From Arithmetic
To Differential Calculus
In Three Semesters

A Course of Study Based on Common Sense
With Inflammatory Notes
For the Mortification of “Educologists”
And the Vindication of “Just Plain Folks”\(^1\)

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July 18, 2006

\(^1\)While, if need be, I take full responsibility for coining the term “Educologist”,
the phrase “Just Plain Folks” first appeared in Calculus Syllabi, Report of the
Content Workshop, J. Goldstein et al. in Toward a Lean and Lively Calculus, Ron.
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Chapter 1

Accounting For Basic Collections Of Money On A Counter

In this chapter we will be dealing with objects sitting on a counter/desk/table/etc and we will represent these objects on a (black)board/notebook/etc. We will then design procedures to be carried on the board to arrive at a representation of the result of what we did on the counter.

But, if the distinction between what sits on the counter and what we write on the board is quite clear in the classroom, it is not as easy to make in a book and here we will have to resort to various devices.

• Inasmuch as possible, we shall use pictures to stand for objects on the counter but, as this is not always possible, we shall also use their usual name but with a particular typeface so as to distinguish them from what we will write to represent them on the board.

For instance, we will use dollar as an alternate for to stand for a dollar-bill sitting on the counter while we will write Dollar to represent it on the board.

• Similarly, we shall use ONE, TWO, ..., TEN, ELEVEN, etc, to stand for the numbers of objects sitting on the counter, with the firm understanding that, on the board, we can write only 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

• We shall use the symbol ♠ to signal that we are in the semantic mode, that is, working on the counter and the symbol ♦ to signal that we are in the syntactic mode, that is, writing on the board.

1If nothing else, this might help remind Educologists that what they write on the board
1.1 Representing Basic Collections with (Counting) Number-Phrases

1. We begin with the issue of representing on the board money sitting on a counter. (Note by the way that banks used to be called “counting houses”.) For instance, given dollars and dimes on the counter, we use the words Dollars and Dimes as denominators, that is as names/symbols/denominations/etc to represent them on the board.

<table>
<thead>
<tr>
<th>♠ On the counter.</th>
<th>♦ On the board.</th>
</tr>
</thead>
<tbody>
<tr>
<td>We have</td>
<td>We write</td>
</tr>
<tr>
<td>Dollar, Dollar, Dollar, Dollar, Dollar, Dollar</td>
<td>Dollar, Dime, Dime, Dime, Dime</td>
</tr>
</tbody>
</table>

2. The first breakthrough in the development of arithmetic was the realization that objects of different kinds have to be accounted for separately because, when all the objects are of the same kind, we can then refer to them collectively, that is as a collection. For instance, we shall refer to dollar, dollar, dollar, dollar, dollar, dollar, dollar on the counter as a collection of dollars. On the other hand, according to this agreement, dollar, dollar, dime, dime, dime, dime, dime will not be a collection. What would it be a collection of?

What this does is to allow us to represent a collection on the counter by writing on the board a (counting) number-phrase, that is a phrase consisting of:

- a numerator to indicate how many objects there are in the collection, which we do for the moment by writing on the board a slash, /, for each object in the collection on the counter, and,
- a denominator to indicate the kind of objects the collection is made of.

For instance,
We will say that the number of objects in a collection is *that* which is represented by the numerator in the number-phrase that represents the collection on the board. Thus, while the denominator represents the kind of objects in a collection, the numerator represents the number of objects in a collection\(^5\).

Observe that, even though a number-phrase is much more economical a way to represent on the board a collection of objects on the counter than writing one denominator for each object, there is no loss of information. Essentially, what we have done was merely to separate quantity from quality but, as it will turn out, this is a very powerful idea\(^6\).

In particular, given, say, dollar, dollar, dollar on the counter, we can ask two very different questions:

- “What is on the counter?” whose answer on the board is the number-phrase /// Dollars
- “How many dollars are on the counter?” whose answer on the board is the numerator ///.

**Note.** We will need to make a distinction somewhat analogous to our use in English of “one dime” versus “a dime”. We will distinguish between a collection consisting of one dime, which we represent on the board by the number-phrase / Dime, and the object that a dime is, which we represent on the board by the denominator Dime. While this will surely appear as beyond nitpicking, not making the distinction would turn the development of board procedures into a nightmare.

3. The second breakthrough in the development of arithmetic occurred when Indian scribes introduced as numerators the digits 1, 2, 3, ... , 9 to be used instead of /, //, ///, ..., //////// so that we now write, say,
3 Dollars instead of /// Dollars.

a. Once we have memorized the succession 1, 2, 3, \ldots, 9, what this does is to give us a procedure to find the numerator of the (counting) number-phrase that represents a given collection of objects on the board: we count the collection, that is we point in turn at each object in the collection, while reciting the succession of digits. The numerator we write on the board is the last digit recited in the count. For instance,

<table>
<thead>
<tr>
<th>♠ On the counter.</th>
<th>♣ On the board.</th>
</tr>
</thead>
<tbody>
<tr>
<td>We have</td>
<td></td>
</tr>
<tr>
<td>![Money Image]</td>
<td></td>
</tr>
<tr>
<td>We count</td>
<td></td>
</tr>
<tr>
<td>1, 2, 3, 4, 5, 6, 7,</td>
<td></td>
</tr>
<tr>
<td>We write 7 Dollars</td>
<td></td>
</tr>
</tbody>
</table>

b. At this point, we can count only up to 9 Dollars because we cannot recite ten as we have no symbol to represent ten on the board. So, by a basic collection, we shall mean a collection with fewer than ten objects which we can therefore count with just the above digits. At some fundamental level, basic collections are thus the only ones we can really represent! Reaching “ten” will be the signal for “bundling” as we shall see in Section 2.2.

Note. There is nothing sacred about ten: it is simply how many fingers (“digit” is just a fancy word for finger) are on our two hands and we could have used just about any number of digits instead of ten. For example, deep down, computers use a machine language based on two digits, 0 and 1, because any electronic device is either off or on. At intermediate levels, computer software may use eight (0, 1, 2, 3, 4, 5, 6, 7) or sixteen digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f). The Babylonians used sixty digits, a historical remnant of which is the fact that there are sixty seconds to a minute and sixty minutes to an hour. The point here is that all that

---

7 Educologists will surely note that, while we take a cardinal viewpoint on the counter, we take an ordinal viewpoint on the board. In other words, we use ordinal counting, a syntactic means, to arrive at a semantic end, the cardinality of a collection. This might indeed be an issue.

8 Moreover, I shall use basic collections to illustrate issues that are independent of the number of objects in the collection(s).
we will do with ten could easily be done with any number of digits\textsuperscript{9}.

c. A small complication is that the numerator 1 often “goes without saying” which has the unfortunate effect of obliterating the difference between denominator and number-phrase. This is often expressed as a default rule:

\textit{When no numerator is given, the numerator 1 is intended and goes without saying.}

\textit{Note.} Unfortunately, this default rule is often abbreviated as “when there is no numerator, the numerator is 1” which is dangerous because, when there is no numerator, it is tempting to think that there is no object either! To be on the safe side, we will avoid letting the numerator 1 go “without saying”.

d. Finally, we note that we have not yet introduced the digit 0. This is only because, so far, we have had no need for it. In fact, historically, the digit 0 was a much later invention. It will be introduced in Section 1.5.

\textit{Note.} Since we refer to, say, dollar, dollar, dollar, dollar, dollar, dollar, dollar as a collection of dollars, it is tempting to “improve” a bit and write “a collection of 7 dollars” but we should resist the temptation because dollars are objects that sit on the counter while 7 is something we write on the board and we don’t want to mix what is written on the board with what sits on the counter. On the other hand, we can speak of a collection of seven dollars.

1.2 Comparing Collections: Equalities and Inequalities

We now want to compare collections—involving the same kind of objects. (We will compare collections involving different kinds of objects in Section 4.5.)

1. We begin with the comparison of two collections on the counter and with the board procedure for getting the result of the comparison. We will deal with the issue of how to represent this result on the board in sub-section 2. below.

\textbullet On the counter, what we do is to match one-to-one the objects in the two collections; the particular relationship that stands between the two collections will depend on which of the two collections the leftover objects are in.

---

\textsuperscript{9}Z. P. Dienes used to start with base-three arithmetic blocks and the digits 0, 1, 2.
count from ... to ... is less numerous than count forward succeed is more numerous than

❖ On the board, we count each one of the two collections and then we count from the numerator of the first number-phrase to the numerator of the second number-phrase, that is, starting after the numerator of the first number-phrase, we count to the numerator of the second number-phrase.

Either way, we then have three possibilities:

a. In the first case, that is

♠ When the leftover objects are in the second collection, we will say that the first collection is less numerous than the second collection\(^\text{10}\).

❖ To count from the first numerator to the second one, starting with the digit after the first numerator, we must count forward, that is, we must call the digits that succeed it in the succession 1, 2, 3, 4, 5, 6, 7, 8, 9 and end with the second numerator.

For instance, \(4, 5, 6, 7\) is a forward count that starts after 3 and ends with 7.

For instance,

<table>
<thead>
<tr>
<th>♠ On the counter.</th>
<th>❖ On the board.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Jack</strong> has ♠</td>
<td>We count Jack’s collection:</td>
</tr>
<tr>
<td></td>
<td>1, 2, 3</td>
</tr>
<tr>
<td><strong>Jill</strong> has ♠</td>
<td>We count Jill’s collection:</td>
</tr>
<tr>
<td></td>
<td>1, 2, 3, 4, 5, 6, 7</td>
</tr>
</tbody>
</table>

We match **Jack**’s collection one-to-one with **Jill**’s collection:

We count from **Jack**’s collection to **Jill**’s collection:

**Jack**’s collection is less numerous than **Jill**’s collection

b. In the second case, that is

♠ When the leftover objects are in the first collection, we will say that the first collection is more numerous than the second collection.

❖ To count from the first numerator to the second one, starting with the

\(^{10}\)Educologists may question this contrived term. Of course, the issue is to have different terms for use on the counter and to write on the board and only experience can tell if the difference is worth making.
1.2. EQUALITIES AND INEQUALITIES

digit before the first numerator, we must count backward, that is, we must call the digits that precede it in the succession 1, 2, 3, 4, 5, 6, 7, 8, 9 and end with the second numerator. For instance, \[ \rightarrow \] is a backward count that starts before 5 and ends with 3.

Note. Thus, the precession 9, 8, 7, 6, 5, 4, 3, 2, 1 should be memorized as well as the succession 1, 2, 3, 4, 5, 6, 7, 8, 9\(^{11}\).

For instance,

<table>
<thead>
<tr>
<th>♠ On the counter.</th>
<th>♦ On the board.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Jack</strong> has</td>
<td>We count Jack’s collection:</td>
</tr>
<tr>
<td></td>
<td>1, 2, 3, 4, 5</td>
</tr>
<tr>
<td><strong>Jill</strong> has</td>
<td>We count Jill’s collection:</td>
</tr>
<tr>
<td></td>
<td>1, 2, 3</td>
</tr>
</tbody>
</table>

We match **Jack**’s collection one-to-one with **Jill**’s collection: We count from **Jack**’s collection to **Jill**’s collection:

\[ \leftarrow 3, 4 \]

**Jack**’s collection is more numerous than **Jill**’s collection.

**c.** In the third case, that is

♠ When there are no leftover objects, we will say that the first collection is as numerous as the second collection.

♦ The two numerators are the same and we must count neither forward nor backward.

For instance,

\(^{11}\)Should Educologists ask children to do so, they might discover that children actually love to count backward.
verb
sentence
<
is smaller than
>
is larger than
=
is equal to

Jack has

We count Jack’s collection:

1, 2, 3

Jill has

We count Jill’s collection:

1, 2, 3

We match Jack’s collection one-to-one with Jill’s collection:

We count from Jack’s collection to Jill’s collection:

Jack’s collection is equal to Jill’s collection.

We must count neither forward nor backward.

2. In order to represent on the board the result of comparing two collections, we first need to expand our mathematical language beyond number-phrases.

a. Given a relationship between two collections, we need a verb that represents the relationship between the two collections. Then we can write a sentence involving the two number-phrases that represent the collections and the verb that represents the relationship between the two collections:

- We will use the verb < to represent the relationship is less numerous than and we will read it is smaller than. For instance, for the first of the above three examples, we will write the sentence 3 Dollars < 7 Dollars which we will read “THREE dollars is smaller than FIVE dollars”.
- We will use the verb > to represent the relationship is more numerous than and we will read it is larger than. For instance, for the second of the above three examples, we will write the sentence 5 Dollars > 3 Dollars which we will read “FIVE dollars is larger than THREE dollars”.
- We will use the verb = to represent the relationship is as numerous as and we will read it is equal to. For instance, for the third of the above three examples, we will write the sentence 3 Dollars = 3 Dollars which we will read “THREE dollars is equal to THREE dollars”.

In other words,
1.2. EQUALITIES AND INEQUALITIES

<table>
<thead>
<tr>
<th>When we must count forward</th>
<th>we write</th>
<th>which is read as</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;</td>
<td>“is smaller than”</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>When we must count backward</th>
<th>we write</th>
<th>which is read as</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt;</td>
<td>“is larger than”</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>When we must not count either way</th>
<th>we write</th>
<th>which is read</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>=</td>
<td>“is equal to”</td>
</tr>
</tbody>
</table>

**Note.** Beware that the symbols < and > go in directions opposite to that of the arrowheads when we count from the first numerator to the second numerator. (If need be, one can think of < as · : with · being “smaller” than : and of > as : · with : being “larger” than ·.)

b. *Sentences* involving the *verbs* > or < are called **strict inequalities** while sentences involving the *verb* = are called **equalities**. For example,

3 Dollars < 7 Dollars and 8 Dollars > 2 Dollars are strict inequalities

3 Dollars = 3 Dollars is an equality

c. In English, when we say that we allow “up to” 5 Dollars, we mean that we allow 1 Dollar, 2 Dollars, 3 Dollars, 4 Dollars but that we do *not* allow the endpoint itself, 5 Dollars. If we do want also to allow the endpoint, 5 Dollars, we say “up to and including” 5 Dollars.

In mathematics we shall also need to make this distinction, that is, to allow or not to allow the endpoint, and, when we do allow it, we will say that the inequality is a **bounded inequality**:

- We will use the verb ≦ to represent the relationship *is less numerous than or as numerous as* and we will read it **less than or equal to**.
- We will use the verb ≧ to represent the relationship *is more numerous than or as numerous as* and we will read it **more than or equal to**.

d. Inasmuch as the sentences that we wrote above represented actual relationships between collections on the counter, they were true but there is of course nothing to prevent us from writing sentences that are false in the sense that there is no way that we could come up with situations that these sentences would represent. For example, the sentences

5 Dollars = 3 Dollars and 5 Dollars < 3 Dollars,

are false because there is no way that we could realize them on the counter, that is come up with actual collections with these relationships.

Observe, by the way, that the sentence

5 Dollars ≦ 3 Dollars
CHAPTER 1. BASIC COLLECTIONS OF MONEY

is true.

e. However, while occasionally useful, it is usually not very convenient to write sentences that are false because then we must not forget to write that they are false when we write them and we may miss that it says somewhere that they are false when we read them. So, inasmuch as possible, we shall write only sentences that are true and we will use the default rule:

*When no indication of truth or falsehood is given, mathematical sentences will be understood to be true and this will go without saying.*

When a sentence is false, rather than writing it and say that it is false, what we shall usually do is to write its negation—which is true and therefore which “goes without saying”. We can do this either in either one of two manners:

- We can place the false sentence within the symbol $\neg$, 
- We can just slash the verb which is what we shall usually do.

For instance, instead of writing that the sentence 5 Dollars = 3 Dollars is false we can either write the (true) sentence $\neg[5 \text{ Dollars} = 3 \text{ Dollars}]$ or the (true) sentence $5 \text{ Dollars} \neq 3 \text{ Dollars}$

3. The (linguistic) duality that exists between < and > must not be confused with (linguistic) symmetry, a concept which we tend to be more familiar with\(^{12}\).

a. Examples of linguistic symmetry include pairs of sentences—which may be true or false—such as the following:

- Jack is a child of Jill versus Jill is a child of Jack
- Jill beats Jack at poker versus Jack beats Jill at poker
- Jack loves Jill versus Jill loves Jack
- 9 Dimes > 2 Dimes versus 2 Dimes > 9 Dimes

In each example, the two sentences represent opposite relationships between the two people/collections because, even though the verbs are the same, the two people/collections are mentioned in opposite order.

Observe that just because one of the two sentences is true (or false) does not, by itself, automatically force the other to be either true or false and that whether or not it does depends on the nature of the relationship.

b. Examples of linguistic duality include:

\(^{12}\)This confusion is a most important linguistic stumbling block for students and one that Educologists utterly fail to take into consideration.
1.3. EQUATIONS AND “INEQUATIONS”

In each example, the two sentences represent the same relationship between the two people/collections because, even though the people/collections are mentioned in opposite order, the two verbs are dual of each other which “undoes” the effect of the order so that only the emphasis is different. Observe that here, as a result, if one of the two sentences is true (or false) this automatically forces the other to be true (or false) and this regardless of the nature of the relationship.

c. The following are examples of simultaneous (linguistic) symmetry and (linguistic) duality because the verbs are the same and the order does not matter.

- Jack is a sibling of Jill versus Jill is a sibling of Jack
- 2 Nickels = 1 Dime versus 1 Dime = 2 Nickels

Observe that, here again, as soon as one sentence is true (or false), by itself this automatically forces the other to be true (or false) and that it does not depend on the nature of the relationship.

1.3 Specifying Collections: Equations and “Inequations”

In real life, we often have to specify things we want by stating some requirement(s) that these things must satisfy.

Here, we will specify collection(s) by the requirement that they stand in a given relationship with a given collection, namely one or the other of the following,

- is more numerous than the given collection,
- is less numerous than the given collection,
- is as numerous as the given collection.

For instance, say that

Jack has three dollars,
Jill has seven dollars,
Dick has three dollars,
Jane has four dollars.

and that we specify the collection(s) that satisfy the requirement that they be more numerous than Jack’s collection.

1. We could of course proceed as we did in Section 1.2:
On the counter, matching Jack’s collection one-to-one with each one of the collections of Jill, Dick and Jane shows that this specifies the collections of Jill and Jane.

On the board, counting from Jack’s collection each one of the collections of Jill, Dick and Jane specifies the same collections. This approach, though, is somewhat short of ideal if only because it would become very time-consuming with large numbers of collections to compare. So, what we want is to develop a board procedure that is more efficient in that the time it requires will not go up appreciably as the number of collections and of objects in the collections goes up.

2. Before we do that, though, we need a way to phrase requirements that lends itself to procedural manipulations.

   a. Essentially, what we will do is to introduce the mathematical version of something common in everyday life, namely forms such as

   "[ ] was President of the United States."

   which, when we fill it it with some data, say,

   Kissinger

   produces a sentence, namely

   "Kissinger was President of the United States."

   which happens to be false while, when we fill it with the data

   Bill Clinton

   it produces the sentence

   "Bill Clinton was President of the United States."

   which happens to be true.

   b. In the case of the above example,

   On the counter, we want the collections of dollars that satisfy the requirement that they be more numerous than three dollars.

   On the board, we want the solutions of the form

   Dollars > 3 Dollars

Thus, from what we did above, we have that

- the data 7 produces the sentence $7\text{ Dollars} > 3\text{ Dollars}$ which is true,
- the data 4 produces the sentence $4\text{ Dollars} > 3\text{ Dollars}$ which is true,
- the data 3 produces the sentence $3\text{ Dollars} > 3\text{ Dollars}$ which is false.
1.3. EQUATIONS AND “INEQUATIONS”

so that 7, 4 are solutions of the form $x$ Dollars $> 3$ Dollars while 3 is a non-solution.

c. Boxes, though, would soon turn out to be impossibly difficult to use and, instead, we will use unspecified numerators, such as for instance the letter $x$, as in

\[ x \text{ Dollars} \]

and, instead of the form $\square$ Dollars $> 3$ Dollars we shall write

\[ x \text{ Dollars} > 3 \text{ Dollars} \]

We shall call:
- **equations** those forms whose verb is $=$,
- **strict inequations** those forms whose verb is either $<$ or $>$,
- **bounded inequations** those forms whose verb is either $\leq$ or $\geq$.

d. Instead of filling the box with the data, say, 3, we replace $x$ by 3 and the instruction to do so will be

\[ |_{\text{where } x:=3} \]

in which the symbol $:=$, borrowed from a computer language called PASCAL, is read as “is to be replaced by”. Thus

\[ x \text{ Dollars} |_{\text{where } x:=3} \]

is a specifying-phrase in that it specifies

\[ 3 \text{ Dollars} \]

The following sentence

\[ x \text{ Dollars} |_{\text{where } x:=3} = 3 \text{ Dollars} \]

is therefore “trivially” true; it is an example of a type of sentence called identity because it identifies the numerator specified by the specifying-phrase.

We also have
- \[ x \text{ Dollars} |_{\text{where } x:=7} > 3 \text{ Dollars}, \]
- \[ x \text{ Dollars} |_{\text{where } x:=4} > 3 \text{ Dollars}, \]
- \[ x \text{ Dollars} |_{\text{where } x:=3} \not> 3 \text{ Dollars}. \]

3. We now turn to the simplest possible instance of a more general problem which is that we shall now want all the collections, if any, that stand in a given relationship with a given collection.

For example,
Say Jack has five dollars on the counter. We then want to find all collections of dollars that satisfy a given one of the following three requirements:

i. is less numerous than Jack’s collection,
ii. is more numerous than Jack’s collection,
iii. is as numerous as Jack’s collection.

(In other words, we are looking here at three distinct problems at once.)

❖ On the board, we are looking for the solution set of the corresponding inequation/equation:

i. $x$ Dollars < 5 Dollars
ii. $x$ Dollars > 5 Dollars
iii. $x$ Dollars = 5 Dollars

We now proceed to do just so. Regardless of which one of the three requirements we are trying to satisfy, we begin by considering the equation

$$x \text{ Dollars} = 5 \text{ Dollars}$$

whose solution set contains of course one, and only one, numerator: 5.

Then,

a. If it was the equation we were trying to solve, we are of course done.
b. If we were trying to solve either one of the inequations

$$x \text{ Dollars} < 5 \text{ Dollars} \quad \text{or} \quad x \text{ Dollars} > 5 \text{ Dollars},$$

it remains to determine which side of the break-even point the solution set of the inequation is. (The break-even point is the solution of their associated equation, $x \text{ Dollars} = 5 \text{ Dollars}$, that is, of the equation obtained from the inequation by replacing the verb, < or >, by the verb =.)

That the solution set must be a whole side of the break-even point is because if the solution set was only part of a whole side, then there would have to be both a solution and a non-solution on the same side of the break-even point and then there would have to be another break-even point in-between the two. But that cannot be since a break-even point is a solution of the associated equation $x \text{ Dollars} = 5 \text{ Dollars}$ and we just saw that it has one and only one solution, namely 5.

So, on each side of the break-even point, all we need to do is to pick one numerator and test it against the wanted requirement, that is to ask whether this test-point is a solution or a non-solution: Then, every numerator on the same side of the break-even point as the test-point will be the same.
For instance, say we are looking at the inequation

\[ x \text{ Dollars} > 5 \text{ Dollars} \]

The associated equation is

\[ x \text{ Dollars} = 5 \text{ Dollars} \]

so that the break-even point of the inequation is 5. Then, on each side of 5, we pick a test-point. Say we pick 3 and 7. Since to count from 3 to 5 we have to count forward, 3 is not a solution and all numerators on the same side of 5 as 3 will not be solutions either. Since to count from 7 to 5 we have to count backward, 7 is a solution and all numerators on the same side of 5 as 7 will also be solutions so that the solution set of the inequation

\[ x \text{ Dollars} > 5 \text{ Dollars} \]

is 6, 7, 8, \ldots.

Note. It is customary, though, to write solutions sets in-between curly brackets as in \{6, 7, 8, \ldots\} and we shall follow the custom.

Observe that the time we spent with the above procedure does not depend anymore on the number of collections we are dealing with.

