Chapter 13

Reasonable Mathematics: A Political Necessity

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This chapter, which was written under a GNU Free Documentation License, grew out of a short email exchange with B. Atweh. The author is also grateful to the anonymous reviewer of the initial submission who pointed out that "[i]t seems that the author only takes on the issue of quality and assumes that equity comes as a result of [quality]." What follows is an attempt at supporting this "assumption".

13.0 Introduction

Mathematics education has been confronting the problem of how to bring "quality" mathematics to "the great unwashed masses" for at least thirty years without any success or even discernible progress. In fact, the only conspicuous thing is that mathematics textbooks during that time have devolved to exposition by way of "template examples" and that the subject matter has been atomized into "independent topics" to facilitate memorization while, typically, instructors deplore that their students cannot remember the simplest things past the test.

And yet, it is not difficult to see how the stress generated by memorization on the scale required by, say, a year of mathematics must necessarily have that result: Sisyphus like, students feel they are starting anew again and again, topic after topic, and realize there is nothing they can do that will help them cope with the next topic. Which, of course is why they constantly ask “Is this going to be on the test?” or complain “You haven’t shown us how to do that!”.

Nevertheless, the operating, if tacit, assumption is that the great majority of students are incapable of learning on any other basis, least of all on the basis of *intelligence*. And by an unfortunate, even if perhaps unavoidable, coincidence, not only has research in mathematical learning also mostly dealt with isolated topics but, even more unfortunately, it too essentially equates *learning* with *memorizing*.

In this chapter, we will argue that the alternative “[q]uality mathematics [...] seen as a reflection of its rigor, formality and generalisability” versus “utilitarian importance” would seem to be rather beside the point given that there does exist a *third* approach in which: i) *mathematical quality* is ensured by the very fact that learning is done on the sole basis of a continuous appeal to “reason”, ii) *utilitarian needs* are much better addressed in that learning is done by constant references to the real world and in that a logical mind is very much an asset in coping with ever-changing practical situations while obsolescence is built-in into “training”.

However, as the above-mentioned reviewer pointed out, what certainly remains to be shown is how this approach can indeed work for “the great unwashed masses”.

We will begin by discussing at some length the background against which the proposed approach was designed and some of the issues that had to be, and were, taken in consideration. After that, we will present an example of how essential a role the manner in which the contents are conceived *mathematically* plays in the construction of any contents architecture lending itself to the only kind of “natural” educational enterprise likely to ensure *equity*. We will then discuss some of the parameters governing any implementation of such an approach and briefly present what there already is by way of—freely available—materials.

13.1 The Only Way Things Can Be?

Quality and *equity* are usually looked upon as being entirely irreconcilable and this perhaps nowhere as much as in the U. S. in which the only schools really open to people of small means, two-year community colleges, do not even come close to affording the“preparation” necessary to transfer to schools such as MIT or CalTech or even Harvard, Princeton, Stanford etc. Considerably worse, there are very large numbers of people whose high school experience prevents them from entering even two-year community colleges.

Starting about thirty years ago, an apparent concern for *equity* had led to the creation of "developmental"—initially known as remedial—programs[1]. But the fact is that these programs never even began to work, if by working we mean affording access to higher education. For instance, at the author's school, the percentage of students entering the development program who eventually complete Calculus I -Differential Calculus, the "mathematics of change", is less than 1%.

In this case, one cause is immediately apparent, namely the very length of the sequence: Starting with, say, Arithmetic, it takes another four semesters, Basic Algebra (8/9th grade Algebra I), Intermediate Algebra (10/11th grade Algebra II), College Algebra, College Trigonometry before one can even aspire to learn Differential Calculus. A study conducted at the author's school showed that more students drop out of the sequence *between* courses than fail in the courses.

A more subtle but in fact much more prevalent cause lies in the already mentioned deep belief in the necessity to "simplify things" for the students by atomizing the subject matter into bits and pieces that can then be "learned" but that can, in fact, only be *memorized*. And it is precisely this total reliance on memory that has devastating consequences for *equity*. Briefly:

- Forcing the students to rely on memory not only deprives them of the opportunity to become aware of *modus ponendo ponens*, that is of the fact that "if I have this—which I do, then I must also have that" which is precisely what is necessary for the *extension* of knowledge, but also of *modus tollendo tollens*, that is of the fact that "I cannot have this because if I did I would then have to have that—which I cannot have" which is precisely what is necessary for the *verification* of knowledge.
- Such a massive reliance on memory leads straight to "math anxiety", so-called even though it is nothing more than the anxiety of having no idea of what it is one is forced to do. Not to dwell on the fact that developmental students generally have little cause to have the trust in the world and in themselves that is conducive to the peace of mind necessary for the *use* of memorized stuff. This alone makes education the privilege of the better off.

