2 + [...]$ and so near ∞, Sign Concavity $\langle \cup, \cup \rangle$

So, f is concave down nowhere.

Xm **II-14.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{\quad f \quad} f(x) = +4x + 5$$

and let g be the function specified by the global input-output rule

$$x \xrightarrow{\quad g \quad} g(x) = +7x + 17$$

find the input(s), if any, for which $g(x) > f(x)$.

Discussion: The inputs for which $g(x) > f(x)$ are the solutions of the inequation

$$+7x + 17 > +4x + 5$$

We solve the inequation by the standard method:

i. We get the boundary points(s) by solving the *associated equation* $+7x + 17 = +4x + 5$

ii. We check the boundary point(s) against the inequation to find out if it (they) is (are) solution(s)

iii. We pick an input in each interval and check it against the inequation to find out if it (they) is (are) solution(s) and then use the **Pasch Theorem** to conclude.

Xm **II-15.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -3x^3 + x^2 - 5x - 11$$

Where is the slope of f equal to 0?.

Discussion: The slope is usually given by the coefficient of h in the local input-output rule. So, we compute

$$x_0 + h \xrightarrow{f} f(x_0 + h) = -3(x_0 + h)^3 + (x_0 + h)^2 - 5(x_0 + h) - 11$$
$$= \Big[\quad\Big] + \Big[-9x_0^2 + 2x_0 - 5\Big]h + \Big[\quad\Big]h^2 + \Big[\quad\Big]h^3$$

We now want the coefficient of h to be equal to 0:

$$-9x_0^2 + 2x_0 - 5 = 0$$

We get the discriminant, $(+2)^2 - 4(-9)(-5)$, which is negative.

So, the slope of the cubic is not equal to 0 anywhere.

Xm **II-16.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = +3x^2 - 18x + 6$$

What is(are) the highest bounded *output(s)*, if any?.

Discussion: Since the local input-output rule near ∞ is

$$x \xrightarrow{f} f(x) = +3x^2 + [...]$$

or, in other words

$$large \xrightarrow{f} large$$

so that there is no input whose output is largest. Also, since the function f is concave up everywhere, $x_{\text{0-slope}}$ is the input whose output is smallest.

Xm **II-17.** Let the function f be specified by the global input-output rule

$$x \xrightarrow{\quad f \quad} f(x) = -x^2 + 4x + 12$$

For which input(s), if any, is the output of f equal to 0?

Discussion: Since this is a quadratic function, we have Discriminant$f$ = $b^2 - 4ac$, and since:

$$\text{Discriminant}f = (+4)^2 - 4(-1)(+12)$$
$$= +16 + 48$$
$$= +64$$

we have that Discriminant$f$ is positive and therefore that there are two 0-height inputs:

$x_{0 \text{ slope}} \pm \frac{\sqrt{\text{Discriminant} f}}{2a}$

that is

$\frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$

which here gives us

$\frac{-(+4)}{2(-1)} \pm \frac{\sqrt{+64}}{2(-1)} = \frac{-4}{-2} \pm \frac{8}{-2} = +2 \pm -4$

So, the output of f is equal to 0 when the input is either $+2+(-4) = -2$ or $+2-(-4) = +6$

Xm **II-18.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = +x^2 - 5x - 5$$

and let g be the function specified by the global input-output rule

$$x \xrightarrow{g} g(x) = -4x + 1$$

For how many input(s), if any, do the functions f and g return the same output?

Discussion: The output of f is $f(x) = +x^2 - 5x - 5$ and the output of g is $g(x) = -4x + 1$. What is wanted is those inputs for which $f(x) = g(x)$ therefore those inputs for which

$$+x^2 - 5x - 5 = -4x + 1$$

that is those inputs for which

$$+x^2 - 5x - 5 + 4x - 1 = 0$$

$$+x^2 - x - 6 = 0$$

We compute the discriminant: $(-1)^2 - 4(+1)(-6) = +1 + 24 = +25$

Since the discriminant is *positive*, we have *two solutions* and the outputs of f and g are equal when the input is either one of these two solutions.

(The solutions are $\frac{-(-1)}{2(+1)} \pm \frac{\sqrt{+25}}{2(+1)} = \frac{+1}{+2} \pm \frac{5}{+2}$ that is either $\frac{+1+5}{+2} = +3$ or $\frac{+1-5}{+2} = -2$ but the question did not ask for that.)

Xm **II-19.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -\frac{2}{3}x - 2$$

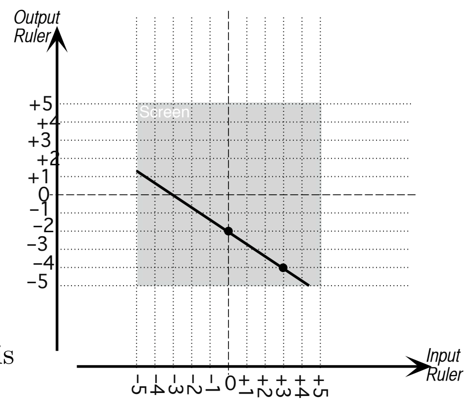
find its global graph.

Discussion: From our knowledge of affine functions, we know that the global graph is a straight line. So, we pick any two inputs, compute their outputs, plot the two input-output pairs and draw a straight line through the two plot points.

Using the inputs 0 and 1 often minimizes the computations but here, since the coefficient of x is a fraction, instead of 1, we use the denominator of the fraction, namely $+3$.

- $0 \xrightarrow{\ f\ } f(0) = -\frac{2}{3}(0) - 2 = 0 - 2 = -2$
- $+3 \xrightarrow{\ f\ } f(+3) = -\frac{2}{3}(+3) - 2 = -2 - 2 = -4$

and the graph is



Xm **II-20.** Let f be the function specified by the global input-output rule

$$x \xrightarrow{\ f\ } f(x) = +3(-x+4)^2 - 2x + 13$$

Find Height-sign of f near ∞.

Discussion: The global input-output rule of f

$$x \xrightarrow{\ f\ } f(x) = +3\,(-x+4)^2\, - 2x + 13$$

is not in *the standard form*

$$x \xrightarrow{\ f\ } f(x) = ax^2 + bx + c$$

a. So, the first thing to do is to put the given global input-output rule in standard form. For that we use the addition formula for quadratics:

$$(-x+4)^2 = (-x)^2 - 8x + 4^2$$

Then, the global input-output rule becomes:

$$\begin{aligned} x \xrightarrow{\ f\ } f(x) &= +3\,(-x+4)^2\, - 2x + 13 \\ &= +3[\,x^2 - 6x + 2^2\,] - 2x + 13 \\ &= +3x^2 - 20x + 25 \end{aligned}$$

b. The local input-output rule near ∞ is then

$$x \text{ near } \infty \xrightarrow{\ f\ } f(x) = +3x^2 + [...]$$

c. So, Height-sign of f near $\infty = (+,+)$