Observe that, here, the break-even point is also an endpoint in that all the numerators on the one side of the break-even point are solutions and all the numerators on the other side of the break-even point are not solutions. This, though, will not be always the case and we will encounter break-even points that will turn out not to be endpoints.

1.4 Aggregating To A Collection. Addition.

Comparing collections is static in that nothing gets created and we now turn to operations on collections which are dynamic in that:

i. Starting from a given initial situation,

ii. We perform some action on the initial situation,

iii. Which results in some terminal situation.

Given an operation, we will be considering different types of problems that can be associated with the operation. In the simplest type, which we shall call direct problems, given an initial situation and an action, we want to find the terminal situation\(^{13}\). (We call this a direct problem because it “goes in the same direction” as the operation.)

\(^{13}\)Educologists will of course recognize this as the problem which consists, given an input, of finding the output under a given function.
In this section, we consider direct problems associated with aggregation, an operation in which the initial situation involves a collection of objects, the action is to aggregate another collection of objects (of the same kind) and the terminal situation then involves the aggregate collection, namely the collection obtained by collecting all the objects in the two collections into one single collection\(^{14}\).

For instance, a direct problem might be

♠ Aggregating FOUR dollars to THREE dollars:

Performing the action of collecting all the objects in the two collections gives the collection in the terminal situation:

On the board,

i. In order to state the problem we use the symbol + to denote addition, the procedure that will give us the numerator of the number-phrase that represents the aggregate collection, and we write

\[3 \text{ Dollars } + 4 \text{ Dollars} \rightarrow 3 \text{ Dollars } + 4 \text{ Dollars}\]

where 3 Dollars + 4 Dollars is the specifying-phrase that represent on the board the aggregate collection before we count it. We shall call it a specifying-phrase because, while it is not a number-phrase, it does specify a number-phrase namely that which will be the result of the addition.

More succinctly, but less transparently, we shall usually write only the specifying phrase

\[3 \text{ Dollars } + 4 \text{ Dollars}\]

\(^{14}\)In other words, we are not introducing addition as a binary operation but as a unary operation, the pro and con of which Educologists will readily see.
ii. In order to **identify** the collection specified by the specifying-phrase\(^{15}\), we **count** the *initial* collection and then **forward count** the collection being aggregated, that is, starting after the count of the *initial* collection, we call the digits that succeed it in the succession 1, 2, 3, 4, 5, 6, 7, 8, 9 while pointing at the objects in the collection being *aggregated*. For instance, \(\frac{4, 5, 6, 7}{3}\) is a *forward count* that starts after 3 and ends with 7. The numerator of the number-phrase that represents the *aggregate* collection is the *end* of the forward count.

iii. In order to **represent** the solution of the direct problem, we write a *sentence* which we will call an **identifying sentence** because it *identifies* the number-phrase that was specified by the specifying-phrase:

\[
3 \text{ Dollars} + 4 \text{ Dollars} \rightarrow 3 \text{ Dollars} + 4 \text{ Dollars} = 7 \text{ Dollars}
\]

or, more succinctly,

\[
3 \text{ Dollars} + 4 \text{ Dollars} = 7 \text{ Dollars}
\]

*Note.* A *specifying-phrase* such as \(3 \text{ Dollars} + 4 \text{ Dollars}\) is of course not to be confused with a *sentence* such as \(3 \text{ Dollars} < 4 \text{ Dollars}\).

Altogether then,

\(^{15}\)Educologists will surely wonder why not just say “compute”. It is a matter of *connotation*: “identify” leaves the question open while “compute” implies that the question is already closed and displaces the student’s attention to “how” to get the “answer”.
CHAPTER 1. BASIC COLLECTIONS OF MONEY

1.5 Removing From A Collection; Subtraction. (A Reverse Problem.)

We saw in Section 1.4 that, given an operation, a direct problem consists in performing a given action on a given initial situation and thus getting to some terminal situation—whatever that may turn out to be. In this section we consider a rather different kind of problem coming out of the fact that we are usually not ready to accept whatever terminal situation may happen to come up but, rather, that we usually have a goal in mind, namely a specific terminal situation that we want.

Generally speaking, we shall call reverse problem any problem involving a wanted terminal situation but there are two types of reverse problem depending on what else is given aside from the wanted terminal situation.

• If it is also the initial situation that is given, then what we must find is what action on this initial situation will get us the wanted terminal situation.

16Educologists will surely appreciate this being “question oriented”.

<table>
<thead>
<tr>
<th>♠ On the counter.</th>
<th>♣ On the board.</th>
</tr>
</thead>
<tbody>
<tr>
<td>We have 3 Dollars + 4 Dollars</td>
<td>We write the specifying phrase(^{16})</td>
</tr>
<tr>
<td>to which we aggregate</td>
<td>We count the 1st collection: (\frac{1, 2, 3}{\phantom{1, 2, 3}})</td>
</tr>
<tr>
<td>We count the 2nd collection forward starting after 3: (\frac{4, 5, 6, 7}{\phantom{4, 5, 6, 7}})</td>
<td></td>
</tr>
<tr>
<td>The aggregate collection is: The numerator of the result is 7.</td>
<td>We write the identifying sentence 3 Dollars + 4 Dollars = 7 Dollars</td>
</tr>
</tbody>
</table>

So, we have the general statement:

When we aggregate on the counter, we add on the board by counting forward.
1.5. SUBTRACTION

situation.
• If it is also the action that is given, then what we must find is for what initial situation will this action get us the wanted terminal situation.

In either case we call this a reverse problem because, since it starts from a wanted terminal situation, it “goes in the opposite direction” from the actual operation.

1. A special instance of a reverse problem arises when we want to undo the result of an action, that is, when we want to “return” from the terminal situation to the initial situation.

For instance, we might want to undo the aggregation of a collection to an initial collection.

The reverse problem associated with the “undo aggregate” operation then is to find what action on the original terminal situation will get us back to the original initial situation:

♠ On the counter, we must remove from the original terminal collection the collection that had been aggregated to the original initial collection.

❖ On the board, we count the terminal collection and then we count backward from the numerator of the terminal collection the collection that
had been *aggregated.* (This makes sense since, in the original operation, we obtained the numerator of the *terminal* collection by counting *forward* from the numerator of the *initial* collection the collection being *aggregated.*)

We shall say that we *subtract* the numerator of the collection being *aggregated*—in the original operation—from the numerator of the *terminal* collection and we use the symbol − to write, for instance,

\[
7 \text{ Dollars } - 4 \text{ Dollars} \rightarrow 7 \text{ Dollars } - 4 \text{ Dollars}
\]

where 7 Dollars − 4 Dollars is the *specifying-phrase* that represent on the board the *leftover* collection before we count it, namely that which will be the *result* of the subtraction.

In other words,

<table>
<thead>
<tr>
<th>♠ On the <em>counter.</em></th>
<th>♦ On the <em>board.</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>From the <em>terminal</em> collection in the original operation, we <em>remove</em> the collection that had been <em>aggregated</em></td>
<td>We write the <em>specifying phrase</em> 7 Dollars − 4 Dollars</td>
</tr>
<tr>
<td>We count the <em>terminal</em> collection: 1, 2, 3, 4, 5, 6, 7</td>
<td></td>
</tr>
<tr>
<td>The <em>leftover</em> collection</td>
<td>We count the <em>aggregated</em> collection <em>backward</em> starting <em>after</em> 7: [3, 4, 5, 6]</td>
</tr>
<tr>
<td>The <em>numerator</em> of the <em>initial</em> collection is 3.</td>
<td>The <em>sentence</em> 7 Dollars − 4 Dollars = 3 Dollars</td>
</tr>
</tbody>
</table>

**2.** We now turn to a reverse problem that is more general in that, given any *initial* collection and any wanted *terminal* collection, we shall now want to find what collection (if possible) we have to *aggregate* to the *given* initial collection to get the wanted *terminal* collection.
1.5. SUBTRACTION

Here, though, because the wanted terminal collection need not have resulted from the aggregation of a collection, but can now be any collection, the reverse problem, as we shall see, may or may not have a solution. There are three cases.

a. The wanted terminal collection is more numerous than the given initial collection:

To find the collection to be aggregated, we proceed essentially as when we wanted to undo aggregation

♠ On the counter, we remove the initial collection from the wanted terminal collection.

❖ On the board, we subtract the numerator of the initial collection from the numerator of the wanted terminal collection, that is, we count the initial collection backward from the numerator of the wanted terminal collection.

partial undo

b. The wanted terminal collection is as numerous as the initial collection.

♠ There is just enough in the wanted terminal situation to remove the given first collection but, rather than to say that there is no leftover collection, we shall say that the leftover collection is empty.

❖ We subtract the initial numerator from the terminal numerator, that is, after we have counted the wanted terminal collection we count the
CHAPTER 1. BASIC COLLECTIONS OF MONEY

initial collection backward but here we must introduce a new digit, 0, to end the backward count. The digit 0 is thus the numerator in the number-phrase that represents any empty collection.\footnote{Educologists will be interested to note that students have no trouble with 0 Dollar being different from 0 Dime while they certainly have trouble with “the empty set”.
}

Note. As a matter of historical fact, 0 was invented much later than the other digits and not for this purpose. We shall see the historical purpose in Section ??

c. The wanted terminal collection is less numerous than the initial collection.

♠ There is not enough in the wanted terminal situation to remove the given first collection. The reverse problem has no solution.

❖ On the board, we cannot subtract the initial numerator from the terminal numerator because we cannot count backward more than we counted in the first place!

Thus, say, specifying-phrases such as 3 Dollars − 5 Dollars make no sense whatsoever.

So, we have the general statement:

When we remove on the counter, we subtract on the board by counting backward.

3. We can now look at more complicated problems in which we would be looking for the solution set of one of the following

i. 3 Dollars + x Dollars < 7 Dollars or 3 Dollars + x Dollars ≤ 7 Dollars

ii. 3 Dollars + x Dollars > 7 Dollars or 3 Dollars + x Dollars ≥ 7 Dollars

iii. 3 Dollars + x Dollars = 7 Dollars

For instance, say the initial situation is that Jack has three dollars that he will donate to Jill but that the wanted terminal situation is that Jill should have a collection more numerous than seven dollars. The question thus is what collection should be aggregated to Jack’s collection.

♠ On the counter, removing the three dollars in Jack’s collection from seven dollars, we find that four dollars are leftover. So, if we aggregate four dollars to Jack’s collection, then the aggregated collection will be as numerous as Jill’s collection and aggregating a collection more numerous than four dollars to Jack’s collection will make Jill’s collection more numerous than seven dollars.

❖ On the board

− Jack’s collection is represented by 3 Dollars and Jill’s collection is represented by 7 Dollars and thus we are trying to find the solution(s),
if any, of the inequation

\[ 3 \text{ Dollars} + x \text{ Dollars} > 7 \text{ Dollars} \]

To obtain the break-even point, that is the solution of the associated equation,

\[ 3 \text{ Dollars} + x \text{ Dollars} = 7 \text{ Dollars} \]

we must identify

\[ 7 \text{ Dollars} - 3 \text{ Dollars} \]

that is we must count from 3 to 7:

\[ 3 \rightarrow 4, 5, 6, 7 \rightarrow 7 \]

which is a forward count of 4. Thus the break-even point is 7 Dollars – 3 Dollars = 4 Dollars.

We pick a test-point on each side of the break-even point, say 2 Dollars and 5 Dollars.
By counting from 3, we get:

\[ 3 \text{ Dollars} + 2 \text{ Dollars} \not\geq 7 \text{ Dollars} \]

and

\[ 3 \text{ Dollars} + 5 \text{ Dollars} > 7 \text{ Dollars} \]

So the solution set of \( 3 \text{ Dollars} + x \text{ Dollars} > 7 \text{ Dollars} \) is \{5, 6, 7, \ldots\}.

### 1.6 Combinations.

Situations in the real world are rarely that simple that they only involve one single kind of objects. As it turns out, though, only a small but far-reaching adjustment needs to be made to what we have done so far.

1. When the objects are not all of the same kind, that is when we do not have a collection and therefore we cannot represent them by a (counting) number-phrase.

For instance, say we have dime, dime, nickel, nickel, nickel, nickel, nickel, nickel, dime, dime, dime, nickel, nickel, nickel?
CHAPTER 1. BASIC COLLECTIONS OF MONEY

In the latter case, for instance, and in accordance with the “second breakthrough in the development of arithmetic” (Section 1.1), we begin by separating the objects into a bunch\(^{18}\) of collections:

- the collection dollar, dollar, which we can represent by the (counting) number-phrase 2 Dollars
- the collection dime, dime dime, dime, which we can represent by the (counting) number-phrase 4 Dimes
- the collection nickel, nickel, nickel, which we can represent by the (counting) number-phrases 3 Nickels

Then, we represent the bunch on the board by writing 2 Dollars & 4 Dimes & 3 Nickels where & means “and”. We will call this a combination\(^{19}\) of Dollars, Dimes and Nickels. Thus combinations represent on the board bunches of collections on the counter.

Combinations are a very powerful concept that comes up again and again and that, in fact, is the subject of a whole part of mathematics called Linear Algebra\(^{20}\).

Note. Here again, it is usual to write, say, 2 Dollars & Dime but while we see of course that Dollars is the denominator in a number-phrase whose numerator is 2, we have to remember that Dime is not a denominator but really stands for a number-phrase whose numerator is 1 and whose denominator is Dime so that 2 Dollars & Dime stands for 2 Dollars & 1 Dime. We will always write, say, 2 Dollars & 1 Dime rather than 2 Dollars & Dime.

2. In the absence of any additional information, we cannot compare bunches of collections. We shall see in Section ?? what kind of information permits what kind of comparison.

3. When two collections consist of different kinds of objects, we cannot aggregate them since the result would not be a collection but a bunch of two collections.

However, we can attach bunches of collections and the result is still just

---

\(^{18}\)I am avoiding the terms set, group and bundle since they are pre-empted.

\(^{19}\)The term has better connotations than vector.

\(^{20}\)There, the vector-phrase \(2\hat{i} + 3\hat{j}\) represents the answer to the question “What is on the counter?” while the numerator-pair \((2,3)\) represents the answer to the question “How many are on the counter?”.

By the way, in Linear Algebra too, the denominators corresponding to the “dimensions” the numerators refer to are usually “understood”. Which may explain why the dual space is never introduced up front. See NotMU, Fall 2003.
another bunch represented by a combination\textsuperscript{21}. For instance,

\[ 2 \text{ Dollars} \& 3 \text{ Nickels} + 4 \text{ Dollars} \& 5 \text{ Dimes} = \text{ Dollar, Dollar, Nickel, Nickel,}
\quad \text{ Dollar, Dollar, Dollar, Dollar, Dime, Dime, Dime, Dime, Dime, Dime, Dime,}
\quad = 6 \text{ Dollars} \& 5 \text{ Dimes} \& 3 \text{ Nickels} \]

4. Neither, when two collections consist of \textit{different} kinds of objects, can we cannot \textit{remove} one from the other. Occasionally, we can \textit{detach} one bunch from another and the result being usually a bunch. For instance,

\[ 7 \text{ Dollars} \& 5 \text{ Nickels} \& 9 \text{ Dimes} - 4 \text{ Dollars} \& 1 \text{ Dime} = 7 \text{ Dollars} - 4 \text{ Dollars} \& 5 \text{ Nickels} \& 9 \text{ Dimes} - 1 \text{ Dime} = 3 \text{ Dollars} \& 5 \text{ Nickels} \& 8 \text{ Dimes} \]

However, most of the time we cannot as, for instance, in

\[ 7 \text{ Dollars} \& 5 \text{ Nickels} - 4 \text{ Dollars} \& 3 \text{ Dimes} \]

\textsuperscript{21}Educologists will of course recognize this as addition in a \textit{vector space}.
Chapter 2

Accounting For
Extended Collections
Of Money On A Counter

We now turn to the case of collections that are extended in that they have more objects than we have digits so that we cannot represent them on the board with just the digits we have\(^1\).

2.1 Bundles and Exchanges

We begin by introducing two ideas relating to collections that we shall use systematically.

1. The first idea is that, given any collection of objects, we can bundle it into one object of a new kind.
   a. For instance, banks wrap collections of fifty dimes into one roll-of-dimes.
   ♠ Similarly, we may thus have on the counter, say, a bundle-of-THREE-dimes as a new kind of object.
   ✦ We shall represent on the board this bundle, a new kind of object, by (3 Dimes), a new kind of denominator in which the parentheses represent the wrapping.

\(^1\)It is not clear, though, that this is the right place for something which, after all, is mostly a syntactic matter. For instance, there might be some advantage in delaying it until after the next chapter since the latter is also, if from a different viewpoint, a natural extension of the preceding chapter. So, which of the two progressions would make the development smoother for the students remains for Educologists to investigate.
b. Then, of course, we can count collections of bundles.
   ♠ For instance, we may have on the counter \textit{bundle-of-TWO-dimes, bundle-of-TWO-dimes, bundle-of-TWO-dimes}.
   ♣ We represent this on the board by writing \((2 \text{ Dimes}), (2 \text{ Dimes}), (2 \text{ Dimes})\) and therefore \(3(2 \text{ Dimes})\) in which 3 is the \textit{numerator} and \((2 \text{ Dimes})\) is the \textit{denominator}.

   c. Now we look at what happens when we \textbf{unpack} the bundles.
   ♠ For instance, unpacking \textit{bundle-of-TWO-dimes, bundle-of-TWO-dimes, bundle-of-TWO-dimes} gives \textit{dime, dime, dime, dime, dime, dime}.
   ♣ On the board, we write, rather naturally
   \[
   3(2 \text{ Dimes}) = (2 \text{ Dimes}), (2 \text{ Dimes}), (2 \text{ Dimes})
   = \text{ Dimes, Dimes, Dimes, Dimes, Dimes}
   = 6 \text{ Dimes}
   \]
   where we obtained the numerator by \textit{counting} as above. But we can also write
   \[
   3(2 \text{ Dimes}) = 2 \text{ Dimes} + 2 \text{ Dimes} + 2 \text{ Dimes}
   = 6 \text{ Dimes}
   \]
   where we used \textit{addition}. And, finally, we can just write
   \[
   3(2 \text{ Dimes}) = 6 \text{ Dimes}
   \]
   where we used the \textbf{multiplication} \textbf{tables} that we memorized as children\footnote{Educologists will note that we studiously avoid speaking of “multiplication”. Indeed, the concept involved here is an entirely different one, namely \textbf{additive power}. The multiplication tables are used here merely as a \textit{syntactic procedure} devoid of any \textit{meaning}.}.

d. There is of course nothing to prevent us to bundle collections that we cannot represent on the board.
   ♠ For instance, we can bundle \textit{dime, dime, dime, dime, dime, dime, dime, dime, dime, dime} as a \textit{bundle-of-TEN-dimes}.
   ♣ And while we \textbf{cannot} represent \textit{dime, dime, dime, dime, dime, dime, dime, dime, dime, dime} on the board, \textbf{we can} represent a \textit{bundle-of-TEN-dimes} as \((\text{Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime})\).
   Instead however, we shall write \((\text{TEN Dimes})\) as a \textbf{shorthand} for the bundle \((\text{Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime})\).
2. The second idea is that since a bundle is an object, we can exchange it for another object, of a different kind. This will be particularly useful as dealing with bundles can be quite cumbersome.

♠ While we cannot represent ten dimes on the board, we can bundle it as bundle-of-TEN-dimes, and then exchange the bundle for a new object, a dollar, which is an object of another kind that we can also represent on the board.

❖ On the board, we will write (ten Dimes) = 1 Dollar.

2.2 Representing Extended Collections With Combinations of (Counting) Number-Phrases

The general idea will be to reduce the number of objects we have to deal with until we need only count basic collections.

We will achieve this by bundling the collections and then counting the bundles of which there are necessarily fewer than there were objects in the original collections. If the collection of bundles is basic, then we can count it. If not, then we bundle the collection of bundles. Etc.

Actually, in the case of money, rather than to deal with the bundles themselves, we usually exchange the bundles for other objects. Then we have collections of these other objects we can bundle. Etc. This avoids having to deal with bundles of bundles of bundles . . . , which, however, is what we shall have to do when, in Section 4.1, we shall count extended collections of discrete goods.

In other words, when dealing with money, we will be using the two ideas that we discussed in Section 2.1.

1. We begin with an example.

♠ Suppose we have, say, the following collection on the counter:

dime, dime, dime, dime, dime, dime, dime, dime, dime, dime,
dime, dime, dime, dime, dime, dime, dime, dime, dime, dime

As we have no digit to represent on the board twenty-three dimes on the counter,

− we bundle ten dimes (out of the twenty-three dimes on the counter),

− we bundle another ten dimes (out of the thirteen dimes left on the counter),

− we then exchange each bundle of ten dimes for one dollar.
And so, instead of the original collection, we now have on the counter a bunch of two basic collections:

dollar, dollar
dime, dime, dime.

which we will be able to represent on the board as a combination.

❖ On the board, we write correspondingly:

\[
\begin{align*}
\text{Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime} \\
&= (\text{ten Dimes}), (\text{ten Dimes}), \text{Dime, Dime, Dime.} \\
&= \text{Dollar, Dollar, Dime, Dime, Dime.} \\
&= 2 \text{ Dollars & 3 Dimes.}
\end{align*}
\]

where we wrote (ten Dimes) as a shorthand for the bundle (Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime). (Section 2.1.)

2. Of course, the problem is that we will have to keep introducing ever-larger denominators. For the time being, we will stick to bills that change at a ten to one exchange rate:

- **Hamiltons:**

- **Franklins:**

- **Cleveland:**

In other words, we shall disregard five-dollar-bills, twenty-dollar-bills and fifty-dollar-bills. (However, we will deal with these later on.)

To emphasize the distinction between the actual bills on the counter and the denominators on the board, Hamiltons, Franklins, Cleveland as denominators for ten-dollar-bills, hundred-dollar-bills and thousand-dollar-bills and also, since consistency is a virtue we will want to practice, Washingtons instead of Dollars. But, for lack of a better word, we will keep writing Dimes as denominator for dimes.

3. We are now in a position to deal with larger and larger collections, such as, for instance, occur when we aggregate bunches of collections.

❖ For instance, when we want to aggregate eight dollars and five dimes with six dollars and three dimes, we bundle ten of the resulting fourteen dollars on the counter and exchange the bundle for a ten-dollar-bill.
The (board) representation of this is:

\[
[8 \text{ Washingtons} & 5 \text{ Dimes}] + [6 \text{ Washingtons} & 3 \text{ Dimes}]
\]
\[= \text{fourteen Washingtons} & 8 \text{ Dimes}\]
\[= 1 \text{ Hamilton} & 4 \text{ Washingtons} & 8 \text{ Dimes}\]

4. As a result of all this, we can now represent collections of all the way up to \text{ninety nine thousand nine hundred ninety nine dimes} which we write on the board as \text{9 Cleveland & 9 Franklin & 9 Hamilton & 9 Washington & 9 Dimes}. In order to go beyond that, we would have to continue to create ever “larger” denominators. Instead, we shall eventually develop another, even more systematic approach.

2.3 (Decimal) Headings

To simplify this representation, a somewhat natural idea would be to write the denominators only \textit{once} and then just write the numerators, “as needed”. The problem, though, is how to indicate which numerator goes with which denominator.

What we do is to write the denominators into a \textit{heading} such as

\begin{tabular}{ccccccc}
Clevelands & Franklins & Hamiltons & Washingtons & Dimes \\
\hline
3 & 1 & 7
\end{tabular}

and then write the numerators directly under the corresponding denominators.

For instance, we write the combination \text{3 Franklin & 1 Washington & 7 Dimes} as follows:

\begin{tabular}{ccccccc}
Clevelands & Franklins & Hamiltons & Washingtons & Dimes \\
\hline
3 & 1 & 7
\end{tabular}

Thus, each \textit{column} corresponds to a collection of a \textit{different kind} of objects. While this may look like going from cumbersome to very cumbersome, we will now see how easy it is to \textit{work} with.