Of course, none of which is to say that memory and even "memorization" should not play any role in whatever "learning" turns out to be, but only

[1]However, it can be argued that a determining factor was the desire of the business world to externalize its training costs.

that the latter should not be reduced to the former.

More generally, and whatever the causes, the educational establishment tends to be extremely elitist in that the prevalent philosophy is that things are the way they are because, everything else being supposed to have been tried and having failed, this very absence of alternative "proves" that this is the only way things can be. Equity is taken to be just a dream. For instance, the inordinate length of the sequence offered to students in need of arithmetic who nevertheless wish to learn differential calculus is considered not only to be an incontrovertible necessity, even if possibly a regrettable one, but somewhat perversely, also a "proof" that some are mathematically more gifted than others. Similarly, an even more perverse consequence of reducing "learning mathematics" to "memorizing disconnected topics" is very rarely deplored: it is that this disenfranchises the students in that, rather than leading them to rely on their own common sense, it habituates them to rely on usually self appointed and hardly disinterested "experts".

Closer to the trenches too, this reliance on memorization is usually seen as just being the necessary result of the conditions in which education is taking place. For instance, such things as lack of parental support, cultural bias, etc are frequently invoked as the root causes for this failure of memorization even though the connections have never been substantiated and this merely seems to be the usual teachers' "moan and groan" at the end of a bad day.

13.2 Nevertheless, Let's Be Real!

Aside from all that, there indeed *are* conditions outside the sphere the profession can hope to act on and these are all very real and must be acknowledged and examined, if only for a chance of finding workarounds.

- One the one hand, there are the concrete conditions due to the nature of our school's administration, our state's educational vagaries, the educational establishment, etc.

  One such constraint is that we can have only three, four, possibly, at the very most, five contact hours a week with our students and this for only one or two developmental semesters and there is absolutely nothing we can do about that. Any proposed solution must explicitly fit within this type of schedule.

  Another constraint is that this is a one-size-fits-all-students schedule. Yet any proposed solution must make sure that a sufficient number of students share enough common reference points most of the time while *streaming* essentially eliminates *equity*. This is actually a most serious issue since it obviously has a direct bearing on the "success rate" and therefore on the tolerance our administration, our state and the educational establishment in general will have for our proposed solution.

And, last but certainly not least, publishers are not about to cease parceling textbooks in order to increase sales by assembling any set of topics in any order the "better to suit customers needs" where, it ought to be remembered, the customer is the faculty but it is the student who foots the bill.

- On the other hand, there are also the constraints due to the nature of the developmental student population. Privately, most teachers will bemoan the lack of intellectual capacity of "these people". They will tell you with bitter glee the various "you are confusing us", "you are not teaching us", "you are not showing us", etc awaiting anyone daring to stray in the least from the "true and tried, show and tell, drill and test" path.

  But even if, per chance, these students should somehow deserve more than memorized indoctrination, even if the shape they are in should only be the result of twelve years of elementary-secondary education, since "show and tell" is now "what they want", this being a free country, "Isn't it what we should give them?"

  A variant is that these are people most of whom, if not all, have job(s), families to support, and for whom, in any case, education is not perceived as really a necessity but, at best, as "what you must get to get a better job". And so, "How dare you intrude in their lives?"

So, to use J. Holt's phrases, the question really is: How do we create a realistic opportunity (equity) for students to become "question oriented" (quality) in a framework completely devoted to the production of "answer oriented" students without infringing on the freedom of the students?

13.3 Meaning, Truth and Consequence

Part of the problem in the U.S. is that it is a country where freedom and individualism are taken to justify one in holding any opinion whatsoever. Other than in court and in graduate school, nowhere is one ever confronted with the need to present one's case for one's opinions. In fact, the very idea of *truth* is completely eschewed and the very idea of *proof* completely eliminated. Even the idea of *meaning* has been held to be "culture bound", thus subjective, and therefore completely worthless. But, as Colin McGinn has pointed out,

> Democratic States are constitutively committed to ensuring and furthering the intellectual health of the citizens who compose them: indeed, they are only possible at all if people reach a certain cognitive level .... [...]. Democracy and education (in the widest sense) are thus as conceptually inseparable as individual rational action and knowledge of the world. [...]. Plainly, [education] involves the transmission of knowledge from teacher to taught. [...]. [Knowledge] is true justified belief that has been arrived at by rational means. [...]. Thus the norms governing political action incorporate or embed norms appropriate to rational belief formation. [...]. The educational system of schools and universities is one central element in this cognitive health service [...].

Thus, once we have decided, for individual or societal reasons, that the main point of mathematics education is that it must provide such a "cognitive health service", it becomes completely unthinkable to keep on teaching bits and pieces of mathematics by "show and tell, drill and test"—and thus, by the way, to use the commercially available texts.

