Can you join five points smoothly?

July 16th, 2008

This **Note** is more of a "progress report" than about issues concerning learning mathematics although, as it happens, the two are deeply linked.

The estimated time of arrival for Reasonable Algebraic Functions (RAF) is now sometimes in Fall 2008 and that for Reasonable Decimal Arithmetic (RDA) sometimes in Spring 2009.

The reason or, if you prefer, the excuse, for the delay. I spent an inordinate amount of time, even for me, trying again and again to redesign the first of the three parts that comprise RAF. The issue was that while Polynomial Functions (Part Two of RAF) and Rational Functions (Part Three of RAF) had long seemed stable enough in terms of their mathematical treatment, there had always been two problems: (1) How to present Power Functions and (2) several issues associated with the methodology used in the treatment of Polynomial Functions and Rational Functions.

1. Essentially, the mathematical approach, due to Lagrange, is to use local polynomial approximation and I discussed in some detail how I used it in a Precalculus-Calculus One sequence. By itself, its implementation does not raise any problem, at least certainly nothing like what is involved when using *limits*:
   - For Affine Functions, all that is needed is replacing x by x_0+h.
   - For Quadratic Functions, all that is needed is an **addition formula** for the expansion of $(x_0+h)^2$.
   - For Cubic Functions, all that is needed is an addition formula for the expansion of $(x_0+h)^3$.
   - For Quartic Functions, etc …
   - For Polynomial Functions of degree n, we need the **Binomial Theorem**.
   - For Rational Functions, all that is needed is **polynomial division** albeit both in *ascending powers* to localize near *bounded inputs* including the *poles*, if any, and in *descending powers* to localize near *infinity*.
   - For Radical Functions, we need to solve a—fortunately sparse—system of equations to find the coefficients of the expansion.
2. While it might not entirely satisfy mathematicians, this approach is, I think, very much in the spirit of mathematics in that cases can be made—and readily turned into proofs—for most of the theorems being used. At the very least and if nothing else, there is none of that "After plotting these points, you can see that they appear to lie on a line, as shown on Figure 1.1. The graph of the equation is the line that passes through the five plotted points." (Larson-Hostetler, Precalculus 6th edition. For the effect this kind of "philosophy" is likely to have on serious students, see Sim Wobpa which was written à propos the 4th edition of the very same opus.) In the rare occasions when, for whatever

reason, a case is not made, this is duly noted and explained.

3. There is, however, an important difference between RAF and Lagrange DC which is that: a) in order to be usable in Precalculus One, the concept of derivative had to go and, b) in order to still make sense as a "mathematical theory", the treatment in RAF had to be, in some sense, "minimum and closed". As it turned out, though, these two requirements were easy to satisfy. Concerning a), the analysis in RAF could be done without *mentioning* derivatives and by just *using* the coefficient of h^n. Concerning b), first RAF does not include Radical Functions on the grounds that the way the localization is obtained brings them closer to Transcendental Functions defined as solutions of differential equations and then, second, since Affine Functions have no turning point and Quadratic Functions have no inflection point with Cubic Functions being able to exhibit both, RAF did not have to deal with any other Polynomial Functions to illustrate these behaviors and the list of contents for RAF is:
   - Power Functions
   - Constant Functions
   - Affine Functions
   - Quadratic Functions
   - Cubic Functions
   - Rational Functions
4.
5. A very important unifying fact is that, in some ways, both Polynomial Functions and Rational Functions are "essentially" little more than Power Functions with some number of **fluctuations**, that is of minimum-maximum pairs, thrown in. To be a bit more precise, though, a big difference is that the fluctuations are bounded in the case of Polynomial Functions whereas they can be infinite in the case of Rational Functions. Specifically, the **essential bounded graph** is defined as the simplest bounded graph compatible with the *local graph near infinity* and the *local graph near the pole*(*s*). Beyond that, of course, the essence of their nature is that Polynomial Functions are locally approximately *polynomial* while Rational Functions are locally approximately *Laurent-polynomial*. Still, an important issue is that of **locating** these fluctuations. Fortunately, in the above list of contents, things are naturally taken care of: after expanding near x_0, we locate the **turning point** by killing the coefficient of h in the localization which, in the case of a *quadratic function*, entails solving an *affine equation* and in the case of a *cubic function* entails solving a *quadratic equation*. Similarly, we locate the **inflection point** of a *cubic function* by killing the coefficient of h^2 which entails solving an *affine equation*. So, in a way, there is a "closure" of sorts.
6. However, in the case of students who need to learn that mathematics has a logical development, as opposed to "math" being a set of "skills", an extremely important issue is that of the order in which to introduce the necessary language and the necessary concepts so as to create an impression of logical flow. The "progression" from Polynomial Functions to Rational Functions thus seemed quite "logical", at least to me. But I realized eventually that it still resulted in a certain "ad hoc" feeling with the students. And another place where the flow was far from being "obvious" occurred