2.4 Adding Under A Heading

In Section 1.4, in the case of \textit{basic} collections, we defined \textit{addition} as the (board) procedure that represents \textit{aggregation} of collections on the counter.
Here, we focus on how addition works out, under a heading, for extended collections.

❖ Suppose, for instance, that we wanted to add the three combinations,

- 8 Hamiltons & 7 Dimes
- 8 Washingtons & 2 Dimes
- 5 Hamiltons & 3 Washingtons & 4 Dimes

First, we rewrite the three combinations under the heading:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Clevelands</td>
<td>Franklins</td>
<td>Hamiltons</td>
<td>Washingtons</td>
<td>Dimes</td>
</tr>
<tr>
<td></td>
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<tr>
<td>8</td>
<td>7</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Adding up the Dimes by “counting on our fingers”:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Clevelands</td>
<td>Franklins</td>
<td>Hamiltons</td>
<td>Washingtons</td>
<td>Dimes</td>
</tr>
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<td>8</td>
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</tr>
<tr>
<td>8</td>
<td>2</td>
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<tr>
<td>5</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

THIRTEEN

gives us THIRTEEN Dimes but we cannot write that. So, we bundle ten of the thirteen dimes and, if we can exchange the bundle for one dollar then we can write³:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Clevelands</td>
<td>Franklins</td>
<td>Hamiltons</td>
<td>Washingtons</td>
<td>Dimes</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Adding up the Washingtons by “counting on our fingers”:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Clevelands</td>
<td>Franklins</td>
<td>Hamiltons</td>
<td>Washingtons</td>
<td>Dimes</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TWOELVE 3

²No Educologist has ever bothered to explain what “carry over” is supposed to evoke in this context.
2.5. Subtracting Under A Heading

In Section 1.5, we defined subtraction, in the case of basic collections, as the (board) procedure that represents what is left on the counter of a collection after we have taken a number of objects. Here, we focus on how addition works out, under a heading, for extended collections.

1. Occasionally, we have enough of each kind of bills to take what we

---

gives us twelve Washingtons but we cannot write that. So, we bundle ten of the twelve dollars and, if we can exchange the bundle for one ten-dollar-bill, then we can write:

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklin</th>
<th>Hamilton</th>
<th>Washington</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>8</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

Finally, adding the Hamiltons by “counting on our fingers”:

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklin</th>
<th>Hamilton</th>
<th>Washington</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>8</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklin</th>
<th>Hamilton</th>
<th>Washington</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

which gives us fourteen Hamiltons but we cannot write that. So, we bundle ten of the fourteen ten-dollar-bills and, if we can exchange the bundle for one hundred-dollar-bill, then we can write:

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklin</th>
<th>Hamilton</th>
<th>Washington</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>8</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

Thus, if we have changing facilities, we can then carry out the addition and the result will be 1 Franklin & 4 Hamiltons & 2 Washingtons & 3 Dimes.

---

2.5 Subtracting Under A Heading

In Section 1.5, we defined subtraction, in the case of basic collections, as the (board) procedure that represents what is left on the counter of a collection after we have taken a number of objects. Here, we focus on how addition works out, under a heading, for extended collections.

1. Occasionally, we have enough of each kind of bills to take what we
want.

♠ Suppose for instance that SEVENTY-EIGHT dollars, in the form of SEVEN ten-dollar-bills and EIGHT one-dollar-bills, are on the counter and that we want TWENTY-FIVE dollars, in the form of TWO ten-dollar-bills and FIVE one-dollar-bills.

❖ On the board, what we want to do is the subtraction

7 Hamiltons & 8 Washingtons − 2 Hamiltons & 5 Washingtons

which we rewrite under a heading as follows:

<table>
<thead>
<tr>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Then, counting 5 steps backward from 8, 3, 4, 5, 6, 7, gives us 3 and counting 2 steps backward from 7, 5, 6, 7, gives us 5.

<table>
<thead>
<tr>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

that is 5 Hamiltons & 3 Washingtons.

2. Most of the time, though, we need to exchange and unpack.

♠ Suppose for instance that, instead of SEVENTY-EIGHT dollars, there is only SEVENTY-THREE dollars on the counter, in the form of SEVEN ten-dollar-bills and THREE one-dollar-bills, and that we want TWENTY-FIVE dollars, in the form of TWO ten-dollar-bills and FIVE one-dollar-bills.

The problem we immediately run into is that THREE one-dollar-bills is not enough from which to take FIVE one-dollar-bills. However, if we can exchange\(^4\) ONE ten-dollar-bill for a bundle of TEN one-dollar-bills, then we can unpack the bundle so that we now have THIRTEEN one-dollar-bills from which we can take the FIVE one-dollar-bills. But now, of course, we have only SIX ten-dollar-bills from which to take the TWO ten-dollar-bills. Altogether, this leaves us with FOUR ten-dollar-bills and EIGHT one-dollar-bills.

❖ On the board, we write:

\(^4\)Educologists will readily explain why they use the term “borrowing” even though it is both completely unfounded and perfectly non-explanatory.
2.6. **DECIMAL NUMBER-PHRASES**

<table>
<thead>
<tr>
<th>Clevelands</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
<td></td>
<td>7</td>
<td>3</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>2</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

and then

<table>
<thead>
<tr>
<th>Clevelands</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>/\ 6</td>
<td>THIRTEEN</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

from which we get

<table>
<thead>
<tr>
<th>Clevelands</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>/\ 6</td>
<td>THIRTEEN</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

and so the result of the subtraction is 4 Hamiltons & 8 Washingtons.

2.6 **Decimal Number-Phrases**

1. While headings are convenient when we want to work with several combinations, they are much too cumbersome if all we want is just to write one combination. So we will now develop yet another way in which it will be convenient both to write a single combination and to add several combinations.

   a. Suppose we wanted to specify by mail a whole lot of combinations. A natural idea would be to mail the heading

<table>
<thead>
<tr>
<th>Clevelands</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>
|            |           | once and for all\(^5\) so that, in later mails, the heading could now go “without saying” and we would just have to mail the numerators. For instance, rather than mailing the combination  

   7 Franklins & 2 Hamiltons & 4 Washingtons

   we would just mail the numerators 7, 2, 4. The problem with that, though, is that the recipient would not be able to reconstruct the specified combination as s/he wouldn’t know under which

---

\(^5\)This would be somewhat similar to what software engineers call a *declaration*. 
denominators in the heading to write these numerators and thus wouldn’t be able to recover the combination.

For instance, s/he wouldn’t know if the numerators 7, 2, 4 should be placed under the heading this way

\[
\begin{array}{cccc}
\text{Clevelands} & \text{Franklins} & \text{Hamiltons} & \text{Washingtons} \\
7 & 2 & 4 & \\
\end{array}
\]

or that way

\[
\begin{array}{cccc}
\text{Clevelands} & \text{Franklins} & \text{Hamiltons} & \text{Washingtons} \\
 & 7 & 2 & 4 \\
\end{array}
\]

or any other way.

So, we cannot just mail the numerators and, along with the numerators, we must also mail some information as to how the numerators are to be placed under the heading.

b. To that purpose, we introduce a new type of number-phrases, (decimal) number-phrase, which consists of:

- the numerators that we want to mail,
- a (select) denominator that we can “pick” from among the denominators in the heading, that is whose choice is entirely ours.
- a (decimal) pointer to indicate which of the numerators is to be placed under the (select) denominator, the agreement being that the other numerators are to be placed “accordingly”.

Note. For a little while, rather than a dot, we shall use _ for a decimal pointer. This is to remind us that, by tradition, a decimal pointer points to its left even though we read from left to right. However, after this section, we shall conform and use a decimal point. (The use of a dot as decimal pointer is not universal and, for instance, many languages use a comma instead of a dot.) The regrettable thing about decimal pointers, as opposed to decimal pointers, is that we will have to remember that the digit being pointed at is to the left of the decimal point.

All the individual numerators, together with the decimal pointer, will be collectively referred to as (decimal) numerator. Thus, just like a (counting) number-phrase consists of a numerator and a denominator, a (decimal) number-phrase consists of a (decimal) numerator and a (select) denominator.

For instance, after we have mailed the heading, we can specify the combination 7 Franklins & 2 Hamiltons & 4 Washingtons by mailing the (decimal)
2.6. **DECIMAL NUMBER-PHRASES**

A *number-phrase* is denoted as

\[ 72_{\ldots}4 \text{ Hamiltons} \]

where we “picked” Hamiltons as our (select) denominator and where 72...4 is the corresponding (decimal) numerator. The recipient would then reconstruct the specified combination as follows:

- since 2 is being pointed at, s/he would place 2 under Hamiltons since it is the (select) denominator,
- since 7 is left of 2, s/he would place 7 under Franklins, the denominator left of Hamiltons, and,
- since 4 is right of 2, s/he would place 4 under Washingtons, the denominator right of Hamiltons.

That is, s/he would write

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

which is indeed 7 Franklins & 2 Hamiltons & 4 Washingtons.

**c.** There is however a problem with combinations such as 5 Franklins & 3 Hamiltons & 8 Dimes which, under a heading, looks like

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Say we “pick” Franklins as our (select) denominator. The problem is that what is immediately to the right of Hamiltons is Washingtons and not Dimes. So, we cannot write 5...38 Franklins because the recipient would reconstruct that (decimal) number-phrase as

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

and thus as specifying 5 Franklins & 3 Hamiltons & 8 Washingtons.

In order to specify 5 Franklins & 3 Hamiltons & 8 Dimes we must indicate that there are no Washingtons. To that purpose, we use the digit 0 which we introduced in Section 1.5 and we write 5...308 Hamiltons and the recipient will reconstruct that as

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>0</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

which is 5 Franklins & 3 Hamiltons & 0 Washingtons & 8 Dimes and indeed the same as 5 Franklins & 3 Hamiltons & 8 Dimes.
Note. This is in fact what 0 was invented for.

2. There are two default rules. (We now use the decimal point . instead of the decimal pointer _.)

- When the decimal point is to the right of the (decimal) numerator, as in 7204 Dimes, it is customary not to write the decimal point at all and just to write 7204 Dimes. The corresponding default rule is:

  When there is no decimal point, it goes without saying that the decimal point is to the right of the (decimal) numerator.

- When there is no doubt as to what the (select) denominator is, say that it is Hamiltons, it is customary not to write it and, for instance, just to write 72.04 instead of 72.04 Hamiltons. The corresponding default rule is:

  When there is no (select) denominator, it goes without saying that it is the (select) denominator that was picked earlier on.

This, though, is extremely dangerous because it depends on us remembering what the (select) denominator is that was picked earlier on\(^6\). So, while it is being done all the time, we shall not.

Note. In the U.S., it is usual to write, for instance, .56 Hamiltons rather than 0.56 Hamiltons. We shall not do so in this text, if only because we don’t like the idea of a pointer pointing, at least apparently, at nothing. In any case, the “energy saving” would not be worth it.

3. To add and subtract (decimal) number-phrases we can just place them back under a heading. When the (decimal) number-phrases have the same (select) denominators, we need not even write the heading but just make sure that the pointed digits are lined up and that, in fact, is exactly what we were taught to do in school.

When the (decimal) number-phrases have different (select) denominators, we may begin by changing to a common (select) denominator but, keeping the heading in mind and with a little bit of practice, it is almost as easy to line up the numerators “as if” they were under the heading.

4. The use of (decimal) number-phrases has several advantages:

a. As we already saw, we can pick as our (select) denominator any denominator we want. For instance, a rich person might pick Clevelands as her/his (select) denominator while the rest of us would probably pick Washingtons as our (select) denominator.

\(^6\)This is perhaps one more occasion to remind Educologists that memory is the weakest part of the mind, that it is the first to go and that, so far, humans can be defined as thinking entities, that is entities amenable to logic, rather than mere memorizing ones.
b. Not only does using (decimal) number-phrases allow us to pick as our (select) denominator any denominator we want but, by placing the (decimal) number-phrase back under the heading, we can easily change the denominator we want to use as our (select) denominator. Of course we then need to adjust the decimal point accordingly.

For instance, placing 85.7 Hamiltons under a heading,

<table>
<thead>
<tr>
<th></th>
<th>Cleveland</th>
<th>Franklins</th>
<th>Hamitons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
<td>5</td>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

we see that the (decimal) number-phrase 85.7 Hamiltons could just as well be written as any of the following (decimal) number-phrases:

- 0.857 Cleveland
- 8.57 Franklins
- 857 Washingtons
- 8570 Dimes

Any of the above means the same as 8 Franklins & 5 Hamiltons & 7 Dimes.

c. Another advantage of (decimal) number-phrases is that if, for whatever reason, we needed to have a “smaller” or a “larger” (decimal) numerator, it would be easy to do. We would “move the decimal point” by placing the number-phrase back under a heading, change the digit being pointed and change the (select) denominator accordingly.

d. We can now count beyond 9: 1, 2, 3, …, 9, 10, 11, …, 99999.

e. Finally, there is another, intriguing, advantage to the use of (decimal) number-phrases. While we can now count all the way up to 9 Cleveland & 9 Franklins & 9 Hamiltons & 9 Washingtons & 9 Dimes, that is all the way up to 99999 Dimes, what if we wanted to add 1 Dime? Let us do it under a heading:

<table>
<thead>
<tr>
<th></th>
<th>Cleveland</th>
<th>Franklins</th>
<th>Hamitons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>1</td>
</tr>
</tbody>
</table>

We have no trouble reaching the following stage (in the presence of exchange facilities):
CHAPTER 2. EXTENDED COLLECTIONS OF MONEY

in the hundreds
in the ones
in the tenths
in the hundredths
in the thousands
principal unit

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklin</th>
<th>Hamilton</th>
<th>Washington</th>
<th>Dime</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>TEN</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

At this point, although we can bundle the ten thousand-dollar-bills, we cannot exchange the bundle for any bill. Nevertheless, as long as we “pick” any of the existing denominators for our (select) denominator, then we can write a number-phrase. For instance, if we “pick” Franklins, we can write the result as 100. Franklins or, if we “pick” Dimes, as 100000. Dimes.

Note. To some extent this would be cheating because we still do not have a denominator corresponding to the leftmost 1 but, other than that, everything looks fine and we could even say that we wrote the (decimal) number phrases pending the creation of that new denominator! But, in case you should worry, we shall in fact develop several ways in Section ?? to get automatically as many denominators as we need.

2.7 Comparing Orders of Magnitude

While both 3 Franklins and 7 Washingtons are larger than 8 Dimes, they are not so in the same manner. We shall say that

- 3 Franklins is in the hundreds,
- 7 Washingtons is in the ones,
- 8 Dimes is in the tenths.

More generally, we shall say that 0.00875 Hamiltons is in the hundredths as we can see from

0.00875 Hamiltons = 8.75 Cents

and that 42187.53 Dimes is in the thousands as we can see from 42187.53 Dimes = 4.218753 Cleveland

All of the above is in terms of Washingtons being the principal unit.
Chapter 3

Accounting For
Money Changing Hands
*Over* The Counter

We now deal with collections that, for whatever reason, are *marked* either one of two ways\(^1\).

The simplest example comes up when we want to undo aggregating *steps* on a path. When we wanted to undo aggregating *apples*, what we did was to remove them. But, when we want to undo aggregating *steps* on a path, we cannot do that by *removing* these steps—which would require going back in time—but only by walking the same number of steps in the opposite direction, that is *backward*. So we have to be able to count both *steps forward* and *steps backward*. But then, for instance, after we have walked forward three steps forward, we can aggregate five steps backward because, altogether, this amounts to walking two steps backward. However, in the case of steps on a path we will need to deal both with positions and walking steps.

### 3.1 States

We will call **state** a collection of objects that, *as a whole*, can be on *this-side* or *that-side* of some *benchmark*.

1. First, a few real-world examples.

\(^1\)It is difficult to understand what causes Educologists to delay the introduction of integers until *after* fractions.
• Being in such and such solar year. Thus, with Christ as benchmark, we can have three hundred forty five years after (345 AD) as well as three hundred forty five years before (345 BC).
• Being at such and such temperature. Thus, we can have $+15^\circ C$ as well as $-15^\circ C$ with the temperature at which water starts freezing as benchmark.
• Being in such and such financial state. Thus, five dollars “ahead of the game” and five dollars “in the hole” are examples of states a gambler can be in while five dollars “in the black” and five dollars “in the red” are examples of states a business can be in.
• Being at such and such point on a path.

2. On the board, we will represent a state by a signed-number-phrase that consists of:

• a (side-) sign to represent the side of the benchmark the collection is,
• the numerator that represents the number of objects in the collection,
• the denominator that represents the kind of objects in the collection.

However, because this will make procedures on the board a lot simpler, we will lump the side-sign together with the numerator of the number-phrase that represents the number of objects in the state and speak of a signed-numerator which we will separate from the denominator.

First, we record on the board, once and for all, which side of the benchmark is to be the standard side. States on the other side of the benchmark will be said to be on the opposite side. Then we need only use, say, ↑ to represent the standard side and ↓ for the opposite side.

For instance, say that in-the-black is on the standard side so that in-the-red is on the opposite side. Then,

<table>
<thead>
<tr>
<th>On the counter, we look at:</th>
<th>On the board, we write:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FIVE dollars in-the-black</strong></td>
<td>(5 ↑) Washingtons</td>
</tr>
<tr>
<td><strong>THREE dollars in-the-red</strong></td>
<td>(3 ↓) Washingtons</td>
</tr>
</tbody>
</table>

where (5 ↑) and (3 ↓) are the signed-numerators and Washingtons is the denominator. Thus, signed-number-phrases will be to states what number-phrases are to collections.

### 3.2 Transactions

We will call transaction a collection of objects that, as a whole, can go this-way or that-way over the counter. Then, just as with states, together
with the number and kind of objects in the collection, we will need to represent the direction of the transaction, that is the way the collection is going over the counter which we do very much in the same manner as with states. First we record on the board, once and for all, which way is to be in the standard direction. Transactions going the other way will be in the opposite direction. Then we need only use, say, \( \rightarrow \) to represent the standard direction and \( \leftarrow \) for the opposite direction.

For instance, say that going from Jack to Jill is in the standard direction so that going from Jill to Jack is in the opposite direction.

\[\begin{array}{c}
\text{♣ Over the counter, we look at:} \\
\text{♦ On the board, we write:}
\end{array}\]

\[
\begin{array}{c|c}
\text{FIVE dollars from-Jack-to-Jill} & (5 \rightarrow) \text{Washingtons} \\
\text{THREE dollars from-Jill-to-Jack} & (3 \leftarrow) \text{Washingtons}
\end{array}
\]

where \( (5 \rightarrow) \) and \( (3 \leftarrow) \) are the signed-numerators and Washingtons is the denominator. Thus, signed-number-phrases will be to transactions the same as what they are to states and what number-phrases are to collections.

3.3 Standard Representations: Signed-Number-Phrases versus T-Accounts

Of course, in practice, we do not use arrows but the ways mathematicians and accountants represent states and transactions are quite different.

1. Instead of arrows, mathematicians “re-use” the + sign for the standard side and the standard direction and the − sign for the opposite side and the opposite direction and write the sign ahead of the numerator. Moreover, the parentheses are usually omitted. Thus,

\[\begin{array}{c}
\text{♣ Over the counter, we look at:} \\
\text{♦ On the board, we write:}
\end{array}\]

\[
\begin{array}{c|c}
\text{FIVE dollars in-the-black} & +5 \text{Washingtons} \\
\text{THREE dollars in-the-red} & -3 \text{Washingtons}
\end{array}
\]

and,

\[\begin{array}{c}
\text{♣ Over the counter, we look at:} \\
\text{♦ On the board, we write:}
\end{array}\]

\[
\begin{array}{c|c}
\text{FIVE dollars from-Jack-to-Jill} & +5 \text{Washingtons} \\
\text{THREE dollars from-Jill-to-Jack} & -3 \text{Washingtons}
\end{array}
\]

direction standard direction opposite direction
CHAPTER 3. MONEY CHANGING HANDS

T-account
double-entry bookkeeping
balance

Note. Actually, the “usual way” is to let the + sign “go without saying” and to mark only, with the − sign, the states on the opposite side and the transactions in the opposite direction. The problem with this practice, though, is that it tends to blur on the board the distinction between states on the standard side or transactions in the standard direction and collections just sitting on the counter. So, we shall always write +5.

2. Accountants use T-accounts. While the rules for operating with T-accounts can, at least initially, appear a bit intricate, double-entry bookkeeping is tremendously powerful and well worth the effort of understanding its basic principles. Here, we will just give a few indications. (For lack of space, we will use here $ instead of Washington as denominator.)

a. A state, called balance in ACCOUNTING, is represented by a line in the corresponding T-account.

In the following examples, money in the black is represented on the left side of the T-account and money in the red is represented on the right side of the T-account.

♠ Over the counter, we look at: ♦ On the board, we write:

<table>
<thead>
<tr>
<th>Jill</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Black</td>
<td>Red</td>
<td>$3</td>
</tr>
</tbody>
</table>

Jill is THREE dollars in the red

and

♠ Over the counter, we look at: ♦ On the board, we write:

<table>
<thead>
<tr>
<th>Jack</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Black</td>
<td>Red</td>
<td>$5</td>
</tr>
</tbody>
</table>

Jack is FIVE dollars in the black

b. A transaction is represented by a line in the T-accounts of the two individuals involved in the transaction.

In the following example, money coming into the account is represented on the black side of the T-accounts while money going out of the account is represented on the red side of the T-accounts.

Of course, Educologists have never shown any interest in such crass matters which is quite regrettable in view of the Grothendieck construction of $\mathbb{Z}$ as $\mathbb{N}^2/\sim$ where $\sim$ is the equivalence relation of debit-credit pairs modulo the balance, that is $(a, b) \sim (c, d)$ iff $a + d = b + c$. Moreover, the “law of money conservation”, $\int_{\text{start}}^{\text{end}} \text{Net Income}(t) = \text{Position}(t)|_{\text{start}}^{\text{end}}$, is a rather nice instance of the Fundamental Theorem of the Calculus.
3.4. ADDITION

_merge string addition of signed-number-phrases ☙_

Over the counter, we look at:

On the board, we write:

<table>
<thead>
<tr>
<th></th>
<th>Jack</th>
<th>Jill</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Black (In)</td>
<td>Red (Out)</td>
</tr>
<tr>
<td>FIVE</td>
<td></td>
<td>$5</td>
</tr>
<tr>
<td>THREE</td>
<td>dollars from-Jack-to-Jill</td>
<td>$3</td>
</tr>
<tr>
<td>dollars from-Jill-to-Jack</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note. Instead of “write the signed-number-phrase for a transaction”, accountants say “enter a transaction” just as, instead of saying “write the number-phrase for a collection”, we say “count a collection”.

However, with the advent of computerized accounting, T-accounts are increasingly giving way to signed-number-phrases.

3.4 Adding Signed-Number-Phrases.

Suppose that, just like we aggregated collections on the counter, we now

- **merge states**, each on this or that side of the same benchmark

or

- **string transactions**, each going one way or the other (but between the same two people).

followed by

Then, just like addition of number-phrases was the board procedure that gave us the number-phrase that represents the result of aggregating collections, addition of signed-number-phrases will be the procedure that will give us the signed-number-phrase that represents the result of merging states or stringing transactions.

We will need a new symbol to distinguish addition of signed-number-phrases from addition of counting-number-phrases. But, as usual, mathematicians dislike introducing new symbols! So, we will try to have it both ways by re-using, yet another time, the symbol + but, at least for the time being, put it within a circle: ☙. Later, we will learn to rely on the context.

1. In order to help us picture things while dealing with signed-numerators, we revert temporarily to the “arrow notation” that we used just above.

In what follows, we deal with transactions but everything applies to states (just use ↑ and ↓ instead of → and ←).

Then, for instance, we look at

3 → as standing for → → →

5 ← as standing for ← ← ← ← ←
In other words, we look at $\rightarrow$ and $\leftarrow$ as if they were denominators that, furthermore, “cancel each other”:

$$
\text{and}
$$

so that

$$
1 \rightarrow \oplus 1 \leftarrow = 0 \leftarrow = 0 \rightarrow \quad \text{and} \quad 1 \leftarrow \oplus 1 \rightarrow = 0 \rightarrow = 0 \rightarrow
$$

2. When we string transactions, we must distinguish two cases.

a. The two transactions go in the same direction.