So, we must now turn to the two generally accepted viewpoints—by mathematicians—from which to consider, and therefore learn, mathematics:

- The *formalist* view, in which mathematics is a game *played* within a given set of rules. Z. P. Dienes in Elementary Education and G. Papy in Secondary Education both essentially took that point of view and implemented approaches that were found to be completely accessible to all children inasmuch as playing is nature's way to enforce learning among higher vertebrates including humans. But, aside from the fact that adults don't really like to play, there is also the problem that this framework is completely dependent on the large amounts of time available in primary-secondary educations and would thus seem difficult to adapt to developmental education.
- The *Platonist* view, in which mathematics is a sort of rarefied physics in which a paper world is created to mirror the real world and paper procedures developed to represent real world processes[2]. *A priori*, this view would appear to be much more adapted to adult education since adults have a much broader knowledge of the real world that can be taken advantage of.

[2]It should of course be noted that this appeal to the real world is completely antithetic to the recourse to so-called "applications" which are *a priori* justifications and whose use has finally been shown to be rather counter-productive in that they are quasi impossible for "unprepared" learners to generalize and abstract from.

However, by itself, the Platonist view does not ensure that students will learn how to recognize that "this is true" or that "this is false", that "this follows from that" or that "this does *not* follow from that", etc. After all, physics too can be taught by "show and tell, drill and test".

13.4 The Model-Theoretic View

In fact, as MGinn noted in the article already quoted,

> people do not really like the truth; they feel coerced by reason, bullied by fact. In a certain sense, this is not irrational, since a commitment to believe only what is true implies a willingness to detach your beliefs from your desires. [...]. Truth limits your freedom, in a way, because it reduces your belief-options; it is quite capable of forcing your mind to go against its natural inclination. This, I suspect, is the root psychological cause of the relativistic view of truth, for that view gives me license to believe whatever it pleases me to believe. [...]. One of the central aims of education, as a preparation for political democracy, should be to enable people to get on better terms with reason—to learn to live with the truth.

But, initially, one certainly cannot simply appeal to "mathematical proofs" if only because, as Edward Thorndike showed a century ago, mathematical proofs do not transfer into convincing arguments. Thus, we have what could be called "the McGinn imperative", namely the necessity in the first stage of a developmental education in mathematics to reconcile the students with the idea that mathematics is the way it is, not because "experts" say so, but because the *real world demands* that it be so.

Now, it happens that *meaning*, *truth* and *logical consequence* were defined by Tarski in a classic 1933 paper in which *proof* becomes the paper representation of *consequence* in the real world. Briefly, *Model Theory*, which was born from Tarski's paper, starts with:

- A *language*, that is a list of nouns and verbs, quantifiers—*i.e.* for all, there is at least one—and connectors—*e.g.* and, and/or, either one but not both, not—together with rules for "well-formedness"[3].
- A number of *structures*, that is, essentially, a number of real world *collections* of *items*, each collection coming with a number of *relationships* and *processes*. (More about this below.)

[3]Wff n Proof is an interesting introduction to this but perhaps better suited for a *formalist* environment.

Sentences in the language can then be *interpreted* in any one of these structures and if the sentence describes things as they are in a structure, then the sentence is said to be *True* under that *interpretation* and, given two sentences, S_1 and S_2, we can say that S_1 *entails* S_2 *as consequence* if any interpretation that makes S_1 True also makes S_2 True. When dealing with *all* possible interpretations of the language, we can then speak of *logical* consequence and Gödel Completeness Theorem then says that, given a deductive system, S_2 can be *proven* from S_1 if and only if S_2 is a *logical consequence* of S_1.

It would thus certainly seem that the *model-theoretic* view can provide a framework for a presentation of mathematics for adult learners in that, in such a framework, by constant reference to the real world, adult learners can both:

- make sense of the various issues involved in paper world representations,
- reconcile themselves with the ideas of *truth* and *proof*.

And then, as noted by Thurston, students can discover that

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics.

and educators that compression is the only way that *equity* can be reconciled with *quality*.

Nevertheless, students in developmental programs are enormously insecure and if a *model theoretic* view would appear to be *necessary* to provide a successful road to quality and equity, it is clearly not *sufficient* and a "reasonable" version of such a *model theoretic* approach thus needs to be specified and developed.

Suffice it to say at this point that model theory can be "softened":

- by way of a mode of *arguing* rather like that used by lawyers in front of a court. (S. E. Toulmin, *The Uses of Argument.*)
- by concentrating on a Coherent View of Mathematics and a Profound Understanding of Fundamental Mathematics. (L. Ma, *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*)

However, before we can really get to any of that, we must do one last thing and examine a number of *linguistic* issues.