when moving from Part Two – Polynomial Functions to Part Three – Rational Functions: there was this sudden, new concern as to whether there might be bounded inputs with infinite outputs. Again, at first, this didn't seem to be a big deal as, after all that is why they were dealt with in two different Parts, but, still, it sure didn't contribute to continuity in the story line. And, of course, there was the issue that the flow in the progression

- Positive-power Functions
- Negative-power Functions
- Polynomial Functions (of degree ≤ 3)
- Rational Functions

7. which, if it certainly works, was somewhat less than felicitous. But then, the flow in the progression

   - Positive-power Functions (Needed to localize Polynomial Functions.)
   - Polynomial Functions (of degree ≤3)
   - Negative-power Functions (Needed to localize Rational Functions.)
   - Rational Functions

8. while it was on an "as needed basis" and thus seemed a bit more logical, didn't deal with the concern about the sudden concern about whether there might be bounded inputs with infinite output.

9. There was also another problem in that I was also introducing the terminology on an "as needed" basis. For instance, the term "turning point" was introduced in the context of Quadratic Functions and the term "inflection point" in the context of Cubic Functions. While it felt right not to start with a massive set of definitions at the beginning of the course, later on, it created problems of reference and ended up exacting a heavier price than I had thought. And then, given that RAF is a standalone, there was the issue of where to introduce the various necessary concepts, those usually thrown-in into a catch all "review chapter. What took me four months to arrive at was the idea of introducing all that was needed on functions defined by graphs. In particular, given the local graph near infinity of a function, a natural question is under what conditions does joining smoothly the local graph near $-\infty$ to the local graph near $+\infty$ result in the **bounded graph**. The condition of course was the answer to the Essential Question—do all bounded inputs have bounded outputs or is there a bounded input with an infinite output?—which until now had come up only with Rational Functions—and Chapter 3 was entirely devoted to the issue of smooth interpolation. But then, finally, I came to realize that it was the notion of **outlying graph**, that is of the *local graph near infinity* together with *the local graph near the poles*, if any, that provided the means to make the contents consistent in that for "all" functions it was the *outlying graph* which controlled the (essential) *bounded graph*.

All of this to say that, as of this Bastille Day, the list of contents for the first five chapters of RAF is:

1 – Introduction

5.2 Normalized Input-Output Rule
5.3 Local Graph Near ∞
5.4 Types of Local Graphs Near ∞
5.5 The Essential Question
5.6 Types of Local Graphs Near 0
5.7 Essential Bounded Graph
5.8 Local Graph near 0
5.9 Notable Inputs
5.10 Essential Global Graph
5.11 Types of Global Graphs

*The Fundamental Problem is the problem of deriving a *graph* from an *input-output rule*. It provides the story line of RFA. Larson-Hostetler and all the others notwithstanding, this cannot be done by "joining five points smoothly".

Hopefully, this architecture will support the rest of RFA in a manner satisfactory to **S**tudents **i**nterested In **m**athematics **W**ith **o**nly a **b**ackground in **p**olynomial **a**lgebra.

As ever, any criticism, critique, feedback, etc is of course welcome, the more detailed, the more welcome.

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P.S. While the original entry was written on Bastille Day, it was slightly modified on July 23.

P.P.S. The navigation in FreeMathTexts is of course perfectly atrocious and, as soon as I understand css better, I will use some open source code I found on the web to remedy this rather unfortunate situation. In other, unrelated news, I am also considering learning about how to set up a forum/listserv.

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