<table>
<thead>
<tr>
<th>♠ Say we have two transactions:</th>
<th>♦ On the board, we write:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FIVE dollars from-Jack-to-Jill</strong></td>
<td>$5 \rightarrow$ Washingtons</td>
</tr>
<tr>
<td><strong>THREE dollars from-Jack-to-Jill</strong></td>
<td>$3 \rightarrow$ Washingtons</td>
</tr>
<tr>
<td>Stringing the transactions</td>
<td>Adding the signed-numerators</td>
</tr>
<tr>
<td>gives</td>
<td>$5 \rightarrow \oplus 3 \rightarrow$</td>
</tr>
<tr>
<td><strong>EIGHT dollars from-Jack-to-Jill</strong></td>
<td>$+8$ Washingtons</td>
</tr>
</tbody>
</table>

*Accountants* would represent this as follows:

<table>
<thead>
<tr>
<th>♠ Over the counter, we look at:</th>
<th>♦ On the board, we write:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Jack</strong></td>
<td><strong>Jill</strong></td>
</tr>
<tr>
<td>Black (In)</td>
<td>Red (Out)</td>
</tr>
<tr>
<td><strong>FIVE dollars from-Jack-to-Jill</strong></td>
<td>$$5</td>
</tr>
<tr>
<td><strong>THREE dollars from-Jack-to-Jill</strong></td>
<td>$$3</td>
</tr>
<tr>
<td><strong>EIGHT dollars from-Jack-to-Jill</strong></td>
<td>$$8</td>
</tr>
</tbody>
</table>

In other words, when we add signed-numerators that have the same sign, we *add* the numerators and the sign of the resulting signed-numerator is of course the sign common to the signed-numerators being added.

b. The two transactions go in opposite directions.

---

3 The lack of syntactic parallel between “in the same direction” and “in opposite directions” can be troublesome. Moreover, “the two transactions are in opposite directions” does not mean the same as “the two transactions are in the opposite direction”.
Say we have the two transactions: On the board, we write:

| THREE dollars from-Jack-to-Jill | 3 → Washingtons |
| FIVE dollars from-Jill-to-Jack | 5 ← Washingtons |

Stringing the transactions:

Adding the signed-numerators:

\[ 3 \oplus 5 \]

\[ \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \]

\[ \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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incorrect. A priori, to obtain the new, corrected total, we have the following three choices.

1. We could strike out the incorrect signed-number-phrase and redo the entire addition:

\[-2 \text{ Dollars} \oplus -7 \text{ Dollars} \oplus +5 \text{ Dollars} \oplus \ldots \oplus +3 \text{ Dollars}\]

Of course, if the addition is really long, this is going to involve a lot of unnecessary work, redoing a lot that had been done correctly.

2. We could subtract the incorrect signed-number-phrase from the incorrect total:

\[-132 \text{ Dollars} \ominus -7 \text{ Dollars}\]

The trouble, though, is that we have no idea what procedure to use for $\ominus$!

3. We can cancel the effect of the incorrect signed-number-phrase on the incorrect total by adding the opposite of the incorrect signed-number-phrase to the incorrect total. Accountants call this entering an adjustment. That this must give us the same correct result as would choice 1. is easy to see by comparing:

- The addition in which $-7 \text{ Dollars}$, the incorrect signed-number-phrase, was struck out:

\[-2 \text{ Dollars} \oplus -7 \text{ Dollars} \oplus +5 \text{ Dollars} \oplus \ldots \oplus +3 \text{ Dollars}\]

- The addition in which $-7 \text{ Dollars}$, the incorrect signed-number-phrase has been left in but has been cancelled by the adjustment $+7 \text{ Dollars}$ that was added at the end:

\[-2 \text{ Dollars} \oplus -7 \text{ Dollars} \oplus +5 \text{ Dollars} \oplus \ldots \oplus +3 \text{ Dollars} \oplus \pm 7 \text{ Dollars}\]

Either way, the signed-number-phrases that are actually involved are:

\[-2 \text{ Dollars} \oplus +5 \text{ Dollars} \oplus \ldots \oplus +3 \text{ Dollars}\]

which makes the case.

Accountants would represent the above as follows:
3.6. CHANGE

We now look at the effect of a transaction on states. Given an initial state and a transaction involving that state, we will call final state the state after the transaction. (This will fit exactly what we did in Section 1.5.)

For instance,

♠ Looking at Jill, suppose that:

• In the initial state, Jill is THREE dollars in-the-red.

In other words,

• Subtracting the incorrect signed-number-phrase (choice 2.):

\[-132 \text{ Dollars} \quad \odot \quad -7 \text{ Dollars}\]

amounts to exactly the same as

• Adding the opposite of the incorrect signed-number-phrase (choice 3.):

\[-132 \text{ Dollars} \quad \oplus \quad +7 \text{ Dollars}\]

Since, as already pointed out, we have no ready-made procedure for subtraction, we will say that “adding the opposite” is the procedure and that, in the case of signed-number-phrases, to subtract something is just short for “to add its opposite”\(^4\).

3.6 Effect of Transactions on States

We now look at the effect of a transaction on states. Given an initial state and a transaction involving that state, we will call final state the state after the transaction. (This will fit exactly what we did in Section 1.5.)

For instance,

♠ Over the counter, we look at:

\[
\begin{array}{c|c|c|c|c|c|c}
\text{Striking out} & \text{Cancelling} \\
\text{Black} & \text{Red} & \text{Black} & \text{Red} \\
\text{(In)} & \text{(Out)} & \text{(In)} & \text{(Out)} \\
\hline
\text{TWO dollars out} & $2 & $2 \\
\text{SEVEN dollars out} & $7 & $7 \\
\text{FIVE dollars in} & $5 & $5 \\
\text{THREE dollars in} & $3 & $3 \\
\text{SEVEN dollars in (Adjustment)} & $7 & \\
\end{array}
\]

subtract
initial state
final state

\[
\begin{align*}
\text{In other words,} \\
\bullet \text{ Subtracting the incorrect signed-number-phrase} \quad \text{(choice 2.)} : \\
-132 \text{ Dollars} \quad \odot \quad -7 \text{ Dollars} \\
\text{amounts to exactly the same as} \\
\bullet \text{ Adding the opposite of the incorrect signed-number-phrase} \quad \text{(choice 3.)} : \\
-132 \text{ Dollars} \quad \oplus \quad +7 \text{ Dollars}
\end{align*}
\]

\[^4\]This is indeed the definition of subtraction in a group. Yet, Educologists usually express this as an operating prescription: “To subtract a signed number, change the sign of the number being subtracted and add”. It does of course work in the short run but not stressing that the term “subtraction” in the case of counting number-phases means something different than in the case of signed number-phrases does nothing for lucidity in the long term.
• Then, a transaction occurs, say FIVE dollars from-Jack-to-Jill.
• Now, in the final state, Jill is TWO dollars in-the-black.

Thus, the effect of a FIVE dollars from-Jack-to-Jill transaction is a FIVE dollars gain on Jill’s state—as well as a FIVE dollars loss on Jack’s state. A transaction in the opposite direction would have the opposite effects.

❖ On the board, to find the change of state, we subtract the initial state from the final state to remove from the final state the effect of all previous transactions.

\[
\text{Change of State} = \text{Final State} \ominus \text{Initial State}
\]

\[
= +2 \text{ Washingtons} \ominus -3 \text{ Washingtons} \\
= +2 \text{ Washingtons} \oplus +3 \text{ Washingtons} \\
= +5 \text{ Washingtons} \\
= \rightarrow 5 \text{ Washingtons}
\]

We thus have that

\[
\text{Effect of a Transaction} = \text{Change of State}
\]

This seemingly trivial statement will have in fact far-reaching generalizations.

### 3.7 Comparing States

Essentially, we compare states just the way we compared collections in Section 1.2: Where we used forward and backward counts, here we use transactions in the standard and opposite directions:

- When it takes a transaction in the standard direction to go from an initial state to a final state, we say that the initial state is algebra-smaller than the final state.
- When it takes a transaction in the opposite direction to go from an initial state to a final state, we say that the initial state is algebra-larger than the final state.
- When it takes no transaction to go from an initial state to a final state, we say that the initial state is algebra-equal to the final state.

### 3.8 Equations and Inequations

We can now return to the problems we encountered in Section 1.5.
Chapter 4

Accounting For *Discrete* Goods

We now turn to the accounting for *goods*. While *essentially* the same as the accounting for *money*, there are a few technical issues and further developments. Here, though, we will deal with the accounting for

- **discrete goods** such as, say, *apples* or *bananas*

as opposed to the accounting for

- **continuous goods** such as, say, *lengths* or *liquids*

because the latter involve the *English* System of denominators which present problems of their own and which, therefore, we shall leave to a later chapter.

4.1 Counting And Adding Collections Of Discrete Goods

1. The main difference between counting *money* and *discrete goods* is that:

   - When counting *money*, when we have more than *TEN* of a kind, we
     - *bundle* *TEN* of a kind, and then we
     - *exchange* the bundle of *TEN* of a kind for 1 of the next kind up—for which we already have a *denominator*.

The question is how to read 23 *Apples*
To count discrete goods such as, say, apples, and while we will still bundle collections of ten objects, we will usually not be able to exchange these bundles the way we did with money, so that the denominators will usually have to represent bundles rather than objects as was the case with money.

Note. Bundling by ten is rather recent. Historical remnants of a time when discrete goods were bundled by twelve include eggs and pencils.

2. As far as aggregating collections of different goods goes, the situation is the same as in Section 1.6, where we already saw that combinations can always be added.

If, say Jack has three apples, five bananas and thirteen cents and Jill has two apples, eight bananas and eleven cents, when they put everything together, they have five apples, thirteen bananas and twenty-four cents.

On the board we proceed as follows:

\[
[3 \text{ Apples } & 5 \text{ Bananas } & 13 \text{ Cents}] \& [2 \text{ Apples } & 8 \text{ Bananas } & 11 \text{ Cents}]
= 5 \text{ Apples } & 13 \text{ Bananas } & 24 \text{ Cents}
\]

4.2 Evaluating Collections of Discrete Goods: Co-multiplication

We seldom deal with goods alone and, often, we will want to know how much money a given collection of goods is worth. For that purpose, we shall need a unit-worth for the objects. On the board, we shall represent a worth by a value and a unit-worth by a unit-value.

Note. Just as we say “To count a collection” as a short for “To find the number-phrase that represents a collection”, we shall say “To evaluate a collection” as a short for “To find the number-phrase that represents the worth of a collection”.

1. For instance, given a collection of apples, we may be interested in how many dimes it is worth and for that we need to know how many cents each apple is worth.

Suppose Jack has three apples with a unit-worth of, say, seven cents-per-apple. Thus, at a unit-worth of seven cents-per-apple, Jack’s worth would be twenty-one cents.
4.3. SIGNED-CO-MULTIPLICATION

❖ On the board, we write \( 7 \, \text{Cents} \, \text{per-apple} \) to represent the unit-value SEVEN cents-per-apple and we co-multiply:

\[
\text{Jack's Value} = [3 \, \text{Apples}] \times [7 \, \text{Cents} \, \text{Apple}]
\]
\[
= [3 \times 7] \, \text{Cents}
\]
\[
= 21 \, \text{Cents}
\]

In other words, the (board) procedure for co-multiplication is as follows:
• Get the denominator by “canceling” the Apples, and
• Get the numerator from the multiplication tables we had to memorize as children.

2. Co-multiplication is in fact an extremely important concept.
   a. The “canceling” of denominators, as that of Apples above, is in fact something that scientists and engineers do all the time and it is in fact at the heart of a part of mathematics called DIMENSIONAL ANALYSIS.
   b. Moreover, co-multiplication is also at the heart of LINEAR ALGEBRA\(^1\), a part of mathematics already mentioned at the very beginning of this text, in which 3 Apples would be a vector while 7 Cents-Apple would be a co-vector. Accordingly, since we call 3 Apples a number-phrase, we will call 7 Cents-Apple a co-number-phrase.
   c. More modestly, co-multiplication arises as at least one aspect of percentage:

\[
[3 \, \text{Dollars}] \times [7 \, \text{Cents} \, \text{Dollar}] = [3 \times 7] \, \text{Cents} = 21 \, \text{Cents}
\]

4.3 Effect of Transactions on States: Signed Co-Multiplication

We now extend the concept of co-multiplication to signed-number-phrases. The idea is of course to obtain the gain or loss caused by a transaction.

For illustration purposes, we will look at transactions occurring in an apple store. Suppose that, not only can apples, for whatever reason to be left to the reader’s imagination, appear in or disappear from the store, but the apples can also be good or bad.

1. We begin with the issue of representing the transactions of apples. This is no different from representing the transactions of dollars which we saw earlier on.

\(^1\)Being the action on an element of a vector space of an element of its dual.
We have apples that can appear or disappear from the store. (Just like, earlier on, we had dollars going from Jack to Jill and/or from Jill to Jack.)

We choose the standard direction, for which we use the + sign, to be that of apples that appear in the store and so the opposite direction, for which we use the − sign, is that of apples that disappear out of the store. Thus, for instance,

<table>
<thead>
<tr>
<th>♠ In the real world, we may have:</th>
<th>♦ We then write:</th>
</tr>
</thead>
<tbody>
<tr>
<td>THREE apples appear in the store,</td>
<td>[+3 Apples]</td>
</tr>
<tr>
<td>or</td>
<td></td>
</tr>
<tr>
<td>FIVE apples disappear from the store.</td>
<td>[−5 Apples]</td>
</tr>
</tbody>
</table>

2. Because the resulting gain or loss of such transactions will of course depend on what unit-worth the apples have, we now look at the kinds of unit-worth apples can have.

The apples may be either good or bad and, if those that are good can be sold at a unit-profit-worth, those that are bad must be disposed of at a unit-loss-worth.

To represent unit-profit-worth and unit-loss-worth on the board, we use co-signed-number-phrase as we did with co-number-phrases and we use a + sign for a unit-profit-value and a − sign for a unit-loss-value. Thus, for instance,

<table>
<thead>
<tr>
<th>♠ In the store, we may have:</th>
<th>♦ We then write:</th>
</tr>
</thead>
<tbody>
<tr>
<td>apples that are good and could be sold at a unit-profit-worth of, say, SEVEN cents-per-apple</td>
<td>+7 Cents Apple</td>
</tr>
<tr>
<td>or</td>
<td></td>
</tr>
<tr>
<td>apples that are bad and must be disposed of at a unit-loss-worth of, say, SEVEN cents-per-apple</td>
<td>−7 Cents Apple</td>
</tr>
</tbody>
</table>

Note. The choice of symbols, + to represent good and − to represent bad, is not really an arbitrary choice because of the way they will interact with the symbols for appearing and disappearing. We leave it as an exercise for the reader to investigate what happens when other choices are made.

3. We are now finally, in a position to look at the four kinds of effect that transactions of goods can have on money states.

make pictures of good in = good, good out = bad, bad in = bad, bad out = good

represent by ++ = +, +− = −, −+ = −, −− = +
4.4. COMMON DENOMINATOR

In the real world, we have

♦ On the board, we write

Case 1.
THREE apples appear in the store. 
They are good, with a unit-profit-worth of SEVEN cents-per-apple.
The result is

\[ [+3 \text{ Apples}] \times [+7 \text{ Cents per Apple}] = [+3] \times [+7] \text{ Cents} \]

a profit-worth of TWENTY-ONE cents.

Case 2.
THREE apples appear in the store.
They are bad, with a unit-loss-worth of SEVEN cents-per-apple.
The result is

\[ [+3 \text{ Apples}] \times [-7 \text{ Cents per Apple}] = [+3] \times [-7] \text{ Cents} \]

a loss-worth of TWENTY-ONE cents.

Case 3.
THREE apples disappear from the store.
They are good, with a unit-profit-worth of SEVEN cents-per-apple.
The result is

\[ [-3 \text{ Apples}] \times [+7 \text{ Cents per Apple}] = [-3] \times [+7] \text{ Cents} \]

a loss-worth of TWENTY-ONE cents.

Case 4.
THREE apples disappear from the store.
They are bad, with a unit-loss-worth of SEVEN cents-per-apple.
The result is

\[ [-3 \text{ Apples}] \times [-7 \text{ Cents per Apple}] = [-3] \times [-7] \text{ Cents} \]

a profit-worth of TWENTY-ONE cents.

4.4 Common Denominator

Usually, in order to deal with collections of goods of different kinds, these must still have “something in common”.

For instance, say Jack has six apples and five bananas while Jill has two apples and eight bananas. The question, say, of who has more and
who has less is meaningless as we cannot compare apples and bananas.

Collections can have “something in common” in two ways.

1. When both kinds of goods can be exchanged for a same kind of goods. ♠ For instance, say Jack has six apples and five bananas while Jill has two apples and eight bananas but that one apple can be exchanged for four nuts and one banana can be exchanged for seven nuts. Jack can then exchange his six apples for twenty-four nuts and his five bananas for thirty-five nuts for a total of fifty-nine nuts while Jill can exchange her two apples for eight nuts and her eight bananas for fifty-six nuts for a total of sixty-four nuts.

❖ On the board we use the multiplication tables to write:

\[
\text{Jack’s Holding} = 6 \text{ Apples} & 5 \text{ Bananas} \\
& = 6(4 \text{ Nuts}) + 5(7 \text{ Nuts}) \\
& = 24 \text{ Nuts} + 35 \text{ Nuts} \\
& = 59 \text{ Nuts}
\]

and

\[
\text{Jill’s Holding} = 2 \text{ Apples} & 8 \text{ Bananas} \\
& = 2(4 \text{ Nuts}) + 8(7 \text{ Nuts}) \\
& = 8 \text{ Nuts} + 56 \text{ Nuts} \\
& = 64 \text{ Nuts}
\]

2. Usually, though, we don’t deal with just collections of goods but also with money. So, even being able to exchange the given kinds of goods for a “common” kind of goods would still leave us with both these goods and money.

♠ Suppose, for instance, that Jack not only has six apples and five bananas as above but also thirteen cents and that Jill not only has two apples and eight bananas as above but also thirty-one cents. Even if both apples and bananas could be exchanged for nuts as above, this would still leave Jack and Jill holding both nuts and cents:

- Jack would be holding fifty-nine nuts and thirteen cents
- Jill would be holding sixty-four nuts and thirty-one cents

In this kind of situations, it will be convenient to be able to evaluate both kinds of goods in terms of money so that, eventually, we will be left only with money. This will require that, corresponding to each good, we have a unit-worth for that good.

For instance, if, say, apples have a unit-worth of seven cents-per-apple and bananas have a unit-worth of four cents-per-banana, then:
4.5. COMPARISON

- **Jack**’s six apples will be worth forty-two cents and his five bananas will be worth twenty cents so that, with the thirteen cents he already had, **Jack’s worth** will be seventy-five cents.
- **Jill**’s two apples will be worth fourteen cents and her eight bananas will be worth thirty-two cents so that, with the thirty-one cents she already had, **Jill’s worth** will be seventy-seven cents.

❖ On the board we proceed as follows

\[
\text{Jack’s Value} = \left[ 3 \text{ Apples} \right] \times \left[ \frac{7 \text{ Cents}}{1 \text{ Apple}} \right] + \left[ 5 \text{ Bananas} \right] \times \left[ \frac{4 \text{ Cents}}{1 \text{ Banana}} \right] + 13 \text{ Cents}^2
\]

\[
= [3 \times 7] \text{ Cents} + [5 \times 4] \text{ Cents} + 13 \text{ Cents}
\]

\[
= 21 \text{ Cents} + 20 \text{ Cents} + 13 \text{ Cents}
\]

\[
= 75 \text{ Cents}
\]

and

\[
\text{Jill’s Value} = \left[ 2 \text{ Apples} \right] \times \left[ \frac{7 \text{ Cents}}{1 \text{ Apple}} \right] + \left[ 8 \text{ Bananas} \right] \times \left[ \frac{4 \text{ Cents}}{1 \text{ Banana}} \right] + 11 \text{ Cents}
\]

\[
= [2 \times 7] \text{ Cents} + [8 \times 4] \text{ Cents} + 11 \text{ Cents}
\]

\[
= 14 \text{ Cents} + 32 \text{ Cents} + 31 \text{ Cents}
\]

\[
= 77 \text{ Cents}
\]

We shall refer to Cents as a **common denominator**.

4.5 Comparing Collections of Different Kinds of Goods

To compare collections of goods, they **must** consist of the same kind of objects. On the board, this means that we must have a common denominator. But, once we do, the comparison proceeds quite smoothly.

1. Indeed, if the collections consist of the same kinds of objects, we need of course concern ourselves only with the numbers of objects in the collections as we did in Section 1.2.

- Say Jack has three apples and Jill has five apples. Who’s **worth** is smaller?

---

\[2\text{In other words, } \text{Jack’s Value} = \left[ 3 \text{ Apples} \right] \circ \left[ \frac{7 \text{ Cents}}{1 \text{ Apple}} \right] + 13 \text{ Cents} \]

Educologists will be sure to note.
After matching their apples one-to-one, we find that Jill is left with un-matched apples which means that Jack has fewer apples than Jill. So, Jack’s worth is smaller than Jill’s worth regardless of the unit-worth of apples.

On the board, since we go from 3 Apples to 5 Apples by counting forward, $3, 4, 5 \rightarrow$, we write that $3 \text{ Apples} < 5 \text{ Apples}$ and therefore

$$\text{Jack’s Value} < \text{Jill’s Value}$$

- Say Jack has eight apples and Jill has two apples. Who’s worth is larger?
- After matching their apples one-to-one, Jack has apples left over and so more apples than Jill. Thus, Jack’s worth is larger than Jill’s worth regardless of the worth of apples.

On the board, since we go from 8 Apples to 2 Apples by counting backward, $2, 3, 4, 5, 6, 7, 8 \leftarrow$, we write that $8 \text{ Apples} > 2 \text{ Apples}$ so that

$$\text{Jack’s Value} > \text{Jill’s Value}$$

2. Usually, though, we need to compare collections of different goods and we first need to get a common denominator.

For instance, in the last example of Section 4.4 we found that Jack’s worth would be seventy-five cents and that Jill’s worth would be seventy-seven cents. So, after matching cents one-to-one, we find that Jack’s worth is smaller than Jill’s worth.

On the board, after we get to the point where we have:

$$\text{Jack’s Value} = 75 \text{ Cents}$$
$$\text{Jill’s Value} = 77 \text{ Cents}$$

we proceed as in Section 1.2 and since we go from 75 Cents to 77 Cents by counting forward, $76, 77 \rightarrow$, we write that

$$\text{Jack’s Value} < \text{Jill’s Value}$$
4.6 Computing Unit-Values: Co-division

Division comes up in at least two different types of situation but, as we shall see, from the procedural viewpoint, these situations will turn out to be really the same. Here we shall deal with the type of situations in which we want to arrive at unit-values. We shall refer to is as co-division.

We begin by looking at basic collections, that is collections involving less than ten objects (See Section 1.1) because our purpose here is only to show how division is connected with co-multiplication. We leave the more technical aspects of the procedure to a later chapter.

1. We begin with the somewhat rare case in which “the division comes out even”.

   a. Suppose we wanted to buy two apples at an auction and that we had eight dimes available. We want to know what is the highest possible bid that we could make.

      ♠ Essentially, we proceed by trial and error.

      • Bidding one dime-per-apple would require two dimes to buy two apples with six dimes left over.
      • Bidding two dime-per-apple would require four dimes to buy two apples with four dimes left over.
      • Bidding three dimes-per-apple would require six dimes to buy two apples with two dimes left over.
      • Bidding four dimes-per-apple would require eight dimes to buy two apples with no dime left over.

So, our highest possible bid would be four dime-per-apple.

   ♦ On the board, we use the “2 times” multiplication table to accelerate the trial and error process. We start with 1

      • $2 \times 1 = 2$ which is less than 8 so we try the next one, 2,
      • $2 \times 2 = 4$ which is less than 8 so we try the next one, 3,
      • $2 \times 3 = 6$ which is less than 8 so we try the next one, 4,
      • $2 \times 4 = 8$ which is equal to 8 so we stop with 4.