13.5 The Language Barrier

In a not too distant past in which there were no textbooks and *communication* only consisted in the transmission of knowledge from teacher to students, the teacher would essentially be dictating a textbook to the students. Per force, the students could thus afford very little thinking during class and there was no additional communication other than the occasional Socratic question from the instructor to the students and certainly no question from the students to the instructor.

Today, things are not really that different and whatever communication there is usually occurs in the following context: the instructor lectures, the students take notes, the bell rings, everyone goes home, the instructor to grade the homework just handed in, the students to do the next homework from the template examples in the textbook. Clearly, another paradigm for communication is necessary to ensure both *quality* and *equity*. No longer can the students be simply enjoined to operate from "template examples" and just told whether the answer is "right" or "wrong".

The next issue is that the kind of communication necessary for any attempt at *quality* conflicts with *equity* inasmuch as the natural language of the developmental student population is completely unadapted to the development of any mathematical *quality* however the latter is defined. So, in addition to the issues already mentioned, mathematical contents and their architecture, we must now examine a third kind of issues, namely what would have to be involved in significant, *two-way* communication on the first two issues.

- The *object language*, that is the language used on paper to represent the real world, must be sufficiently clear to ensure against misunderstandings, that is not only refer to the aspect of the real world intended by the *sender* but also exclude from consideration by the *receiver* any other aspect of the real world. This is of course usually done by way of formal definitions but is not likely to be appropriate with students much given to "like . . . you know what I mean". Thus, just as in all the "trades", a specialized language must be developed.

  A most important issue here is that the meaning of mathematical symbols usually depends on the *context* while students feel of course a lot more comfortable with *context-free* terminology, that is with an entirely *one-to-one* correspondence between *terms* and *concepts*.

  An example of a specific issue is that the passage from the active voice, *e.g.* Jill beat Jack, to the passive voice, Jack was beaten by Jill, is almost invariably confused with the more familiar passage to the *symmetric* sentence, Jack beat Jill. This is a most important *linguistic* stumbling block if only because the passage to the passive voice preserves truth while the passage to the symmetric sentence need not.

Another linguistic stumbling block is the use of “whose”. For instance, while Dollar can be defined *directly* in terms of Quarter by saying “One Dollar is equal to four Quarters”, the definition of Quarter in terms of Dollar is an *indirect* one in that we must say “A Quarter is that kind of coin *of which* we need four to change for one Dollar”. The same stumbling block occurs in dealing with roots since $\sqrt{5}$ is to be understood as “that number *whose* square is 5”.

Finally, since, as already noted, developmental students are extremely insecure, once they get started on the road to precision, every single thing suddenly becomes potentially important to them while they have not yet reached the point where they can decide what is and what is not significant and/or what was left unsaid but goes without saying. So, initially, everything must be completely spelled out including default rules to deal with what is left unsaid and goes without saying in conventional mathspeak.

- The *meta-language* is the language in which we talk *about* issues of mathematical contents and issues regarding the way these contents are related, as seen through paper representations. In contrast with the *object language*, the *meta-language* not only can be a lot closer to the students' natural language but, in fact, must be so.

  But, even there, initially, constant consideration must be given to the fact that, quite naturally, students cannot be entirely counted on to realize when they don't understand something because for them understanding something means having committed it to memory and, when asked to explain something, are equally naturally given to taking refuge in vagueness. Interestingly, though, students almost invariably get eventually to a point where, with great delight, they nitpick the instructor to death.

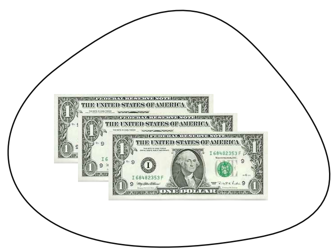
  One last aspect of communication in mathematics is that it needs to be *adversarial* and this is something difficult to arrive at with students who initially take it as “disrespect”. Still, the students have to be made to feel that loose discourse is not only useless but often dangerous.

One difficulty in designing the two languages is in keeping them firmly apart because, after all, there are only so many words in any natural language and we need *two* words for each thing to be named: one for the meta-language and one for the object-language.

We are now ready to give an idea of how things might work in the classroom. We begin with the description, as seen in a model theoretic setting, of some mathematical contents and will then offer some comments about how the way mathematical contents are seen *mathematically* affects the natural learning flow.

13.6 Not So Basic Arithmetic

We start with *bunches* of *items* in the real world and pose the problem of how to represent these on paper. It becomes rapidly obvious that if all the items are of the same kind, in which case we will use the term *collection* instead of bunch, things become considerably simpler as, in that case, we need only a *number-phrase* consisting of a *denominator* to represent the common *kind* of items and a *numerator* to indicate the *number* of items. For instance, we might represent the collection



by the number-phrase

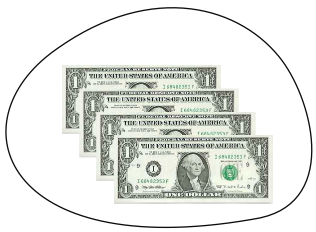
3 Washingtons

where 3 is the numerator and **Washingtons** is the denominator.

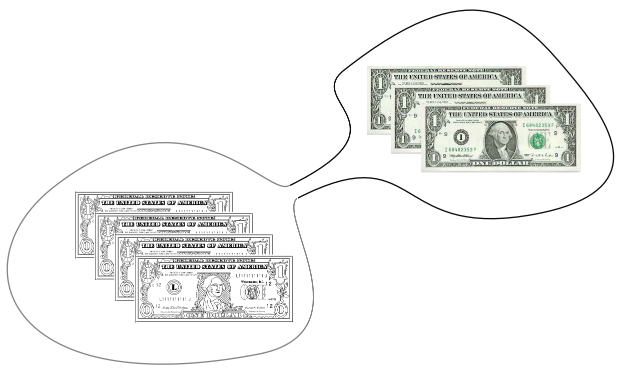
We get the numerator by *counting* the items in the collection, that is by reciting a pre-memorized *litany*, "one, two, three, ..." while touching each one of the items in the collection. Counting is thus a bridge between the real world and the paper world. By themselves, though, collections and their paper representations are not very interesting and pretty soon we start wondering what kind of things we can do *with* them.

The first thing one is likely to do is to *compare* collections which, when the numbers of items are small, is readily achieved in the real world by matching the two collections one to one, that is *cardinally*. Very soon, though, this turns out to be unbearable and, instead, we start dealing with the number-phrases that represent the collections in the paper world and which we compare by checking the positions of the numerators in the litany, that is *ordinally*. This is where the standard *verbs*, $<$, $>$, $=$, $\neq$, $\leqq$, $\geqq$ are introduced, characterized in terms of the real world and getting familiarized with—which they usually very much need to. The *negation* of sentences involving these verbs provides the ground for fertile discussions by way of references to the real world.

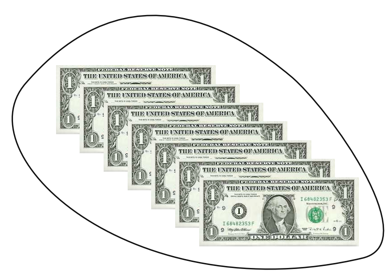
The second thing we are likely to do, given an *initial collection*, is to *act* on it. For instance, given an *initial collection* of four dollar bills,



we may want to *attach* a collection of three dollar bills



and thus get a *final collection* of seven dollar bills.



Again, soon enough, we move to the paper world and try to specify the paper procedure for the *unary operator* that represents attachment and which we call *addition to*[4]. For instance, to represent the above attachment, we would

[4] Looking at addition as a *unary* operator is ancient as shown by a now defunct terminology in which the input was called the *augend*, the numerator involved in adding to operator was called the *addend* and the output was called the *sum*. It still corresponds to standard everyday usage. The view as a binary law of composition, that numbers are

use the following *addition to* operator:

$$\xrightarrow{Add_{3\ \mathsf{Washingtons}}}$$

which is, starting in the litany from the *input*, to count "three" steps, possibly keeping track on one's fingers, and then to represent the above attachment, we would write

$$4\,\mathbf{Washingtons} \xrightarrow{Add_{3\ \mathsf{Washingtons}}} 7\,\mathbf{Washingtons}$$

where 4 **Washingtons** is the input and 7 **Washingtons** is the *output*.

At this point we can already let the students investigate a number of real world situations and represent these in the paper world. They can attach a given collection to various initial collections, they can attach various collections to a given initial collection, they can *chain* various addition operators and find to what single operator an operator followed by another operator is *equivalent* to, etc.

The next thing of course is to look at *reverse actions* namely, here, *detachments*, and their paper representations, *subtractions from*. Immediately, though, we run into the fact that subtractions from cannot always be carried out which corresponds to the fact that neither can detachments which fits the idea of the representation of the real world by a paper world.

We then continue with another type of real world situation involving now *two* kinds of items from which collections can be made but with a *cancellation effect*. For instance, we may look at a banking situation in which we have *opening balances* onto which we can either *deposit* or *withdraw* and thus get a *closing balances*. Naturally, we represent the balances by *plain* numerators as above but rather that dealing separately with plain addition operators and plain subtraction operators, we now introduce *signed operators*. And, if we are rich enough that our bank will permit us to run deficits, we represent the balances by signed number-phrases too.