The quotient, namely what we write on the board to represent our highest possible bid, is thus 4 \text{dimes \ Apples}.

b. From the algebra viewpoint, we are trying to solve the equation

   $$2 \text{ Apples} \times ? = 8 \text{ Dimes}.$$

   that is, to figure out what co-number-phrase would fit.
solution divided by

- \( \frac{\text{Apple}}{	ext{Dimes}} \) must be Apple to allow for the cancellation and ??? must be Dimes because of the right hand side:

\[
2 \text{ Apples} \times ? \frac{\text{Dimes}}{\text{Apple}} = 8 \text{ Dimes.}
\]

\[
(2 \times ?) \text{ Dimes} = 8 \text{ Dimes}
\]

- Since the denominators are now the same, we saw in Section ?? that we must solve:

\[
2 \times ? = 8
\]

Then, using the “2 times” multiplication table, we find that 4 fits in the place of ?.

The procedure to solve the equation \( 2 \text{ Apples} \times ? \frac{\text{Dimes}}{\text{Apple}} = 8 \text{ Dimes} \) is thus to “divide both sides by \( 2 \text{ Apples} \)"

\[
\frac{2 \text{ Apples}}{2 \text{ Apples}} \times ? \frac{\text{Dimes}}{\text{Apple}} = \frac{8 \text{ Dimes}}{2 \text{ Apples}}
\]

and, since, \( \frac{2 \text{ Apples}}{2 \text{ Apples}} = 1 \), we get

\[
? \frac{\text{Dimes}}{\text{Apple}} = \frac{8 \text{ Dimes}}{2 \text{ Apples}}
\]

\[
= 4 \frac{\text{Dimes}}{\text{Apple}}
\]

So, 4 \( \frac{\text{Dimes}}{\text{Apple}} \) is the solution of the equation \( 2 \text{ Apples} \times ? \frac{\text{Dimes}}{\text{Apple}} = 8 \text{ Dimes} \) and we will say that it is “given” by

\[
8 \text{ Dimes} \div 2 \text{ Apples} = 4 \frac{\text{Dimes}}{\text{Apple}}
\]

which we read as “8 Dimes divided by 2 Apples is equal to 4 \( \frac{\text{Dimes}}{\text{Apple}} \)” and where the procedure for \( \div \) is the one above.

In other words,

\[
2 \text{ Apples} \times 4 \frac{\text{Dimes}}{\text{Apple}} = 8 \text{ Dimes.}
\]

and

\[
8 \text{ Dimes} \div 2 \text{ Apples} = 4 \frac{\text{Dimes}}{\text{Apple}}
\]

are dual sentences.

c. If, instead of writing \( 8 \text{ Dimes} \div 2 \text{ Apples} \) we move \( 8 \text{ Dimes} \) in place of the upper dot and \( 2 \text{ Apples} \) in place of the lower dot, we get

\[
\frac{8 \text{ Dimes}}{2 \text{ Apples}}
\]
as another way to write \(8 \text{ Dimes} \div 2 \text{ Apple}\) and thus

\[
\frac{8 \text{ Dimes}}{2 \text{ Apples}} = 4 \frac{\text{Dimes}}{\text{Apple}}
\]

\(d\). In anticipation of the more complicated cases to be investigated later on, we set up the division as follows:

\[
\begin{array}{c|cc}
 & 4 \frac{\text{Dimes}}{\text{Apple}} \\
\hline
2 \text{ Apples} & 8 \text{ Dimes} \\
 & 8 \text{ Dimes} \\
\hline
 & 0 \text{ Dime}
\end{array}
\]

which we read as divide 2 Apples into 8 Dimes.

\textbf{Note.} Observe that the order of the two number-phrases in "divide 2 Apples into 8 Dimes" is the opposite of their order in "8 Dimes divided by 2 Apples" but that both result in the same co-number-phrase, namely \(4 \frac{\text{Dimes}}{\text{Apple}}\). This is another instance of \textit{linguistic duality}.

\(2\). We now look at a case where “the division does not come out even”

\(a\). Suppose we wanted to buy three apples but that we had only seven dimes. We want to know what would be our highest possible bid.

\(\blacklozenge\) Again, we proceed by trial and error.

\(\cdot\) Bidding one \textit{dime-per-apple} would require three \textit{dimes} to buy three \textit{apples} with four \textit{dimes} leftover.

\(\cdot\) Bidding two \textit{dime-per-apple} would require six \textit{dimes} to buy three \textit{apples} with one \textit{dime} leftover.

\(\cdot\) Bidding three \textit{dimes-per-apple} would require nine \textit{dimes} to buy three \textit{apples} which is more than the seven \textit{dimes} we have.

So, our \textit{highest possible bid} is two \textit{dime-per-apple} at which unit-value we would be able to buy two \textit{apples} with one \textit{dime} left over.

\(\blacklozenge\) On the board, we use the “3 times” \textit{multiplication table} to accelerate the trial and error process. We start with 1.

\(\cdot\) \(3 \times 1 = 3\) which is \textit{less} than 7 so we try the next one, 2.

\(\cdot\) \(3 \times 2 = 6\) which is \textit{less} than 7 so we try the next one, 3.

\(\cdot\) \(3 \times 3 = 9\) which is \textit{more} than 7 so we stop with 2.

Since 2 \(\frac{\text{Dimes}}{\text{Apple}}\) represents the \textit{highest possible bid}, we will again call it the \textbf{quotient} and, since 1 \textit{Dimes} represents the \textit{lowest possible leftover}, we will call it the \textbf{remainder} of the division.

\(b\). From the \textit{algebra} viewpoint, we are trying to solve the equation

\[
3 \frac{\text{Apples}}{?} = 7 \text{ Dimes}
\]
However, even though the quotient is $2 \frac{\text{Dimes}}{\text{Apple}}$, we cannot just replace $? \frac{\text{?}}{\text{?}}$ by $2 \frac{\text{Dimes}}{\text{Apple}}$ because the way division is related to co-multiplication is now complicated by the remainder and we can only write

$$7 \text{ Dimes} \approx 3 \text{ Apples} \times 2 \frac{\text{Dimes}}{\text{Apple}}$$

where $\approx$ means that $7 \text{ Dimes}$ is approximately equal to $3 \text{ Apples} \times 2 \frac{\text{Dimes}}{\text{Apple}}$ by which we mean that the difference is “too small to matter”. However, we shall not use the symbol $\approx$ because we shall prefer to keep track of this difference, even though it is “too small to matter” and we shall write

$$7 \text{ Dimes} = 3 \text{ Apples} \times 2 \frac{\text{Dimes}}{\text{Apple}} + (...) \text{ Dimes}$$

where $(...)$ Dimes represents a number of dimes “too small to matter”. In fact, there will be situations where we will have to be more precise about what is being ignored and, in particular, we may even replace $(...)$ by the remainder:

$$7 \text{ Dimes} = 3 \text{ Apples} \times 2 \frac{\text{Dimes}}{\text{Apple}} + 1 \text{ Dimes}$$

Since the denominators are the same, we can focus on the numerators and write:

$$7 = 3 \times 2 + (...)$$

as well as

$$7 = 3 \times 2 + 1$$

Note. More generally, keeping track of things even though they are too small to be taken into account, one way or the other, will turn out to be an extremely powerful approach called asymptotic analysis, one that we shall make extensive and systematic use of in Part 2 and 3 of this proto-text.

c. It is imperative to realize how this extends to the use of $\div$.

First recall that, in the previous case, we wrote

$$8 \text{ Dimes} \div 2 \text{ Apples} = (4 \frac{\text{Dimes}}{\text{Apple}})$$

as just the dual of

$$8 \text{ Dimes} = 2 \text{ Apples} \times (4 \frac{\text{Dimes}}{\text{Apple}})$$

Educologists will surely have recognized in “(...)” a precursor of “little ohs”, the only difference being that the latter give the order of magnitude of what is not taken into account.

---

4Educologists will surely have recognized in “(...)” a precursor of “little ohs”, the only difference being that the latter give the order of magnitude of what is not taken into account.
4.6. CO-DIVISION

But, here, we have

\[ 7 \text{ Dimes} = 3 \text{ Apples} \times (2 \frac{\text{Dimes}}{\text{Apple}}) + (\ldots) \text{ Dimes} \]

so that

\[ 7 \text{ Dimes} \neq 3 \text{ Apple} \times (2 \frac{\text{Dimes}}{\text{Apple}}) \]

and therefore

\[ 7 \text{ Dimes} \div 3 \text{ Apple} \neq 2 \frac{\text{Dimes}}{\text{Apple}} \]

However, we *can* write

\[ 7 \text{ Dimes} \div 3 \text{ Apples} = 2 \frac{\text{Dimes}}{\text{Apple}} + (\ldots) \frac{\text{Dime}}{\text{Apple}} \]

and, if we want to be more specific,

\[ 7 \text{ Dimes} \div 3 \text{ Apples} = 2 \frac{\text{Dimes}}{\text{Apple}} + 1 \frac{\text{Dime}}{3 \text{ Apple}} \]

where the 3 Apples under the remainder 1 Dime are to remind us that the remainder is yet to be divided among 3 Apples.

**d.** If we rewrite \(7 \text{ Dimes} \div 3 \text{ Apples}\) as \(\frac{7 \text{ Dimes}}{3 \text{ Apples}}\), then we can rewrite the above as

\[ \frac{7 \text{ Dimes}}{3 \text{ Apple}} = 2 \frac{\text{Dimes}}{\text{Apple}} + 1 \frac{\text{Dime}}{3 \text{ Apple}} \]

and, with a slight abuse of language,

\[ \frac{7 \text{ Dimes}}{3 \text{ Apple}} = 2 \frac{\text{Dimes}}{\text{Apple}} + 1 \frac{\text{Dimes}}{3 \text{ Apple}} \]

and therefore:

\[ \frac{7 \text{ Dimes}}{3 \text{ Apple}} = \left(2 + \frac{1}{3}\right) \frac{\text{Dimes}}{\text{Apple}} \]

whence the so-called *mixed-numbers notation*:

\[ \frac{7}{3} = 2 \frac{1}{3} \]

which we shall encounter in a later chapter, in a somewhat different context.

**e.** In anticipation of the more complicated cases to be investigated later on, we set up the division as follows:

\[
\begin{array}{c|cc}
\text{3 Apples} & 7 \text{ Dimes} \\
\hline
& 6 \text{ Dimes} \\
\end{array}
\]
f. In an attempt to save time and energy, though, it is usual not to write the denominators and to write only the numerators\(^4\)

\[
\begin{array}{c}
3 \bigg) \ 7 \\
6 \\
1
\end{array}
\]

4.7 Signed Co-division

4.8 Equations and Inequations For Collections of Goods

We saw in Section 2.5 that, given collections of money, a type of problem one often encounters in the real-world is not only how they compare but how they differ. Here we shall investigate how collections of goods differ in worth.

1. We begin with the case where Jack has a collection of apples and Jill has money.

♠ Say that Jack has two apples and that Jill has eight dimes. We want to know for which unit-worth will Jack’s value be the same as Jill’s value.

This is exactly the problem we dealt with in Section 4.6: If one apple is worth four cents, then Jack’s worth will be the same as Jill’s worth.

❄ On the board, we write, again as in Section 4.6,

\[
\begin{array}{c|c}
\text{Jack’s Value} & \text{Jill’s Value} \\
2 \text{ Apples} \times ? \frac{\text{cents}}{\text{apple}} = 8 \text{ Cents} \\
\end{array}
\]

\(^4\)While, of course, students must eventually come to the “short” manner, Educologists may well consider the benefit of spending at least some time getting there, if only because the “long” manner, by the very fact that it uses denominators, reflects more immediately “real life” situations and thus helps providing meaningful checks.
4.8. EQUATIONS AND INEQUATIONS FOR COLLECTIONS

Dividing both sides by 2 Apples gives

\[
\frac{2 \text{ Apples}}{2 \text{ Apples}} \times \frac{8 \text{ Cents}}{2 \text{ Apples}} = 1 \times \frac{8 \text{ Cents}}{2 \text{ Apple}}
\]

and therefore

\[
\frac{8 \text{ Cents}}{2 \text{ Apple}} = 4 \frac{\text{ Cents}}{\text{ Apple}}
\]

2. We now look at the case where both Jack and Jill hold both a collection of apples and money. What we will do is to change Jack’s holding and Jill’s holding so as to obtain a situation such as the previous one, that is where Jack has only a collection of apples and Jill has only money. Of course, we will have to proceed in an equitable manner, that is, whatever we do onto either one, we must do onto the other.

Since this is in fact a very general approach called Separation of Variables, we will say that we separate the apples from the money.

♠ Say that Jack has seven apples and three cents and that Jill has five apples and eleven cents. We want to know for which unit-worth will Jack’s worth be the same as Jill’s worth.

– Since we do not want Jack to have any money and since he has three cents, we take them away from him and, to be equitable, we also take three cents away from Jill which leaves her with only eight cents.

– Since we do not want Jill to have any apple and since she has five apples, we take them away from her and, to be equitable, we also take five apples away from Jack which leaves him with only two apples.

❖ On the board, we can proceed in either one of two ways:

– We can start by separating the Combinations

<table>
<thead>
<tr>
<th>Jack’s Combination</th>
<th>Jill’s Combination</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 Apples &amp; 3 Cents</td>
<td>5 Apples &amp; 11 Cents</td>
</tr>
</tbody>
</table>

Subtracting 3 Cents from each Combination leaves

| 7 Apples          | 5 Apples & 8 Cents |
Subtracting 5 **Apples** from each **Combination** leaves

\[
\begin{align*}
2 \text{ Apples} & \quad 8 \text{ Cents} \\
\end{align*}
\]

Moving to **Values** now gives the *equation*

\[
2 \text{ Apples} \times \frac{??}{???} = 8 \text{ Cents}
\]

– Alternatively, we can start with **Values** up front:

<table>
<thead>
<tr>
<th>Jack’s Value</th>
<th>Jill’s Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$7 \text{ Apples} \times \frac{??}{???} + 3 \text{ Cents}$</td>
<td>$5 \text{ Apples} \times \frac{??}{???} + 11 \text{ Cents}$</td>
</tr>
</tbody>
</table>

Subtracting 3 **Cents** from each side of the *equation* gives

\[
7 \text{ Apples} \times \frac{??}{???} = 5 \text{ Apples} \times \frac{??}{???} + 8 \text{ Cents}
\]

Subtracting 5 **Apples** $\times \frac{??}{???}$ from each side of the *equation* gives

\[
2 \text{ Apples} \times \frac{??}{???} = 8 \text{ Cents}
\]

So, either way, by *separating* the **apples** from the **money**, we have reduced the problem to the preceding one.

3. Inequation

Test

**Test at infinity**

### 4.9 Equations and Inequations For States

The exact same approach works for **states** as long as one keeps in mind that:

- “*to subtract*” means “*to add the opposite*” Section 3.5
- “law of signs for signed co-multiplication” Section 4.3
- “law of signs for signed division” Section 4.7
1. $-2 \text{ Apples} \times \frac{???}{???} = -14 \text{ Cents}$

2. Say that Jack is seven apples in-the-red and three cents in-the-black and that Jill is five apples in-the-red and eleven cents in-the-red. We want to know for which profit/loss-unit-worth will Jack’s worth be the same as Jill’s worth.

   – Since we do not want Jack to have any money, and since he is three cents in-the-black, we take them away from him and, to be equitable, we also take three cents away from Jill which, since she already was eleven cents in-the-red, puts her now fourteen cents in-the-red.

   – Since we do not want Jill to have any apple, and since she is five apples in-the-red, we give her five apples and, to be equitable, we also give five apples to Jack which, since he was seven apples in-the-red, puts him now only two apples in-the-red.

   – Again, on the board, we can proceed in either one of two ways:

     – We can separate the Combinations and then move to Values.

         Jack’s Combination: $-7 \text{ Apples} \& +3 \text{ Cents}$

         Jill’s Combination: $-5 \text{ Apples} \& -11 \text{ Cents}$

         To subtract $+3 \text{ Cents}$ from each side, we add the opposite, $-3 \text{ Cents}$, to each side which results in

         $-7 \text{ Apples} \& -5 \text{ Apples} \& -14 \text{ Cents}$

         To subtract $-5 \text{ Apples}$ from each side, we add the opposite, $+5 \text{ Apples}$, to each side which results in

         $-2 \text{ Apples} \& -14 \text{ Cents}$

         Moving to Values now gives the equation

         $-2 \text{ Apples} \times \frac{???}{???} = -14 \text{ Cents}$

   – Alternatively, we can move to Values up front:

3. Inequation

   Test

   Test at infinity
Chapter 5

Accounting For Continuous Goods on the Counter (I)

• When we count money, what we do each time we have more than ten of a kind is two things (See Chapter I):
  – We bundle ten of a kind
  – We then exchange the bundle of ten of a kind for 1 of the next kind up—for which we usually already have a denominator.

• When we count goods, what we will do will very much depend on the kind of goods we are counting.
  – When we count discrete goods such as, say, apples, and while we will still bundle collections of ten objects, we will usually not be able to exchange these bundles for other objects, as we do when dealing with money, and the denominators will usually have to represent bundles rather than objects as was the case with money.
  – When we count continuous goods such as lengths or liquids, what we will do will not involve any bundling but will involve changes of denominator. In most of the world, because of the metric system, accounting for this kind of goods is thus essentially the same as accounting for money and just as easy. However, under the English system, the process, while it remains essentially the same, involves much memorization.
CHAPTER 5. CONTINUOUS GOODS ON THE COUNTER

5.1 Counting Goods on the Counter

In the U.S., accounting for goods is usually much more difficult than accounting for money because, contrary to what is the case with money, when dealing with goods, we still use English denominators and the English denominators do not change at the rate of ten to 1.

1. Consider for instance the problem presented by, say, the number-phrase 27. Inches which corresponds to 2 ten-Inches & 7 Inches which however changes to 2 Feet & 3 Inches. There are two ways to look at it. One way would be to deplore that the English did not match our ten digits with a denominator for a collection of ten inches. The other way would be to deplore that we do not have six fingers on each hand because then we would probably be using twelve digits which would match the fact that Foot is a denominator corresponding to a collection of twelve inches. However, and to make the problem even worse, English denominators do not even all change at the same rate with the result that there is no way that numerators and denominators could ever be matched. For instance, while 1 Foot changes for twelve inches, 1 Yard changes for 3 Feet, 1 Furlong changes for two-hundred and twenty Yards, 1 Mile changes for 8 Furlongs, etc.

2. Compare counting money
1 Dime, 2 Dime, ..., 9 Dime,
1 Dollar, 1 Dollar & 1 Dime, 1 Dollar & 2 Dime, ..., 1 Dollar & 9 Dime,
2 Dollar, 2 Dollar & 1 Dime, 2 Dollar & 2 Dime, ..., 
...
..., 9 Dollar & 9 Dime,
1 DekaDollar, 1 DekaDollar & 1 Dime, ..., 

with counting lengths

1 Inch, 2 Inch, ..., 9 Inch, ten Inch, eleven Inch, 
1 Foot, 1 Foot & 1 Inch, 1 Foot & 2 Inch, ..., 1 Foot & eleven Inch, 
2 Foot, 2 Foot & 1 Inch, ..., 2 Foot & eleven Inch, 
1 Yard, 1 Yard & 1 Inch, ..., 1 Yard & eleven Inch, 1 Yard & 1 Foot, 1 Yard & 1 Foot & 1 Inch, 
...
..., two-hundred-nineteen Yard & 2 Foot & eleven Inch, 
1 Furlong, 1 Furlong & 1 Inch, ...
5.2 Adding Goods on the Counter

1. Since, regardless of the denominators, we work with number-phrases that are based on ten digits, this makes addition very awkward, even quite tricky.

Say we want to weld the two pipes in Figure 5.1.

![Figure 5.1: A 2 yrd, 1 ft, 9 in pipe and a 1 yrd, 2 ft, 5 in pipe.](image)

When we measure the resulting pipe we find that its length is four yards, one foot, two inches. (Although we are of course much more likely to say that it is thirteen feet, two inches long.) On the board, we want to add 2 Yard & 1 Foot & 9 Inch and 1 Yard & 2 Foot & 5 Inch under the heading

<table>
<thead>
<tr>
<th>Yards</th>
<th>Feet</th>
<th>Inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

and to conclude that the result of the addition is 3 Yard & 4 Feet & 4 Inches which of course does not represent what we found in the real world. The reason again is that it takes twelve inches instead of ten to get one foot and it takes three feet instead of ten to get one yard. So, of course, the addition should really proceed with the English rates of exchange, as follows:

<table>
<thead>
<tr>
<th>Yards</th>
<th>Feet</th>
<th>Inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

which indeed gives 4 Yard & 1 Foot & 2 Inches. This can be confusing
particularly if one does not write the denominators\textsuperscript{1}.

It is no wonder then that even the English gave up on English denominators! Note that, in the U. S., convenience prevailed over tradition in only a very few cases: money is exchanged \textit{ten} to \textit{one} and surveying tapes are marked in \textit{tenths of a foot} rather than \textit{inches}. And of course, systematic denominators prevail in all scientific matters.

2. In contrast, here is an example of how \textit{addition} would go in the metric system.

\section*{5.3 Subtracting Goods \textit{on} the Counter}

text

\section*{5.4 Multiplication Goods \textit{on} the Counter}

We now come to \textit{multiplication} which will turn out to be quite a bit more difficult than \textit{addition} to introduce and to discuss.

The problems come from the fact that multiplication occurs in the representation of at least three very different environments which therefore need to be clearly differentiated.

- Multiplication as \textit{additive power} of number-phrases. We saw in Chapter I that when counting the collection of objects we get from unpacking a collection of bundles, the numerator

- Multiplication as \textit{co-multiplication} of number-phrases. In this case, we are multiplying \textit{goods} by they \textit{unit-price} to get their \textit{money} equivalent.

- Multiplication as \textit{multiplication} of number-phrases. While there are a lot of real life situations in which \textit{addition} of number-phrases occurs naturally, there are a lot fewer real life situations in which \textit{multiplication} of number-phrases does\textsuperscript{2}. In the case of \textit{addition} of number-phrases, we were able to start from its \textit{meaning}, the aggregation of collections, and there was thus no doubt as to what the \textit{result} was to be. This then allowed us to focus on developing the (board) procedure. By contrast, in

\textsuperscript{1}This one is dedicated to the very special attention of those Educologists who claim that dispensing with denominators makes it all so much simpler and hence so much easier.

\textsuperscript{2}Educologists will surely have noted that, while there are infinitely many vector spaces, there are only four (division) algebras—even if this was proven not that long ago! Bott and Milnor (1958)
the case of the multiplication of number-phrases, we must start by finding situations in which multiplying number-phrases will mean something. If and when it does mean something, then this will tell us what the resulting number-phrase might be and only then will it make sense to look for a (board) procedure that will give this resulting number-phrase.

5.4.1 Can Money Be Multiplied By Money?

First, and independently of whether or not multiplying counts might or might not mean anything, we introduce the symbol for multiplication that we will be using for it when writing on the board. We recall that, when we were dealing with addition and subtraction, we would write expressions involving two number-phrases with an addition symbol or a subtraction symbol in-between. For instance, we might have written $3 \text{ Dimes} + 2 \text{ Dimes}$ or $3 \text{ Dimes} - 2 \text{ Dimes}$.

Similarly, multiplying counts would have to involve writing expressions involving two number-phrases with the multiplication symbol $\times$ in-between. For instance, we might write $3 \text{ Dimes} \times 2 \text{ Dimes}$.

At this point, though, we must clear up a frequent confusion: an expression like $3 \text{ Dimes} \times 2 \text{ Dimes}$ is absolutely not the same as the expression $3 (2 \text{ Dimes})$.

Now, we saw in Section ?? on page ?? that the expression $3 (2 \text{ Dimes})$ is nothing more than a number-phrase whose numerator is 3 and whose denominator $(2 \text{ Dimes})$ represents a collection of two dimes so that, when we unpack, we get:

$$3(2 \text{ Dimes}) = (2 \text{ Dimes}), (2 \text{ Dimes}), (2 \text{ Dimes})$$

$$= \text{ Dime, Dime, Dime, Dime, Dime, Dime}$$

$$= 6 \text{ Dimes}$$

However, the fact that an expression on the board such as $3 (2 \text{ Dimes})$ makes perfect sense, that is, represents something on the counter, does not imply that an expression such as $3 \text{ Dimes} \times 2 \text{ Dimes}$ also makes sense since they are expressions of a different kind.

For an expression such as $3 \text{ Dimes} \times 2 \text{ Dimes}$ to make sense it would have to represent the result of doing something with THREE dimes and TWO

---

3Educolists, enamored of abstraction as they are, obviously have no trouble distinguishing $3 \times 2$ from the additive power $3(2)$ but it certainly does not seem to carry over very well to their students and, of course, even less to fractions.
dimes and coming up with a number of dimes the same way as 3 Dimes + 2 Dimes represented the result of aggregating THREE dimes and TWO dimes and the way 3 Dimes − 2 Dimes represented the result of removing TWO dimes from THREE dimes.

The question then is: what could an expression on the board such as $3 \text{ Dimes} \times 2 \text{ Dimes}$ possibly represent on the counter? The answer is: Absolutely nothing and expressions of the form $3 \text{ Denominator} \times 2 \text{ Denominator}$ are usually completely meaningless.

5.4.2 Multiplying Certain Goods on the Counter

In the case of certain goods, though, expressions of the form $2 \text{ Denominator} \times 3 \text{ Denominator}$ can represent the result of doing something with the collections represented by 2 Denominator and 3 Denominator. The objects with which this can be done are quite particular. The example we shall use is that of length, as in “a length of material” and we will thus draw our inspiration from “building materials” in which people speak, for instance, of a “four-by-eight” sheet of plywood.

We will begin with a very simple case and work our way up. At each stage, we will start with the more familiar English denominators and then look at the “same” example with metric denominators. The first stage will not involve any carryover because, as we already saw in the case of addition, English denominators do not lend themselves easily to computation since the English exchange rates are not always the same as is the case in the metric system. Only after we will have figured out what multiplying number-phrases might mean and what the resulting number-phrase then is, will we deal with the technical issue of “carryovers”.

1. The point of this example is to observe that, contrary to what was the case with addition and subtraction, where the denominator in the result of the operation was the same as the denominator in the number phrases being operated on, in the case of multiplication, the denominator in the result is different from the denominator in the number phrases being operated on.

   a. Here it is with English denominators.

   ♠ Given a two inch length and a three inch length,

   • We can construct on the counter (Figure 5.2 on the facing page) a two-by-three rectangle, that is a rectangle that is two inch long one way and three inch long the other way:
5.4. MULTIPLICATION

We may then want to tile this rectangle (Figure 5.3) with one-inch-by-one-inch mosaics:

Figure 5.3: The two by three rectangle tiled with mosaics

Counting the mosaics shows that we will need six one-inch-by-one-inch mosaics.

- The expression 2 inch × 3 inch then represents on the board the mosaics that will be needed to tile the rectangle.

Since, as children, we are usually enjoined to memorize the multiplication tables, the (board) procedure for multiplication in this case consists in looking up the relevant multiplication table. We find that

\[ 2 \text{ inch} \times 3 \text{ inch} = 6 [\text{inch} \times \text{inch}] \]

where 6 is the numerator and where \([\text{inch} \times \text{inch}]\) is the denominator that represents one-inch-by-one-inch mosaics on the board.

b. We now look at the “same” example but with metric denominators.

For instance, given a two meters length and a three meters length, we can construct on the counter (Figure 5.4 on the following page) a two-meter-by-three-meter rectangle, that is a rectangle that is two meters long one way and three meters long the other way and then we can tile it with one-meter-by-one-meter tiles See Figure 5.4 on the next page:

---

4Educologists will surely deplore that this definition is robust in the sense that it will withstand generalization.
CHAPTER 5. CONTINUOUS GOODS ON THE COUNTER I

Counting the tiles shows that we will need six one-meter-by-one-meter tiles.

The expression \(2 \text{ Meter} \times 3 \text{ Meter}\) then represents on the board the tiles that will be needed to tile the rectangle. Looking up the relevant multiplication table gives

\[2 \text{ Meter} \times 3 \text{ Meter} = 6 \left[\text{Meter} \times \text{Meter}\right].\]

where 6 is the numerator and where \([\text{Meter} \times \text{Meter}]\) is the denominator that represents one-meter-by-one-meter tiles on the board.

2. The point of this example is to show that, also contrary to what was the case with addition, where the two denominators in the number-phrases being added had to be the same, in the case of multiplication, the denominators of the number-phrases being multiplied can be different. Indeed, the two sides of a rectangle are often measured with different denominators.

a. We begin with an example involving the more familiar English denominators.

For instance, given a three-inch length and a two-foot length, we can cut on the counter a three-inch-by-two-foot plank, that is a rectangle that is three inches long one way and two feet long the other way. We may then want to tile this plank with one-inch-by-one-foot strips.
5.4. MULTIPLICATION

Three inches by two feet

Figure 5.5: A three inch by two foot rectangle tiled with one-inch-by-one-foot strips

Counting the strips shows that we will need six one-inch-by-one-foot strips.

❖ The expression $3\text{ Inch} \times 2\text{ Foot}$ then represents on the board the strips that will be needed to tile the rectangle on the counter. We find that

$$3\text{ Inch} \times 2\text{ Foot} = 6\text{ [Inch} \times \text{Foot]}.$$  

where 6 is the numerator and where $[\text{Inch} \times \text{Foot}]$ is the denominator that represents one-inch-by-one-foot strips on the board.

b. We now look at the “same” example but with metric denominators.

♠ For instance, given a three meters length and a two dekameters length, we can cut on the counter a three-meters-by-two-dekameters rectangle, that is a rectangle that is three meters long one way and two dekameters long the other way. We may then want to tile this rectangle with one-meter-by-one-dekameters strips (Figure 5.6)

Figure 5.6: A three meters by two dekameters rectangle tiled with one meters by one dekameters strips

❖ The expression $3\text{ Meter} \times 2\text{ DekaMeter}$ then represents on the board the strips that will be needed to tile the rectangle on the counter. We find that

$$3\text{ Meter} \times 2\text{ DekaMeter} = 6\text{ [Meter} \times \text{DekaMeter]}.$$  

where 6 is the numerator and where $[\text{Inch} \times \text{Foot}]$ is the denominator that represents one dekameters strips on the board.

3. The point of this example is to show that, essentially in the same manner, we can multiply combinations of lengths.

a. We begin with English denominators.
For instance, given a two foot, two inch length and a three foot, one inch length, we can construct and tile the rectangle as in Figure 5.7)

![Figure 5.7: The three-foot, one-inch by two-foot, three-inch rectangle tiled with four different kinds of tiles](image)

Counting the tiles shows that we will need:
the following square tiles
- six one-foot-by-one-foot tiles,
- two one-inch-by-one-inch tiles,
and the following rectangular tiles
- two one-foot-by-one-inch tiles,
- six one-inch-by-one-foot tiles,

The one-inch-by-one-foot tiles and the one-foot-by-one-inch tiles are counted separately if only because of the different ways they are striped.

The expression \([3 \text{ Foot} \& 1 \text{ Inch}] \times [2 \text{ Foot} \& 3 \text{ Inch}]\) then represents on the board the tiles that we will need to tile it.

The (board) procedure for multiplication in this case is a bit more complicated. First we set up:

\[
\begin{array}{c}
3 \text{ Foot} & \& 1 \text{ Inch} \\
\times & 2 \text{ Foot} & \& 2 \text{ Inch}
\end{array}
\]
The next step is to get the different kinds of tiles using the appropriate multiplication tables. Observe that we are handling Inch × Foot and Foot × Inch separately:

\[
\begin{array}{ccc}
3 \text{ Foot} & \& & 1 \text{ Inch} \\
\times & 2 \text{ Foot} & \& & 2 \text{ Inch} \\
\hline
6 \text{ Inch} \times \text{Foot} & \& & 2 \text{ Inch} \times \text{Inch}
\end{array}
\]

Altogether, we thus find:

\[
3[3 \text{ Foot} & 1 \text{ Inch}] \times [2 \text{ Foot} & 2 \text{ Inch}] = 6 \text{ Foot} \times \text{Foot},
\]

\[
\& 2 \text{ Foot} \times \text{Inch}
\]

\[
\& 6 \text{ Inch} \times \text{Foot}
\]

\[
\& 2 \text{ Inch} \times \text{Inch}
\]

which is the (board) representation of the above.

b. We now look at the “same” example but with metric denominators.

Given a two dekameter, two meter length and a three dekameter, one meter length, we can construct and tile a rectangle as in Figure 5.8.

Figure 5.8: The two-dekameter, two-meter by three-dekameter, one-meter rectangle tiled with four different kinds of tiles
Counting the tiles shows that we will need:

The following square tiles

- **six** one-dekameter-by-one-dekameter tiles,
- **two** one-meter-by-one-meter tiles,

The following rectangular tiles

- **two** one-dekameter-by-one-meter tiles,
- **six** one-meter-by-one-dekameter tiles,

The rectangular tiles, one-meter-by-one-dekameter tiles and one-dekameter-by-one-meter tiles, are counted separately if only because of the different ways they are striped.

❖ Exceptionally, for reasons of space, here we abbreviate Meter as M and DekaMeter as DekaM.

The expression \[3 \text{ DekaM} \& 1 \text{ M} \times 2 \text{ DekaM} \& 3 \text{ M}\] then represents on the board the tiles that we will need to tile the rectangle. We get:

\[
\begin{array}{c}
3 \text{ DekaM} \quad \& \quad 1 \text{ M} \\
\times \quad 2 \text{ DekaM} \quad \& \quad 2 \text{ M}
\end{array}
\]

\[
6 \text{ DekaM} \times \text{ DekaM} \quad \& \quad 2 \text{ DekaM} \times \text{ M}
\]

Altogether, we thus find:

\[
[3 \text{ DekaM} \& 1 \text{ M}] \times [2 \text{ DekaM} \& 3 \text{ M}] = 6 \text{ DekaM} \times \text{ DekaM},
\]

\[
\& 2 \text{ DekaM} \times \text{ M}
\]

\[
\& 6 \text{ M} \times \text{ DekaM}
\]

\[
\& 2 \text{ M} \times \text{ M}
\]

which is the (board) representation of the above.

4. The point now is to see what happens in the preceding example if we don’t care about the way the tiles are striped or if they are striped the same way.

a. We begin with English denominators

♠ First, observe (Figure ?? on page ??) that a one-inch-by-one-foot rectangle and a one-foot-by-one-inch rectangle can both be tiled with TWELVE one-inch-by-inch-inch mosaics.
5.4. MULTIPLICATION

Figure 5.9: Both a one-inch-by-one-foot rectangle and a one-foot-by-one-inch rectangle can be tiled with twelve one-inch-by-inch-inch mosaics.

Thus, from that viewpoint, the six one-foot-by-one-inch tiles and the two one-inch-by-one-foot tiles in Figure 5.7 on page 78 are of the same kind and we can aggregate them.

❖ We set up in the way we learned in elementary school because it will make it easier to add Inch × Foot and Foot × Inch.

Which denominator to use, Inch × Foot or Foot × Inch, is up to us but we need to agree on it.

\[
\begin{array}{c}
3 \text{ foot} & 1 \text{ inch} \\
\times & \text{2 foot} & \text{2 inch}
\end{array}
\]

\[
\begin{array}{c}
6 \text{ Foot} \times \text{Foot} & 6 \text{ Inch} \times \text{Foot} \\
\& & \text{2 Inch} \times \text{Inch}
\end{array}
\]

if we agree on Foot × Inch as common denominator, or

\[
\begin{array}{c}
6 \text{ Foot} \times \text{Foot} & \text{8 Inch} \times \text{Foot} \\
\& & \text{2 inch} \times \text{Inch}
\end{array}
\]

if we agree on Inch × Foot as common denominator.

b. We look at the “same” example but with metric denominators.

♠ First, observe (Figure 5.10 on the next page) that a one-meter-by-one-dekameter rectangle and a one-dekameter-by-one-meter rectangle can both be tiled with ten one-meter-by-inch-meter tiles.
CHAPTER 5. CONTINUOUS GOODS ON THE COUNTER

Thus, from that viewpoint, the six one-dekameter-by-one-meter tiles and the two one-meter-by-one-dekameter tiles in Figure ?? on page ?? are of the same kind and we can aggregate them.

❖ We set up again in the way we learned in elementary school because it will make it easier to add \( \text{Meter} \times \text{DekaMeter} \) and \( \text{DekaMeter} \times \text{Meter} \).

\[
\begin{array}{c}
3 \text{DekaMeter} & \times & 1 \text{Meter} \\
2 \text{DekaMeter} & \times & 2 \text{Meter}
\end{array}
\]

\[
\begin{array}{cc}
6 \text{DekaMeter} \times \text{DekaMeter} & \times & 6 \text{Meter} \times \text{DekaMeter} \\
& \times & 2 \text{DekaMeter} \times \text{Meter}
\end{array}
\]

\[
\begin{array}{cc}
6 \text{DekaMeter} \times \text{DekaMeter} & \times & 8 \text{DekaMeter} \times \text{Meter} \\
& \times & 6 \text{Meter} \times \text{Meter}
\end{array}
\]

if we agree on \( \text{DekaMeter} \times \text{Meter} \) as common denominator, or

\[
\begin{array}{cc}
6 \text{DekaMeter} \times \text{DekaMeter} & \times & 8 \text{Meter} \times \text{DekaMeter} \\
& \times & 2 \text{Meter} \times \text{Meter}
\end{array}
\]

if we agree on \( \text{Meter} \times \text{DekaMeter} \) as common denominator.

5. To see how multiplication works when we have “carryovers”, we will only use metric denominators because, as we already saw in Section xxx, English denominators do not lend themselves easily to computation since the English exchange rates are not always the same. (For instance, 1 Foot = TWELVE Inch while 1 Yard = 3 Feet.)

a. First we look at an example where the carryover will occur in the addition.
Given a thirty-two meter length and a twenty-three meter length, we look at them as being made-up as follows:

- We look at the thirty-two meter length as being made up of three dekameters and two meters
- We look at the twenty-three meter length as being made up of two dekameters and three meters

We then construct a thirty-two meter by twenty-three meter rectangle, that is a rectangle that is three dekameters and two meters long one way and two dekameters and three meters long the other way.

![Figure 5.11: The tiling of a thirty-two meter by twenty-three meter rectangle.](image)

Counting the tiles shows that we will need:

- six one-dekameter-by-one-dekameter tiles
- four one-dekameter-by-one-meter tiles
- nine one-meter-by-one-dekameter tiles
- six one-meter-by-one-meter tiles

Since we don’t distinguish the one-dekameter-by-one-meter tiles from the one-meter-by-one-dekameter tiles, we can aggregate them and we get
CHAPTER 5. CONTINUOUS GOODS ON THE COUNTER I

THIRTEEN one-dekameter-by-one-meter tiles

or

THIRTEEN one-meter-by-one-dekameter tiles

depending on how we want to see them.

However, since we are not going to be able to write THIRTEEN, we must change THIRTEEN one-dekameter-by-one-meter tiles (or THIRTEEN one-meter-by-one-dekameter tiles) and the question is for what?

Figure 5.12 shows that TEN one-meter-by-one-dekameter tiles tile ONE one-dekameter-by-one-dekameter tiles

![Diagram](image)

Figure 5.12: Changing TEN one-meter-by-one-dekameter tiles

❖ Here again, the (board) procedure reflects what we just did.

\[
\begin{array}{c}
3 \text{ Dekam} & \times & 2 \text{ Dekam} \\
\times & 2 \text{ Dekam} & \times & 3 \text{ M} \\
\hline
6 \text{ Dekam} \times \text{ Dekam} & \times & 9 \text{ M} \times \text{ Dekam} & \times & 6 \text{ M} \times \text{ M} \\
\end{array}
\]

or

\[
\begin{array}{c}
6 \text{ Dekam} \times \text{ Dekam} & \times & \text{ THIRTEEN Dekam} \times \text{ M} & \times & 6 \text{ M} \times \text{ M} \\
\end{array}
\]

And we complete the (board) procedure as follows
5.4. MULTIPLICATION

\[ \begin{array}{c}
3 \text{ Dekam} \times 2 \text{ M} \\
\times 2 \text{ Dekam} \times 3 \text{ M}
\end{array} \]

\[ \begin{array}{c}
1 \text{ Dekam} \times \text{ Dekam} \\
9 \text{ M} \times \text{ Dekam} \times 6 \text{ M} \times \text{ M} \\
6 \text{ Dekam} \times \text{ Dekam} \times 4 \text{ Dekam} \times \text{ M} \\
7 \text{ Dekam} \times \text{ Dekam} \times 3 \text{ Dekam} \times \text{ M} \times 6 \text{ M} \times \text{ M}
\end{array} \]

or

\[ \begin{array}{c}
7 \text{ Dekam} \times \text{ Dekam} \times 3 \text{ M} \times \text{ Dekam} \times 6 \text{ M} \times \text{ M}
\end{array} \]

b. Now we look at an example where the carryover will occur in one of the multiplications:

First we proceed as we did before:

\[ \begin{array}{c}
1 \text{ Dekam} \times 5 \text{ M} \\
\times 1 \text{ Dekam} \times 3 \text{ M}
\end{array} \]

\[ \begin{array}{c}
1 \text{ Dekam} \times \text{ Dekam} \times 5 \text{ Dekam} \times \text{ M} \\
3 \text{ M} \times \text{ Dekam} \times \text{ Fifteen M} \times \text{ M}
\end{array} \]

if we agree on \text{ Dekam} \times \text{ M} as common denominator, or

\[ \begin{array}{c}
1 \text{ Dekam} \times \text{ Dekam} \times 8 \text{ Dekam} \times \text{ M} \times \text{ Fifteen M} \times \text{ M}
\end{array} \]

if we agree on \text{ M} \times \text{ Dekam} as common denominator.

Now we must change \text{ Fifteen M} \times \text{ M} for \text{ Dekam} \times \text{ M} for 1 \text{ Dekam} \& 5 \text{ M} \times \text{ M}:

\[ \begin{array}{c}
1 \text{ Dekam} \& 5 \text{ M} \\
\times 1 \text{ Dekam} \& 3 \text{ M}
\end{array} \]

\[ \begin{array}{c}
1 \text{ M} \times \text{ Dekam} \\
3 \text{ M} \times \text{ Dekam} \& 5 \text{ M} \times \text{ M}
\end{array} \]

if we agree on \text{ Dekam} \times \text{ M} as common denominator, or

\[ \begin{array}{c}
1 \text{ Dekam} \times \text{ Dekam} \& 9 \text{ Dekam} \times \text{ M} \times 5 \text{ M} \times \text{ M}
\end{array} \]

if we agree on \text{ Dekam} \times \text{ M} as common denominator, or

\[ \begin{array}{c}
1 \text{ Dekam} \times \text{ Dekam} \& 9 \text{ M} \times \text{ Dekam} \& 5 \text{ M} \times \text{ M}
\end{array} \]

if we agree on \text{ M} \times \text{ Dekam} as common denominator.

Of course, we shouldn’t wait and we should do the change immediately rather than write \text{ Fifteen M} \times \text{ M}

6. We shall now see how the above multiplication looks under a heading.
CHAPTER 5. CONTINUOUS GOODS ON THE COUNTER I

square denominator

a. First, we recall that the metric heading for lengths is:

<table>
<thead>
<tr>
<th>KILO Meter</th>
<th>HECTO Meter</th>
<th>DEKA Meter</th>
<th>Meter</th>
<th>DECI Meter</th>
<th>CENTI Meter</th>
<th>MILLI Meter</th>
</tr>
</thead>
</table>

and that the rate of change is ten for 1.

b. Corresponding to each of these denominators, we have the corresponding square denominator:

- MilliMeter × MilliMeter also called Square MilliMeter
- CentiMeter × CentiMeter also called Square CentiMeter
- DeciMeter × DeciMeter also called Square DeciMeter
- Meter × Meter also called Square Meter
- DekaMeter × DekaMeter also called Square DekaMeter
- HectoMeter × HectoMeter also called Square HectoMeter
- KiloMeter × KiloMeter also called Square KiloMeter

Note that the rate of change from one denominator to the next is still ten to 1 and that the empty spaces correspond to the non-square denominators. For instance, the empty space between Square Meter and Square DekaMeter is for

\[ \text{DekaMeter} \times \text{Meter} \]

or

\[ \text{Meter} \times \text{DekaMeter} \]

Note that the rate of change from one square denominator to the next is hundred to 1.

Figure 5.13: Changing hundred one-meter-by-one-meter tiles ONE one-dekameter-by-one-dekameter tile

HUNDRED one-meter-by-one-meter tiles ONE one-dekameter-by-one-dekameter tile
c. We now write the above multiplication under the heading:

<table>
<thead>
<tr>
<th>Square DekaMeter</th>
<th>Square Meter</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3 2</td>
</tr>
<tr>
<td></td>
<td>2 3</td>
</tr>
<tr>
<td>6</td>
<td>9 6</td>
</tr>
<tr>
<td>7</td>
<td>3 6</td>
</tr>
</tbody>
</table>
Chapter 6

Fractions

We now deal with the case when the two collections consist of different kinds of objects.

1. In that case, we just cannot aggregate the two collections and, on the board, we can only write them as a combination and that is that!

For instance, given dollar, dollar and dime, dime, dime, on the counter, we can only write the combination 2 Dollars & 3 Dimes. (Even if we were to “put together” dollar, dollar and dime, dime, dime as dollar, dollar, dime, dime, dime, we would not have a collection: what denominator would we use in the number-phrase?)

2. However, if we can exchange the objects in the two collections for collections of identical objects, then we have a common denominator and the combination can then be “turned” into an addition.

Cards: For instance, if we can change quarter for nickel, nickel, nickel, nickel, nickel and dime for nickel, nickel, then we can change quarter and dime, dime for nickel, nickel, nickel, nickel, nickel, nickel, that is for nickel, nickel, nickel, nickel, nickel, nickel, nickel, nickel.

Diamond: On the board, we proceed exactly in the same manner as on the counter and we write:
If: \(1 \text{ Quarter} = 5\) Nickels and \(1 \text{ Dime} = 2\) Nickels, then:
\[
1 \text{ Quarter} \& 2 \text{ Dime} = 5 \text{ Nickels} + 2(2 \text{ Nickels}) = 9 \text{ Nickels}
\]

However, while changing facilities will often be available, we should not expect that they always will.

1. We already saw above that the same ideas continue to apply even when the rate of exchange from one denominator to the next is not always the same or does not exist as for instance when we deal with goods in the British units. But, as we also saw already there, these ideas do not work as smoothly. We return to money because, even there,

\[\begin{array}{c}
\text{Hundred-dollar-bill} \\
\text{Fifty-dollar-bill} \\
\text{Twenty-dollar-bill} \\
\text{Ten-dollar-bill} \\
\text{Five-dollar-bill}
\end{array}\]

with \textit{Hundred-dollar-bill, Fifty-dollar-bill, Twenty-dollar-bill, Ten-dollar-bill, Five-dollar-bill, One-dollar-bill}, we observe that \(1 \text{ Hundred-dollar-bill}\) exchanges for \(2 \text{ Fifty-dollar-bill}\) but that \(1 \text{ Fifty}\) does not exchange for any number of \textit{Twenty-dollar-bill}. However, both \textit{Fifty-dollar-bill} and \textit{Twenty-dollar-bill} exchange for any of the following common denominators: \textit{Ten-dollar-bill, Five-dollar-bill, One-dollar-bill.}

It is a mildly interesting exercise to learn how to add and subtract under such headings but we shall turn our attention to something more fruitful which is how to introduce ever-smaller units, that is going \textit{downward}.