By refering to the real world, developmental students have no difficulty figuring out, *by themselves*, the procedures for comparing signed numerators, for signed addition to and realizing that they involve all the procedures developed earlier in the case of plain numerators: plain comparison, plain addition to and plain subtraction from. For instance, it does not take them long to realize that *signed subtraction from* is nothing but signed addition

"added together", is fairly recent, indeed central to abstract algebra but, from the point of view of this text, most inconvenient and nothing more than, apparently, a leftover of the "new math".

to of the *opposite* since, when a bank needs to *remove* a transaction from a balance, what it actually does is to *enter* the opposite transaction:

$$\xrightarrow{\quad Subtract_{-3\ \mathsf{Washingtons}}\quad} \quad = \quad \xrightarrow{\quad Add_{+3\ \mathsf{Washingtons}}\quad}$$

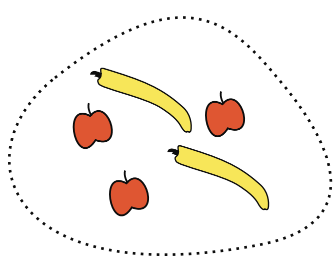
Multiplication is introduced via another kind of real world situation: Given a collection of items and a *unit-price*, that is a collection of, say, coins that can be exchanged for one item, multiplication gives us the *total price* of the collection at the unit-price. For instance, we have

$$3\,\cancel{\mathsf{Apples}} \times 5\frac{\mathsf{Cents}}{\cancel{\mathsf{Apple}}} = 15\ \mathsf{Cents}$$

This kind of situation can be modified in a natural manner to lead to the "rule of signs": Let collections of apples coming into a warehouse be represented on paper with positive numerators and collections of apples getting out of the warehouse by negative numerators. Similarly, let the profits to be derived from selling good apples be represented on paper by positive numerators and the costs to be incurred by disposing of bad apples by negative numerators. No student has ever been found who did not say, *instantly*, that getting rid of bad apples was a good thing or, for that matter, who had any trouble with

$$-3\,\cancel{\mathsf{Apples}} \otimes -5\frac{\mathsf{Cents}}{\cancel{\mathsf{Apple}}} = +15\ \mathsf{Cents}$$

The case in which the items are *not* all of the same kind leads to Linear Algebra and this is the first occasion in which we *reduce* a new problem to an already solved one: The only new idea is that a bunch has first to be *decomposed* into collections but, after that, we can represent the bunch by a *combination*. For instance, a bunch with two kinds of items but *no* cancellation effect such as



can be represented in the paper world by the combination

$$3\ \mathsf{Apples}\ \&\ 2\ \mathsf{Bananas}$$

(where & stands in for the more usual but very unfortunate +) or, better yet, by the *basket* (aka *vector*)

$$\Big(3\,\textsf{Apples}, 2\,\textsf{Bananas}\Big)$$

After that, essentially, things can be left *entirely to the students*: Baskets can be added to and subtracted from in the obvious manner, the passage to signed-numerators is obvious and multiplication naturally becomes

$$\Big(\ +2\,\textsf{Apples}\ ,\quad -7\,\textsf{Bananas}\ \Big) \boxtimes \begin{bmatrix} +5\,\frac{\textsf{Cents}}{\textsf{Apple}} \\ \\ -3\,\frac{\textsf{Cents}}{\textsf{Banana}} \end{bmatrix} = +31\,\textsf{Cents}$$

We would say that the *lists* as well as the *co-lists* just got to be of dimension 2 with the *total price* remaining a *scalar* and get on our *linear* way.

13.7 Comments

The previous was mostly an attempt at giving the reader some sense of how a model theoretic view can provide the students with a *systematic* approach to things mathematical as opposed to our own occasional recourse to "illustrations" invariably and correctly dismissed by the students as *ad hoc* gimmicks. The perceptive reader will of course have already seen the many further implications of, and the glaring omissions in, the preceding section and so, given the space limitations, we will discuss only a few of these.

1. At least inititally, many symbols must be used so as to be *context-free*, which, for instance, is why we use $\lessdot$, $\oplus$, $\ominus$, etc to distinguish them from $<$, $+$, $-$, etc: these symbols all stand for distinct procedures. Somewhere along the line, though, we need to accustom the students to the idea that notations are not hard-wired and that this is in fact fortunate because there is no single notation good for every usage. For instance, at some point, we must fully discuss the identification of positive numerators with plain numerators and appropriate *default rules* must be explicitly written down. We must also go from

$$-7\,\textsf{Washingtons} \xrightarrow{Add_{+3\ \textsf{Washingtons}}} -4\,\textsf{Washingtons}$$

to just

$$-7\,\textsf{Washingtons} \oplus +3\,\textsf{Washingtons} = +4\,\textsf{Washingtons}$$

and then

$$-7\,\textsf{Washingtons} + 3\,\textsf{Washingtons} = 4\,\textsf{Washingtons}$$

The point, though, is that the longhand will still be there for whenever something needs to be clarified and also for when a reference point is needed later on, as when *functions*—in which the underlying procedure becomes an explicit *input-output rule*—such as, for instance,

$$x \xrightarrow{\quad Affine_{-3,+5} \quad} Affine_{-3,+5}(x) = -3x + 5$$

are being introduced.