This is of course what we just did above, in Systematic Arithmetic, when we defined \textit{dime, cent} and \textit{mill}. For instance, starting with \textit{One-dollar-bill} as origin, \(1 \text{ quarter}\) is defined as \(1\) of-which-4-will-exchange-for-1-\textit{One-dollar-bill}, where the phrase of-which-4-will-exchange-for-1-\textit{One-dollar-bill} is the denominator , and, say, \(3 \text{ quarter}\) is read as \(3\) of-which-4-will-exchange-for-1-\textit{One-dollar-bill}.

Unfortunately, such denominators are traditionally coded in a manner that, if historical, is not very evocative of what they represent. The best way to deal with this issue is to morph one into the other:

i. \(3 \text{ quarter}\) (In which \textit{quarters} is the denominator.)
ii. \(3\) of-which-4-will-exchange-for-1-\textit{One-dollar-bill} (In which of-which-4-will-exchange-for-1-\textit{One-dollar-bill} is the denominator.)
iii. $3 \ 4 \ ?\ 1$-One-dollar-bill (In which $4 \ ?\ 1$-One-dollar-bill is the denominator.)

iv. (in which is the denominator.)

v. (in which is the denominator.)

vi. One-dollar-bill (in which One-dollar-bill is the denominator.)

Note that, here, the horizontal line in iv, v, and vi is merely a separator called fraction bar. Note that in v and vi there ought to be a 1 before One-dollar-bill but it goes without saying.

Unfortunately, two things happened that made the use of fractions a lot less easy than it should be. The first is that, as usual, the unit, here One-dollar-bill, almost always goes without saying. The second is that the 4 in the above code is called the denominator while it is only part of the denominator.

2. However, from this we can now deal easily with the usual topics regarding fractions.

a. For some reason lost in history, a fraction whose numerator is more than the denominator is deemed to be improper and to some extent inadmissible. What one is supposed to do is to change a maximum of parts for units. For instance, given 7 of-which-4-will-exchange-for-1-One-dollar-bill, we can exchange 4 of-which-4-will-exchange-for-1-One-dollar-bill for 1 One-dollar-bill after which we have the combination 1 One-dollar-bill & 3 of-which-4-will-exchange-for-1-One-dollar-bill. Similarly, given 23 of-which-4-will-exchange-for-1-One-dollar-bill, we want to exchange as many of them for One-dollar-bills. So we must find out how many One-dollar-bills we will be getting and how many parts will remain. In other words, we divide 23 by 4 to get the number of One-dollar-bill and the number of those of-which-4-will-exchange-for-1-One-dollar-bill that will remain.

Since One-dollar-bills = 5 One-dollar-bills & 3 of-which-4-will-exchange-for-1-One-dollar-bill, it is natural to exchange 5 One-dollar-bills for 20 of-which-4-will-exchange-for-1-One-dollar-bill after which they have 23 of-which-4-will-exchange-for-1-One-dollar-bill. And vice versa.

b. When the denominators are the same, they get One-dollar-bill + One-dollar-bill = One-dollar-bill in exactly the same manner as 7 nickel + 9 nickel = 16 nickel.

c. When the denominators are not the same, they must change to a common denominator before they can add: One-dollar-bill & One-dollar-bill = One-dollar-bill + One-dollar-bill = One-dollar-bill in exactly the same manner as 7 dime & 3 quarter = 14 nickel + 15 nickel = 29 nickel.
3. In the case of multiplication and division, the difficulty is, again, with the denominators but a picture readily shows that, for instance, \( = 24 \text{ 1-inch } \times 1\text{-inch} \) & \( 12 \text{ 1-inch } \times 1\text{-of-which-5-will-exchange-for-1-inch} \) & \( 6 \text{ 1-of-which-2-will-exchange-for-1-inch } \times 1\text{-inch} \) & \( 3 \text{ 1-of-which-2-will-exchange-for-1-inch } \times 1\text{-of-which-5-will-exchange-for-1-inch} \), that is, more familiarly, \( = 24 + 12 + 6 + 3 = 27 \text{ + + +} \).

*Note.* While one might want at this point to make the students realize that \( , \) it is of no use here and, by the time they will have to use it, that is when investigating functions, they will have forgotten. Anyhow, it will be much more satisfying to do with the notion of nearness, in conjunction with \( . \)
Chapter 7

Multiplicative Powers

7.1 Repeated multiplication/division

Given a number $a$, we shall often have to multiply or divide it by a number of copies of some other number $x$

1. We begin by discussing the corresponding language.
   - $a (x)^{+3}$ is to be read as $a$ multiplied by 3 copies of $x$
   - $a (x)^{-3}$ is to be read as $a$ divided by 3 copies of $x$
   where
     - the number $a$ is called the coefficient,
     - the number $x$ is called the original
     - the number, $+3$ or $-3$, is called the exponent where
       * the counting-number 3 indicates the number of copies to be made of the original
       * the sign, $+$ or $-$, indicates whether the coefficient is to be multiplied or divided by the copies

Occasionally, the exponent will turn out to be 0, but, even in that case, we will continue to have

- $a (x)^0$ is to be read as $a$ multiplied/divided by 0 copies of $x$

In this last case, we thus have

\[ a (x)^0 = a \]

But then, by comparison with

\[ a \cdot (+1) = a \]

we can conclude that

\[ (x)^0 = +1 \]
2. When replacing $a$ and $x$ by signed numbers, it is safer to enclose them within parentheses. For instance, with $a = +7$ and $x = +5$, we write:

\[
(+7)(+5)^3 = +7 \text{ multiplied by 3 copies of } +5 = (+7) \cdot (+5) \cdot (+5) \cdot (+5) = (+7) \cdot (125) = +825
\]

and

\[
(+7)(+5)^{-3} = +7 \text{ divided by 3 copies of } +5 = \frac{+7}{(+5) \cdot (+5) \cdot (+5)} = \frac{+7}{+125} = +0.056
\]

and

\[
(+7)(+5)^0 = +7 \text{ multiplied by 0 copies of } +5 = +7
\]

Altogether, we have

<table>
<thead>
<tr>
<th>The code:</th>
<th>to be read as:</th>
<th>says to write:</th>
<th>gives:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(+7)(+5)^3$</td>
<td>$+7$ multiplied by 3 copies of $+5$</td>
<td>$(+7) \times (+5)(+5)(+5)$</td>
<td>$(+7) \cdot (+125)$</td>
</tr>
<tr>
<td>$(+7)(+5)^{-3}$</td>
<td>$+7$ divided by 3 copies of $+5$</td>
<td>$(+7) \div (+5)(+5)(+5)$</td>
<td>$\frac{+7}{+125}$</td>
</tr>
<tr>
<td>$(+7)(+5)^0$</td>
<td>$+7$ multiplied by 0 copy of $+5$</td>
<td>+7</td>
<td>+7</td>
</tr>
</tbody>
</table>

### 7.2 Powers

Of course, $+$ and $+1$ and, even more so, $+1$ tend to go without saying.

1. In particular, the coefficient $+1$ usually goes without saying and then

- $(x)^3$ is called the positive third power of $x$
- $(x)^{-3}$ is called the negative third power of $x$
- $(x)^0$ is called the zeroth power of $x$
7.2. **POWERS**

In other words, powers can be looked upon as repeated multiplications involving the coefficient +1 with the latter going without saying.

However, it will be more convenient for us to use both the coefficient +1 and the coefficient −1. Thus,

- 
  - \((+5)^{+3}\) will mean \((+1) \cdot (+5)^{+3}\)
  - \((-5)^{+3}\) will mean \((-1) \cdot (+5)^{+3}\)
  - \((+5)^{-3}\) will mean \((+1) \cdot (+5)^{-3}\)
  - \((-5)^{-3}\) will mean \((-1) \cdot (+5)^{-3}\)

and we will call these **gauge powers**.

2. When it is the exponent which is equal to +1, it is less of an issue to let it go without saying since

- \((+7)(+5)^{+1}\) is to be read as +7 multiplied by 1 copy of +5, that is \((+7)(+5)\),

while

- \((+7)(+5)\) is to be read as +7 multiplied by +5, that is \((+7)(+5)\)

which is exactly the same as above.

On the other hand, the exponent −1 can never go without saying.

3. When replacing \(x\) by a negative number, for example \(-5\), in a power, one should be careful that

\(-5^{+4}\) does not work out to the same number as \((-5)^{+4}\)

Indeed,

- in \(-5^{+4}\), the − stands for the coefficient −1 and the 5 stands for +5 and is the original of which the copies are to be made. In other words, we have:

\[
-5^{+4} = (-1) \cdot (+5)^{+4} \\
= (-1) \text{ multiplied by 4 copies of } (+5) \\
= (-1) \cdot (+5) \cdot (+5) \cdot (+5) \cdot (+5) \\
= -625
\]

- in \((-5)^{+4}\) the coefficient is +1, going entirely without saying, and −5 is the orginal of which the copies are to be made. In other words, we have:

\[
(-5)^{+4} = (+1) \cdot (-5)^{+4} \\
= (+1) \text{ multiplied by 4 copies of } (-5) \\
= (+1) \cdot (-5) \cdot (-5) \cdot (-5) \cdot (-5) \\
= +625
\]

On the other hand,

\(-5^{+3}\) does work out to the same number as \((-5)^{+3}\)
Indeed,

• in \(-5^3\) the \(-\) stands for the coefficient \(-1\) and \(5\) standing for \(+5\) is the original of which the copies are to be made. In other words, we have:

\[
-5^3 = (-1) (+5)^3
= (-1) \text{ multiplied by 3 copies of } (+5)
= (-1) \cdot (+5) \cdot (+5) \cdot (+5)
= -125
\]

• in \((-5)^3\) the coefficient is \(+1\), going entirely without saying, and \(-5\) is the original of which the copies are to be made. In other words, we have:

\[
(-5)^3 = (+1) (-5)^3
= (+1) \text{ multiplied by 3 copies of } (-5)
= (+1) \cdot (-5) \cdot (-5) \cdot (-5)
= -125
\]

This has nothing to do with the sign of the exponent and the same would hold true with a negative exponent.

What is involved here is the **parity** of the exponent, that is whether the exponent indicates an **odd** number or an **even** number of copies. This will turn out to be extremely important when we investigate power functions.
7.3 Roots
Chapter 8

Real Numbers And Their Graphic Representation

Engineers like to say that the only real Real Numbers are the Decimal Numbers.

8.1 Decimal Approximations

Since, other than counting-numbers that were defined directly, all other numbers were specified indirectly, that is as solutions of equations that we can usually solve only approximately, we will use real numbers only as code to designate collectively all the many Decimal Numbers that are approximate solutions of the equation.

After whatever decimal number we use, though, we shall have to write + (...), read “plus a little bit too small to matter in the current situation” because the equality will not be exact. We look at a couple of examples.

Example 1. We view \( \frac{11}{7} \) as a specifying-phrase, namely the solution of the equation \( 7x = 11 \), that is \( \frac{11}{7} \) stands for whatever decimal number 7 copies of which will add to 11 within the approximation required by the situation. So, we might use any one of the following.

---

1 As opposed to Educologists, “Just plain folks”, as well as engineers, have no particular reason to distinguish repeating from non-repeating decimals.

2 In other words, we look at Real Numbers are just equivalence classes, of Dedekind cuts from the order point of view, or of Cauchy sequences from the metric point of view! Even non-standard infinitesimals, should one wish to develop them eventually, can be looked upon as an extension of the decimal numbers.

3 In other words, (...) is a “precursor” of the little \( o \) notation.
• \( \frac{11}{7} = 1 + (\ldots) \)
  because
  \[ 7 \cdot \left[ 1 + (\ldots) \right] = 7 \cdot 1 + 7 \cdot (\ldots) = 7 + (\ldots) \]
• \( \frac{11}{7} = 1.5 + (\ldots) \)
  because
  \[ 7 \cdot \left[ 1.5 + (\ldots) \right] = 7 \cdot 1.5 + 7 \cdot (\ldots) = 10.5 + (\ldots) \]
• \( \frac{11}{7} = 1.57 + (\ldots) \)
  because
  \[ 7 \cdot \left[ 1.57 + (\ldots) \right] = 7 \cdot 1.57 + 7 \cdot (\ldots) = 10.99 + (\ldots) \]
• \( \frac{11}{7} = 1.571 + (\ldots) \)
  because
  \[ 7 \cdot \left[ 1.571 + (\ldots) \right] = 7 \cdot 1.571 + 7 \cdot (\ldots) = 10.997 + (\ldots) \]
• \( \frac{11}{7} = 1.5714 + (\ldots) \)
  because
  \[ 7 \cdot \left[ 1.5714 + (\ldots) \right] = 7 \cdot 1.5714 + 7 \cdot (\ldots) = 10.99987 + (\ldots) \]

Thus, in practice, depending on the circumstances, we might replace \( \frac{11}{7} \) by any one of the following:
1 + (\ldots)
1.5 + (\ldots)
1.57 + (\ldots)
1.571 + (\ldots)
1.5714 + (\ldots)

**Example 2.** Similarly, we view \( \sqrt[3]{13} \) as a specifying-phrase standing for whatever decimal number is a solution of the equation \( x^3 = 13 \) that is, \( \sqrt[3]{13} \) stands for whatever decimal number 3 copies of which will multiply to 13 within the approximation required by the situation. So, we might use any one of the following.
• \( \sqrt[3]{13} = 2 + (\ldots) \)
  because
  \[ [2 + (\ldots)]^3 = 8 + (\ldots) \]
• \( \sqrt[3]{13} = 2.3 + (\ldots) \)
  because
  \[ [2.3 + (\ldots)]^3 = 12.167 + (\ldots) \]
• \( \sqrt[3]{13} = 2.4 + (\ldots) \)
  because
  \[ [2.4 + (\ldots)]^3 = 13.824 + (\ldots) \]
• \( \sqrt[3]{13} = 2.35 + (\ldots) \)
  because
  \[ [2.35 + (\ldots)]^3 = 12.977875 + (\ldots) \]
8.2 Sign and Size of a Number

Given a type of function, in order to find out under what conditions we can “join smoothly” a plot into a graph, we shall investigate the kind of outputs this type of functions returns for various kinds of inputs.

Thus, before we can start on our program, we need to look at numbers from a new point of view.

1. The **Sign** of a number is the side of 0 that the number is on the ruler. For instance,
   - Sign of \(-7\) is *negative* because \(-7\) is *left of* 0 on the ruler.
   - Sign of \(+3\) is *positive* because \(+3\) is *right of* 0 on the ruler.

   ![Sign Diagram]

   **Sign** means Which Side of 0.

The **Size** of a number is *how far from* 0 the number is on the ruler. For instance, \(-5\) and \(+5\) are the **same-size**, namely 5, because they are both 5 away from 0:

![Size Diagram]

**Size** means How Far From 0.
2. While $-5$ is *algebra-smaller* than $+3$ (see Section 3.7), $-5$ is *larger-in-size* than $+3$ because $-5$ is *farther away from 0* than $+3$: $-5$ is 5 away from 0 while $+3$ is only 3 away from 0. Similarly, while $+3$ is *algebra-larger* than $-5$, (see Section 3.7), $+3$ is *smaller-in-size* than $-5$ because $+3$ is *closer from 0* than $-5$: $+3$ is 3 away from 0 while $-5$ is 5 away from 0.

![Larger/Smaller In Size](image)

Larger/Smaller In Size *means* Farther from/Closer to 0.

3. We will have to distinguish:
   - Numbers that are *small-in-size*, as opposed to numbers that are *algebra-small*. We shall use the letter $h$ to stand for inputs that are *small-in-size*.
   - Numbers that are *large-in-size*, as opposed to numbers that are *algebra-large*. Unfortunately, there is no generally accepted letter to stand for inputs that are *large-in-size*.

To an extent, *small-in-size* and *large-in-size* are relative concepts. For instance, whether a *gain* or a *loss*, ten-thousand dollars is a small sum of money for people like Bill Gates or George W. Bush while to “the rest of us” ten-thousand dollars is large enough. However, the words have one definite meaning that is the same for everybody. For instance, nobody likes losing a large sum of money. Bill Gates or George W. Bush wouldn’t like to lose a million dollars just the way “the rest of us” wouldn’t like to lose ten-thousand dollars. It is just the *cutoff point* that varies from people to people.

4. Still, for our purpose, it will be necessary to make the concepts a bit more precise. We observe that:
   - Any number of copies of an original whose Size is *equal* to 1, and that is only either the number $-1$ or the number $+1$, multiply to a result which is the exact *same-size* as the original.

On the other hand,
   - Any number of copies of an original whose Size is *larger* than 1, that is numbers either *algebra-smaller* than $-1$ or *algebra-larger* than $+1$, multiply to a result which is *larger-in-size* than the original.
   - Any number of copies of an original whose Size is *smaller* than 1, that is numbers that are *algebra-between* $-1$ and $+1$, multiply to a result which is *smaller-in-size* than the original.
8.2. SIGN AND SIZE OF A NUMBER

We thus have:

\[ \begin{array}{c|c|c}
\text{All these numbers are large in size} & \text{All these numbers are small in size} & \text{All these numbers are large in size} \\
\hline 
-5 & -4 & -3 \\
-2 & -1 & 0 \\
+1 & +2 & +3 \\
+4 & +5 \\
\end{array} \]

Small In Size and Large In Size (Strictly).

For instance, we could treat \(-1.1\) as a \textit{large-in-size} number because, say, three copies of \(-1.1\) multiply to \(-1.331\) which is \textit{larger-in-size} than the original, \(-1.1\).

Similarly, we could treat that \(+0.2\) as a \textit{small-in-size} number because, say, three copies of \(+0.2\) multiply to \(+0.008\) which is \textit{smaller-in-size} that the original, \(+0.2\).

Strictly speaking, this is the only thing we will need to know about a number to decide whether to treat it as \textit{large-in-size} or \textit{small-in-size}.

5. To make things easier, though, we will stay way away from \(-1\) and \(+1\) as in

\[ \begin{array}{c|c|c}
\text{All these numbers are large in size} & \text{All these numbers are small in size} & \text{All these numbers are large in size} \\
\hline 
\text{Small In Size and Large In Size (Safely).} \\
\end{array} \]

where the parentheses indicate that the endpoints are “open to change”, according to the circumstances.

6. In fact, practically, we will “think” of the endpoints as being \(-10\), \(-0.1\), \(+0.1\), \(+10\) and this for the following reason.

Consider \(-7\) and \(+2\). They are both \textit{in-the-ones} and therefore of the same \textbf{order of magnitude}. Multiplying, say, three copies of each shows that they are both \textit{large-in-size}.

However,

- \(+8\), the third power of \(+2\), is \textit{in-the-ones} and therefore of the \textbf{same order} of magnitude as the original.
- \(-343\), the third power of \(-7\), is \textit{in-the-hundreds} and therefore of a \textbf{larger order} of magnitude than the original.

So, there is a \textbf{qualitative} difference between \(-7\) and \(+2\): While both are \textit{large-in-size}, their third powers are \textit{not} of the same order of magnitude. We
shall say that while \(-7\) and \(+2\) are both *large-in-size*, they are so *by different orders of magnitudes*.

Similarly, consider \(-0.7\) and \(-0.2\). They are both *in-the-tenths* and therefore of the same *order of magnitude*. Multiplying, say, two copies of each shows that they are both *small-in-size*.

However,

- \(+0.49 = +0.5 + (...)\), the second power of \(-0.7\) is *in-the-tenths* and therefore of the same order of magnitude as the original.
- \(+0.04\), the second power of \(-0.2\), is *in-the-hundredths* and therefore of a *smaller* order of magnitude than the original.

So, there is a *qualitative* difference between \(-0.7\) and \(-0.2\): While both are *small-in-size*, their second powers are *not* of the same order of magnitude.

We shall say that while \(-0.7\) and \(-0.2\) are both *small-in-size*, they are *so by different orders of magnitudes*.

Thus, finally, the endpoints that we shall use in practice are:

\[
\begin{align*}
\text{Small In Size and Large In Size (Practically).} \\
+10 & -10 \\
+0.1 & -0.1
\end{align*}
\]

What this will do is to ensure that:

- Copies of a large-in-size original will multiply to results that are not only larger-in-size than the original but are *so by an order of magnitude*.
- Copies of a small-in-size original will multiply to results that are not only smaller-in-size than the original but are *so by an order of magnitude*.

Thus for instance, consider \(+20\) and \(-70\) which are both *in-the-tens*. When we multiply 2 copies, we get \((+20)(+20) = +400\) which is *in-the-hundreds* while \((-70)(-70) = +4900\) which is *in-the-thousands*. Of course, we still have a qualitative difference but, at least, both are *larger* than the originals *by an order of magnitude*.

Similarly, consider \(-0.02\) and \(-0.07\) which are both *in-the-hundredths*. When we multiply 2 copies, we get \((-0.02)(-0.02) = +0.0004\) which is *in-the-ten-thousandths* while \((-0.07)(-0.07) = +0.0049 = +0.005 + (...)\) which is *in-the-thousandths*. Again, we still have a qualitative difference but, at least, both are *smaller* than the originals *by an order of magnitude*.

From now on, we shall use:
8.3. THE ARITHMETIC OF SIGNS AND SIZES

- small as short for small-in-size
- large as short for large-in-size

We shall also use the following language:
- $x$ is near 0 will mean the same as $x$ is small (in size).
- $x$ is near $\infty$ will mean the same as $x$ is large (in size).

<table>
<thead>
<tr>
<th>Near 0</th>
<th>and</th>
<th>Near $\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+10</td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>−10</td>
<td></td>
<td>−0.1</td>
</tr>
</tbody>
</table>

All these numbers are near $\infty$

and, when we want to include the sign,
- $x$ is near $0^+$ will mean the same as $x$ is small (in size) and positive.
- $x$ is near $0^-$ will mean the same as $x$ is small (in size) and negative.
- $x$ is near $+\infty$ will mean the same as $x$ is large (in size) and positive.
- $x$ is near $-\infty$ will mean the same as $x$ is large (in size) and negative.

<table>
<thead>
<tr>
<th>Near $0^+$</th>
<th>and</th>
<th>Near $0^-$</th>
<th>and</th>
<th>Near $+\infty$</th>
<th>and</th>
<th>Near $-\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+10</td>
<td></td>
<td>−10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+0.1</td>
<td></td>
<td>−0.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

All these numbers are near $-\infty$

All these numbers are near $+\infty$

For most practical purposes, we can think of
- large numbers as positive powers of 10, possibly dilated by a single digit number such as, for instance, $3 \cdot 10^5 = 300000$ but also even $3 \cdot 10^1 = 30$,
- small numbers as negative powers of 10, possibly dilated by a single digit number such as, for instance, $3 \cdot 10^{-5} = 0.00003$ but also even $3 \cdot 10^{-1} = 0.3$,
- finite numbers as the zero power of 10 such as, for instance, $3 \cdot 10^0 = 3$.

8.3 The Arithmetic of Signs and Sizes

It will be crucial to be able to operate with numbers in relation to multiplication and division on the sole basis of their sign or size.
1. Signs behave according to the so-called “rule of signs”:

\[
\begin{align*}
+ \text{ times } + &= + \\
+ \text{ times } - &= - \\
+ \text{ times } + &= - \\
- \text{ times } - &= + \\
\end{align*}
\]

and

\[
\begin{align*}
+ &= + \\
+ &= - \\
- &= + \\
- &= - \\
\end{align*}
\]

2. Sizes behave logically and we must be careful about that logic.

a. Sizes behave obviously in the following cases:

finite \times \text{ large} = \text{ large} \\
\frac{\text{ finite}}{\text{ large}} = \text{ small} \\
\text{ finite} \times \text{ small} = \text{ small} \\
\frac{\text{ finite}}{\text{ small}} = \text{ large} \\
\text{ large} \times \text{ large} = \text{ large} \\
\frac{\text{ large}}{\text{ small}} = \text{ large} \\
\text{ small} \times \text{ small} = \text{ small} \\
\frac{\text{ small}}{\text{ large}} = \text{ small} \\
\text{ large} \\

b. However, sizes do not behave that simply in the following cases:

\text{ large} \times \text{ small} = ??? \\
\frac{\text{ large}}{\text{ large}} = ??? \\
\frac{\text{ large}}{\text{ small}} = ???
This is because here the matter of how small is small and how large is large comes in. For instance, the following three are all instances of large $\times$ small.