2. The first glaring omission is that of *fractions* which actually give us a further example of the perhaps surprising use of number-phrases. Given that "A Quarter is that kind of coin *of which* we need four to change for one Dollar", it is natural to represent a quarter on paper by the denominator

$$\Big[4 \;\rightarrow 1\;\textsf{Washingtons}\Big]$$

Then, for instance, when the denominators are the same, we have

$$\begin{aligned} 3\Big[4 \;\rightarrow 1\;\textsf{Washingtons}\Big] + 2\Big[4 \;\rightarrow 1\;\textsf{Washingtons}\Big] \\ = 5\Big[4 \;\rightarrow 1\;\textsf{Washingtons}\Big] \end{aligned}$$

and, if desired, from the built-in definition of a quarter:

$$= 1\;\textsf{Washingtons}\;\&\;1\Big[4 \;\rightarrow 1\;\textsf{Washingtons}\Big]$$

When the denominators are not the same as with, say, three quarters and seven dimes, to be able to add we naturally need to *change* the denominators to a *common denominator*:

$$\begin{aligned} &3\Big[4 \;\rightarrow 1\;\textsf{Washingtons}\Big]\;\&\;7\Big[10 \;\rightarrow 1\;\textsf{Washingtons}\Big] \\ &\qquad = (3\cdot 5)\Big[(4\cdot 5) \;\rightarrow 1\;\textsf{Washingtons}\Big]\;\&\;(7\cdot 2)\Big[(10\cdot 2) \;\rightarrow 1\;\textsf{Washingtons}\Big] \\ &\qquad = 15\Big[20 \;\rightarrow 1\;\textsf{Washingtons}\Big] + 14\Big[20 \;\rightarrow 1\;\textsf{Washingtons}\Big] \\ &\qquad = 29\Big[20 \;\rightarrow 1\;\textsf{Washingtons}\Big] \end{aligned}$$

Note that with a bit of "preparation" such as changing apples and bananas for strawberries, a notation inspired from co-lists might have made the changes easier to follow. In any case, at this point, all that remains to do is: i) to develop a less ponderous notation, and ii) to focus on how to get a least common multiple

3. A second glaring omission is that of the *decimal numbers*. They are introduced together with the *metric system* with *money* serving as real world situation. But, aside from the fact that, as engineers are fond to say, "The *real* real numbers are the *decimal* numbers", the reason decimal numbers are of the utmost importance to us is that they will be at the core of the *local polynomial approximations* which are all we will need to develop a "limit-free" differential calculus in the manner of Lagrange. See xxx for more on this.

4. And in fact, for our purposes, *fractions* are not much more than code for division which bring us to a third glaring omission, namely that of the idea of *approximation*. Suffice it to say here that we look upon $\frac{123}{37}$ as a shorthand for "Divide 37 into 123". This in turns brings the question "What is $\frac{123}{37}$ equal to?" the response to which is, naturally, "It depends". For instance, and depending on the situation, we can write:

$$\begin{aligned}\frac{123}{37} &= 0 + [...] \\ &= 3.3 + [...] \\ &= 3.32 + [...] \\ &= 3.324 + [...] \\ &= 3.3234 + [...] \\ \text{etc} &\end{aligned}$$

where [...] stands for *something too small to be taken into consideration in the present situation*. It is inspired by, and a first step towards, $o[h^n]$, Landau's "little oh".

13.8 Now What?

We will now conclude with a few practical remarks for anyone intrigued enough to consider the matter a bit further.

1. A most important thing to keep in mind is that any such developmental course has zero chance of being successful if the students are then thrown into "conventional" courses. There is no way that twelve years of education can be undone in even a whole year. A longer convalescence is needed. Thus, the reinsertion in a conventional course of study must occur later and any developmental sequence will require that further courses be also designed and developed for a smooth transition.

Because of his own mathematical interests and competencies, the particular course of study that the author chose to implement was that which culminates with Differential Calculus. An additional reason was that, essentially, the author had already designed, under a 1988 NSF Calculus Grant, a two semester sequence as an alternative to the conventional, Precalculus I, Precalculus II and Calculus I (Differential) sequence. The 1992 report of the author's school's Office of Institutional Research read in part:

> Of those attempting the first course in each sequence, 12.5% finished the [conventional three semester 10 hour] sequence while 48.3% finished the [integrated two semester 8-hour] sequence, revealing a definite association between the [integrated two semester 8 hour] sequence and completion (?2 (1) = 82.14, $p < .001$).

The report also said that the passing rates in Calculus II (Integral) for the students coming from the above two sequences were almost identical but that this was not significant because, in both sequences, most students did not continue into Calculus II (Integral).

What made the two-semester sequence work is directly relevant to the theme of this chapter in that it was the systematic use of (Laurent) polynomial approximations (Lagrange's approach) and that these are of course nothing but an extension of decimal approximations so that a "profound understanding of fundamental mathematics", in this case functions, decimal approximations, equations and inequations, and (Laurent) polynomials, is all that is necessary and is likely achievable in one four-hour semester designed along the lines suggested here.

2. But then, following Hestenes dictum,

> Early in my career, I naively thought that if you give a good idea to competent mathematicians or physicists, they will work out its implications for themselves. I have learned since that most of them need the implications spelled out in utter detail.

and the whole project being a rather radical departure from current practices, it was imperative, if anybody was ever to try such an approach, to develop materials beyond what might have been needed for just a proof of concept. The author is currently engaged in the development of the materials for the whole three-semesters sequence.

Some of the materials are available online at http://www.freemathtexts.org. Having been written in LaTeX under a GNU Free Documentation License, they are therefore freely downloadable, printable, distributable, and modifiable.

However, at this time, they are only standalone forms of parts of the materials for the three-semester sequence and come in "bundles" currently formatted for a fifteen week semester meeting twice a week but should not be difficult to reformat for other schedules. Presumably, they can be used in Developmental Basic Algebra and Precalculus I - Algebraic Functions.
A bundle consists of a rather long-winded text divided into eighteen chapters together with the following "ancillaries":

- One homework per chapter
- One quiz per chapter
- One Review Discussion for each one of the three parts of the text
- One Review Test for each one of the three parts of the text
- One Exam for each one of the three parts

All the ancillaries are built from a single Question Base of "checkable items" and keys can be generated. While the ancillaries come out of the box with lists of checkable items, these lists can be modified and checkable items added to the Question Base. The format of the ancillaries can be customized to a significant degree. For instance, the questions can be presented in random or list order, as open or multiple choice questions.

3. The author uses these material as follows: The students are to read a chapter before each class and to do a homework on that chapter. The class then starts with students questions about the text and the homework. After about half the time has been thus spent, the instructor hands out but does not collect a very short quiz, spends some more time on the chapter and then "introduces" the next chapter. A couple of weeks before each one of the three exams, the instructor hands out a Review Discussion with instances of the checkable items to be on the exam, each being "discussed" at some length. The class before the exam, the instructor hands out a multiple-choice Review Test consisting of the exact same questions and checks them on the spot. The next class is for the exam and, at the end of the semester, instead of a final, the students can make up any and all exams with the make up score replacing, for the better or for the worse, the exam score. Contrary to what one might think, because of the way the questions are built, this is not a giveaway but there really is no mathematical reason why not. But this is another story.

4. The whole approach is thus heavily predicated on the willingness and the ability of the students to read the materials. Given that, initially, reading is not seen by the students as a way to acquire knowledge or even know-how, the first thing is to get the students used to the idea that they *must* reach for the text in order to "understand". But they need help with that and one way is to have the course linked with a Remedial English Reading course in

which the reading materials is the text in the mathematics course.
The link was tried once and the instructor who had taught the Remedial English Reading course later wrote that

> The students that stayed to the end also appreciated [the approach] whether they passed or not. If we pursue another link, the English teacher should definitely read the math text with the students. Unfortunately, because I had my own reading to do, we did not read the math in English class as we should have done.

5. At this point, both the developmental course and the PreCalculus course are only standalone versions of part of the materials for a three-semester sequence ending with Differential Calculus and no statistics exist. Nevertheless, there are good reasons to think that a three semesters sequence would produce much better results than the current ones: Given the success-rate mentioned at the outset, an overall success rate from Arithmetic to Differential Calculus ought, no matter what, be considerably higher and so it is hoped that people will consider the approached proposed here.
But then, as Mark Twain [almost] wrote:

> When an entirely new and untried *educational* project is sprung upon the *faculty*, they are startled, anxious, timid, and for a time they are mute, reserved, non-committal. The great majority of them are not studying the new doctrine and making up their minds about it, they are waiting to see which is going to be the popular side.

(The words he actually used were *political* and *people*. Not that much of a difference after all.)