We have

$$2000000 \times 0.0003 = 600.$$  

but

$$20000 \times 0.00000 = 0.06$$  

and

$$200000 \times 0.00003 = 6.$$  

That is, this is a matter of order of magnitude and we will deal with this before we start investigating RATIONAL FUNCTIONS.

### 8.4 Rulers

Up until now, we have represented numbers by writing signed-number-phrases but, quite often, it will be very convenient also to picture numbers. To this purpose, we shall use rulers by which we mean essentially what goes by that name in the real world, like, for instance,
which, for a variety of reasons, is not a ruler.

*Note.* In school environments, rulers are usually called *number lines*.

### 8.5 Windows

In order to *picture* a *number-pair*, we will need a *window* which consists of:

- a *screen*
- a *first ruler* placed under the *screen*,
- a *second ruler* placed left of the *screen*,
- *grey-space* between the *screen* and the *rulers*. We shall see in what it is for.)

*Note.* The above arrangement is quite *arbitrary*. In particular, the rulers need not go from left to right and from bottom to top. More generally, for instance, the rulers need not be at a 90 degree angle.

Then, for instance, to represent the *number-pair* \((-2, +5)\), (see Figure below),

1. we represent the *first number* in the pair, \(-2\), by a *first-number point* on the *first ruler*,
ii. we draw the **first-number level line**—a *vertical* line, through the first-number point,

iii. we represent the *second number* in the pair, +5 by a **second-number point** on the *second* ruler,

iv. we draw the **second number level line**—a *horizontal* level line, through the second-number point,

v. Then, the **plot point**, that is the point that represents the *pair* \((-2, +5)\) on the *screen* is where the first number level line and the second-number level level line intersect.

Observe that a *plot-point* is a **good picture** of a *number-pair* because, once we have drawn the plot-point, we can erase the two number points as well as the two level lines *without loss of information*. In other words, given, for instance, the following *plot-point*
we can recover the number-pair of which this plot-point is the picture. All we have to do is to go backwards through the above steps:

i. we draw the first-number level line (vertical) through the given plot point,

ii. the point that represent the first number in the pair is where the first-number level line intersects the first-number ruler,

iii. we draw the second-number level line (horizontal) through the given plot point,

iv. the point that represent the second number in the pair is where the second-number level line intersects the second-number ruler.
Sometimes, to facilitate plotting, the window comes as graph-paper, that is already equipped with level lines that form a grid:
Note. Very often, the level lines for 0 are singled out as for instance on the graph paper and are called first-axis and second-axis. Sometimes, only the axes are drawn without the rest of the graph paper and/or the rulers. We shall always draw the rulers but, when doing qualitative investigations, we will only draw those levels lines that are relevant to the investigation.

We will distinguish:

- **infinite** is what cannot be seen because it is outside the window regardless of the extent of the ruler. This is most of the ruler.
- **finite** is what can be seen. So, first it has to be in the window and then it has to appear as a tick-mark on the corresponding ruler. Thus, whether something is finite depends on the extent of the ruler and then on the resolution of the ruler. See Section 8.4
- **infinitesimal** is what cannot be seen in the window regardless of the resolution of the ruler.
Chapter 9

Introduction to Functions

Recall that, from the Arithmetic viewpoint, numbers were specified directly while, from the Algebra viewpoint, we looked for numbers that were specified indirectly by being solutions of given equations.

We will now investigate the processes, devices, procedures, agencies, converters, translators, etc, by which new numbers can be obtained from given numbers.

In this Part II. Algebraic Functions, the processes, devices, procedures, agencies, converters, exchangers, translators, etc,, ... , etc, that we will investigate will be specified directly, just like numbers were in Arithmetic. We shall do this by way of input-output rules.

In Part III. Transcendental Functions, the processes, devices, procedures, agencies, converters, exchangers, translators, etc, that we shall investigate will be specified indirectly, just like numbers were in algebra, that is by being required to be solutions of given equations.

9.1 What are relations and functions?

To perceive that something is changing, it is necessary to look at it against something else that either does not change at all or that changes differently. For instance, the amount of income tax changes in terms of income, the amount of property tax changes in terms of assets, the amount of sales tax changes in terms of expenses.

More precisely, in order to observe something changing, we must pair each of the stages that it goes through in terms of the stages that some reference thing goes through, if only a clock or a calendar!

For instance, we might say that, in 2003, someone’s income tax was
$6,753. Just to have said that someone’s income tax was $6,753 would not say much since $6,753 is a lot less money now than in 1913 when income tax was first created.

The reference stages will be called inputs and the stages of what we are investigating will be called outputs.

We will call relation the pairing of input numbers with output numbers that results from any process, device, procedure, agency, converter, exchanger, translator, etc. An input number together with an output number it is paired with make up what we shall call an input-output pair. (Note that we will be using parentheses to enclose input-output pairs and that this is yet another use of parentheses.)

Note. Eventually, we shall just use the word "input" instead of the phrase "input number" and the word "output" instead of the phrase "output number".

After this introduction, we shall investigate only those relations, called functions, that meet the single requirement that no input be paired with more than one output. In terms of something that we are observing, this means that once a reference stage has been picked, no more than one stage can be observed.

In other words, given any input, a function may return either no output at all or one output but no more than one output.

Beyong that functions need not meet any other requirement and, in particular, a function may return the same output for different inputs.

For example, a parking meter is a function because, given an input, say 1 Quarter, the parking meter returns a definite amount of parking time, say 30 Minutes. We would then say that (1 Quarter, 30 Minutes) is an input-output pair. Note that any amount of money above the maximum will return the same amount of parking time.

On the other hand, a slot machine is not a function because, given an input, say 1 Quarter, a slot machine could return any number of Quarters.
9.2 Relations specified by an input-output table.

The simplest way to specify a relation is just to write out all the input-output pairs. This is often done by writing the inputs in an input-row (or in an input-column) and the outputs returned by the relation directly underneath the inputs in an output-row (or directly next to the inputs in an output-column). We shall then say that the relation is specified by an input-output table.

For example, consider the input-output table

<table>
<thead>
<tr>
<th>Inputs</th>
<th>−6</th>
<th>−5</th>
<th>−3</th>
<th>−2</th>
<th>0</th>
<th>+1</th>
<th>+2</th>
<th>+6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outputs</td>
<td>−4</td>
<td>+5</td>
<td>+5</td>
<td>−1</td>
<td>0</td>
<td>−5</td>
<td>−5</td>
<td></td>
</tr>
</tbody>
</table>

in which, for instance, the input −3 is paired with the output +5. Since, here, no input is paired with more than one output, the input-output table specifies in fact a function.

Note that this function does not return any output for −5. For that matter, neither does it return any output for, say, −4, −1 or for +1.7 or −647 or, indeed, for any input other than those listed. Also, observe that for the inputs −3, −2 and +6, the function returns the same output −5.

On the other hand, consider the following input-output table

<table>
<thead>
<tr>
<th>Inputs</th>
<th>−42</th>
<th>−20</th>
<th>+2</th>
<th>+24</th>
<th>+57</th>
<th>+173</th>
<th>+345</th>
<th>+359</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outputs</td>
<td>+765</td>
<td>+56</td>
<td>+3</td>
<td>−7</td>
<td>+56</td>
<td>+56</td>
<td>−111</td>
<td>0</td>
</tr>
</tbody>
</table>

It pairs the input +2 with more than one output, namely the two outputs +3 and −7, and, even though it pairs all the other inputs with at most one output each, this is enough to prevent this table from specifying a function.

However, note that, other than for the input +2, this input-output table would specify a function. In particular, the fact that the inputs −20, +24 and +57 all give the same output +56 would not, by itself, prevent the table from specifying a function.

9.3 Plot of a relation specified by an input-output table

There are many ways to represent a relation by a picture but since, here, input-output pairs are number-pairs, we shall use the windows that we introduced in Section 8.5.

1. For instance, to represent the input-output pair (−2, +5), (see Figure below),
i. we represent the input-number $-2$, by an input-point on the input ruler,

ii. we draw the input level line—a vertical line, through the input-point,

iii. we represent the output-number, $+5$ by an output-point on the output ruler,

iv. we draw the output level line—a horizontal line, through the output-point

v. Then, the plot-point, that is the point that represents the input-output pair $(-2, +5)$ on the screen is at the intersection of the input level line and the output level line.

2. Observe that a plot point is a good picture of an input-output pair because, given a plot-point, we can erase the input-point, the output-point as well as the level lines without loss of information. In other words, given a plot-point we can recover the input-output pair of which the plot point is the picture. All we have to do is to go backwards through the above steps as in Section 8.5:
3. Now we define a **plot** as a collection of plot points for all the input-output pairs in the table.

For example, given the table

<table>
<thead>
<tr>
<th>Inputs</th>
<th>−6</th>
<th>−5</th>
<th>−3</th>
<th>−2</th>
<th>0</th>
<th>+1</th>
<th>+2</th>
<th>+6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outputs</td>
<td>−4</td>
<td>+5</td>
<td>+5</td>
<td>−1</td>
<td>0</td>
<td>−5</td>
<td>−5</td>
<td></td>
</tr>
</tbody>
</table>

we can plot each and every input-output pair which gives us the **plot** of the above **table**:
Observe that, since each plot-point is a good picture of an input-output pair, the plot is a good picture of the input-output table. In other words, the plot contains exactly the same information as the input-output table so that a relation can be specified by a plot just as well as by an input-output table and we can go back and forth between the two.

Moreover, most of the time, only the plot-points are shown since the input level-lines and the output level-lines can always be recovered and thus can “go without saying”.
9.4 Relations specified by a graph

We saw above that, when we represent input-output pairs by a plot-point in a window, the picture is a good one because, from the plot-point, we can recover the input-output pair. So, a graph, that is any line drawn on the screen, whether curved or not, will specify a relation.

For instance, given the following graph,

we can pick the following graph-points
and, proceeding as in Section 9.3, get the following input-output table

<table>
<thead>
<tr>
<th>Inputs</th>
<th>−6</th>
<th>−3</th>
<th>−2</th>
<th>0</th>
<th>+1</th>
<th>+2</th>
<th>+6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outputs</td>
<td>−4</td>
<td>+5</td>
<td>+5</td>
<td>−1</td>
<td>0</td>
<td>−5</td>
<td>−5</td>
</tr>
</tbody>
</table>

However, by picking the following graph-points on the same graph

we would get a different input-output table

<table>
<thead>
<tr>
<th>Inputs</th>
<th>−5</th>
<th>−4</th>
<th>−1</th>
<th>+1</th>
<th>+2</th>
<th>+4</th>
<th>+6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outputs</td>
<td>−1</td>
<td>+2</td>
<td>0</td>
<td>0</td>
<td>−5</td>
<td>−2</td>
<td>−5</td>
</tr>
</tbody>
</table>

In other words, since we can go on forever picking graph points, a graph carries a lot more information than we can put in any input-output table.

9.5 Functions specified by an input-output rule.

While in some sciences, such as psychology and sociology, functions are usually specified by input-output tables, in other sciences, such as physics and electronics, functions are usually specified by input-output rules as follows:

i. We use a letter, usually \(x\), as place holder for inputs. In other words, we will be able to replace \(x\) by any number we want.

ii. We must either have or give a name to the function. In the absence of any given name, we shall usually use the letter \(f\).
iii. Then, $f(x)$ will stand for the output returned for the input $x$ by the function $f$.

For instance, if a function called, say, $FUN$ doubles the input and adds 5 to give the output, then the input-output rule of $FUN$ is:

$$x \xrightarrow{FUN} FUN(x) = 2x + 5$$

Then, to find the output for a specific input, say 7, we just replace all the occurrences of $x$ by 7:

$$7 \xrightarrow{FUN} FUN(7) = 2 \cdot 7 + 5$$

and then we compute the output underneath:

$$7 \xrightarrow{FUN} FUN(x) = 2 \cdot 7 + 5$$
$$= 14 + 5$$
$$= 19$$

We can then write

$$7 \xrightarrow{FUN} 19$$

which says that $(7, 19)$ is an input-output pair under the function $FUN$.

However, instead of the above and because it will make it easier later on to keep track of what we are doing, we shall use the input-output rule as follows:

$$\text{When } x = 7 \xrightarrow{FUN} FUN(x)|_{\text{when } x=7} = 2x + 5|_{\text{when } x=7}$$
$$= 2 \cdot 7 + 5$$
$$= 14 + 5$$
$$= 19$$

9.6 The Fundamental Problem

We now address the problem of picturing by a graph a function given by an input-output rule.

The first idea that comes to mind is to pick a number of inputs, create an input-output table and then plot the input-output table. Students are then usually advised to “join smoothly” the plot-points to get the graph of the function but this clearly cannot work because, on the basis of the outputs
we have already computed, we cannot find all the other outputs which is what the graph would do.

Consider, for example, the function given by the input-output rule

\[ x \xrightarrow{\text{RAT}} RAT(x) = \frac{3x + 6}{x^2 - 10} \]

We can find the output for, say, the input +2

When \( x = +2 \) \( \xrightarrow{\text{RAT}} \) \( RAT(x) \bigg|_{\text{when } x = +2} = \frac{3x + 6}{x^2 - 10} \bigg|_{\text{when } x = +2} = \frac{3 \cdot (+2) + 6}{(+2)^2 - 10} = \frac{12}{-6} = -2 \)

Proceeding as above, we can get, for instance, the following input-output table:

<table>
<thead>
<tr>
<th>Inputs</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>+1</th>
<th>+2</th>
<th>+4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outputs</td>
<td>-1</td>
<td>+3</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
<td>+3</td>
</tr>
</tbody>
</table>

and, from it, the following plot:
But now, how are we to “join” the plot points? The first thing that usually comes to the mind is something like this

But now the graph gives the output $+\frac{1}{2}$ for the input $+3$ and it is not likely that this is what the input-output rule would give. Beyond that, it is not likely that the extrapolation given by the input-output rule is just as given by the graph. Nor that the plot-points we picked should happen to be precisely the peaks and valleys of the graph. Etc.

On second thought then, we might “join the plot-points smoothly”, somewhat as below.
But again, since we drew the graph more or less at random, it is very unlikely that for, say, the input $+3$ the input-output rule would give the output $+1$ which is the output given by the graph. Neither is it likely that the plot should happen to be precisely the highs and lows of the graph.

In fact, given the above plot, anyone of the following is just as likely or unlikely to be the graph:
The advice usually given at this point is “just get more plot points” but the next question is then “how many plot-points is enough to guarantee that we have the graph?”

In fact, having very many plot points may make it impossible to “join them smoothly” as demonstrated by Strang’s Famous Example:
Yet this plot turns out to be a plot of a well-known function, called \textit{SINE}, whose “smooth” graph will be found in the next course.

The following then will be our “program” to cope with this “fundamental problem”:

\begin{itemize}
  \item[i.] Rather than try to find a universal way to find the graph of any function, we shall investigate \textbf{types of functions}, one type at a time, and find out for each type under what conditions we can “join smoothly” a plot into a graph. Here are the types of functions that we shall investigate in this part.
  \begin{itemize}
    \item[a.] Power Functions
    \begin{itemize}
      \item[i.] Affine Functions
      \item[ii.] Quadratic Functions
      \item[iii.] Cubic Functions
    \end{itemize}
  \end{itemize}
  \item[b.] Polynomial Functions
  \begin{itemize}
    \item[i.] Quadratic Functions
    \item[ii.] Cubic Functions
    \item[iii.] Cubic Functions
  \end{itemize}
  \item[c.] Rational Functions
  \begin{itemize}
    \item[ii. By the way, the reverse problem of going from a \textit{graph} to an \textit{input-output rule} can be a surprisingly difficult one and, other than in a very few cases, we shall not deal with here.
    \item[iii. On the other hand, the problem of going from a \textit{plot} to an \textit{input-output rule} is completely different in nature and central to the subject called \textbf{Data Analysis}.}
Chapter 10

Power Functions

Power functions are functions that multiply or divide a finite coefficient by a number of copies of the input according to the exponent:

- The counting number in the exponent is the number of copies of the input that are to be used.
- If the sign of the exponent is $+$, the coefficient is to be multiplied by the copies of the input,
- If the sign of the exponent is $-$, the coefficient is to be divided by the copies of the input.

The input-output rule of a power function has three features:

- The sign of the coefficient,
- The sign of the exponent,
- The parity of the exponent (the fact that it is even or odd.)

For instance, the function $BLIP$ whose input-output rule is

$$x \xrightarrow{BLIP} BLIP(x) = (-7)x^{+3} = (-7) \cdot x \cdot x \cdot x$$

is a power function whose input-output rule has the following features:

- The coefficient is negative,
- The exponent is positive,
- The exponent is odd.

and the function $FLOP$ whose input-output rule is

$$x \xrightarrow{FLOP} FLOP(x) = (+6)x^{-4}$$

is a power function whose input-output rule has the following features.
The reasons we start with the investigation of power function are that:

i. Power function are the simplest possible functions,

ii. Power function are the simplest embodiment of fundamental features,

iii. Power function are the building blocks in terms of which we will de-construct other functions.

10.1 Size of Outputs

Given a power function and given the size of an input, we want to find out if we can determine the size of the output in terms of just the features of the input-output rule.

Note. We shall represent large numbers by large and small numbers by small but it will be most important to be completely specific and not to confuse the size of the input and the size of the output.

We begin with an example. Given the function $KIT$ whose input-output rule is

$$x \xrightarrow{KIT} KIT(x) = (-8)x^5$$

we want to find out the size of the outputs for a large input that we will represent by large.

Since the coefficient $-8$ is “in the ones” and since we are multiplying the coefficient by copies of inputs that are large, the actual coefficient does not matter so we normalize the coefficient $-8$ to $-1$, that is we represent its size by 1.

Altogether, we have

$$large \xrightarrow{KIT} KIT(large) = (1)(large)^5 = (1) \cdot (large) \cdot (large) \cdot (large) \cdot (large) \cdot (large)$$

and, since 5 copies of large multiply to large,

$$= (1) \cdot (large) = large$$
10.1. SIZE OF OUTPUTS

Next, proceeding as in the example, we shall consider the following question:

**QUESTION 1.** Which features of the input-output rule control the size of the output in terms of the size of the input?

1. Given a function $POSEXP$ whose input-output rule is of the form
   \[ x \xrightarrow{POSEXP} POSEXP(x) = (-7)x + \text{number} \]
   we want to find out the size of the outputs (since $-7$ is “in the ones”, we will represent its size by 1).
   - When the inputs are *large*, we have
     \[
     \text{large} \xrightarrow{POSEXP} POSEXP(\text{large}) = (1)(\text{large})^{+\text{number}} = (1) \cdot (\text{large}) \cdot (\text{large}) \cdot \ldots \cdot (\text{large})
     \]
     and, since any number of copies of *large* multiply to *large*,
     \[
     = (1) \cdot (\text{large}) = \text{large}
     \]
     We see that the size of the output is *large*, that is the *same as* the size of the *input*.
   - When the inputs are *small* we have
     \[
     \text{small} \xrightarrow{POSEXP} POSEXP(\text{small}) = (1)(\text{small})^{+\text{number}} = (1) \cdot (\text{small}) \cdot (\text{small}) \cdot \ldots \cdot (\text{small})
     \]
     and, since any number of copies of *small* multiply to *small*,
     \[
     = (1) \cdot (\text{small}) = \text{small}
     \]
     We see that the size of the output is *small*, that is the *same as* the size of the *input*. 

So, when the exponent is positive, the size of the output is the same as the size of the input.

2. Given a function \( \text{NEGEXP} \) whose input-output rule is of the form
\[
x \xrightarrow{\text{NEGEXP}} \text{NEGEXP}(x) = (-7)x^{-\text{number}}
\]
we want to find out the size of the outputs.

- When the inputs are large, we have
\[
\text{large} \xrightarrow{\text{NEGEXP}} \text{NEGEXP}(\text{large}) = (1)(\text{large})^{-\text{number}} = \frac{1}{\text{(large) \cdot (large) \cdot \ldots \cdot (large)}} \quad \text{number copies of large}
\]
and, since any number of copies of large multiply to large,
\[
= \frac{1}{\text{large}} = \text{small}
\]
We see that the size of the output is small, that is the opposite from the size of the input.

- When the inputs are small, we have
\[
\text{small} \xrightarrow{\text{NEGEXP}} \text{NEGEXP}(\text{small}) = (1)(\text{large})^{-\text{number}} = \frac{1}{\text{(small) \cdot (small) \cdot \ldots \cdot (small)}} \quad \text{number copies of small}
\]
and, since any number of copies of small multiply to small,
\[
= \frac{1}{\text{small}} = \text{large}
\]
So, when the exponent is negative, the size of the output is the reciprocal of the size of the input.

3. Altogether then, the answer to QUESTION 1 is given by

THEOREM 1. The size of the outputs in terms of the size of the inputs depends only on the sign of the exponent:
10.2. **SIGN OF OUTPUTS**

- When the *exponent* is *positive*, the size of the output is the *same* as the size of the input.
- When the *exponent* is *negative*, the size of the output is the *reciprocal* of the size of the input.

## 10.2 Sign of Outputs

Given a power function and given the sign of an input, we want to find out if we can determine the *sign* of the output in terms of just the *features* of the input-output rule.

*Note.* We shall represent *positive* numbers by + and *negative* numbers by − but it will be most important to be completely specific and not to confuse the sign of the *input* and the sign of the *output*.

We begin with an example. Given the function KIT whose input-output rule is

\[ x \xrightarrow{KIT} KIT(x) = (-8)x^5 \]

we want to find out the sign of the outputs for a positive input that we will represent by +. We have

\[ + \xrightarrow{KIT} KIT(+) = (-8)(+)^5 \]

\[ = (-) \cdot (+) \cdot (+) \cdot (+) \cdot (+) \cdot (+) \]

and, since 5 copies of + multiply to +,

\[ = (-) \cdot (+) \]

\[ = - \]

Next, proceeding as in the example, we shall investigate separately
- the sign of the outputs for *positive* inputs,
- the sign of the outputs for *negative* inputs.

### Sign of the outputs for positive inputs

We consider here the following question:

**QUESTION 2.** Which features of the input-output rule control the sign of the output for a *positive* input.
1. We start with the case where the exponent is positive and we investigate separately the sub-case where the exponent is even and the sub-case where the exponent is odd.

a. Given a function \( \text{POSEVEXP} \) whose input-output rule is of the form
\[
x \xrightarrow{\text{POSEVEXP}} \text{POSEVEXP}(x) = (-7)x^{+\text{even}}
\]
we want to find out the sign of the outputs for positive inputs. We have
\[
+ \xrightarrow{\text{POSEVEXP}} \text{POSEVEXP}(+) = (-)(+)^{+\text{even}}
= (-) \cdot (+) \cdot (+) \cdot \cdots \cdot (+)
\text{even number of +}
= (-) \cdot (+)
= -
\]
We see that, since an even number of + multiply to +, the sign of the output is equal to the sign of the coefficient multiplied by + and so the sign of the output is the same as the sign of the coefficient.

b. Given a function \( \text{POSODEXP} \) whose input-output rule is of the form
\[
x \xrightarrow{\text{POSODEXP}} \text{POSODEXP}(x) = (-7)x^{+\text{odd}}
\]
we want to find out the sign of the outputs for positive inputs. We have
\[
+ \xrightarrow{\text{POSODEXP}} \text{POSODEXP}(+) = (-)(+)^{+\text{odd}}
= (-) \cdot (+) \cdot (+) \cdot \cdots \cdot (+)
\text{odd number of +}
= (-) \cdot (+)
= -
\]
We see that, since an odd number of + multiply to +, the sign of the output is equal to the sign of the coefficient multiplied by + and so the sign of the output is the same as the sign of the coefficient.

So, we see that when the exponent is positive the parity of the exponent does not matter and we have that, for positive inputs, the sign of the output is the same as the sign of the coefficient.

2. We now continue with the case where the exponent is negative and we investigate again separately the sub-case where the exponent is even and the sub-case where the exponent is odd.
10.2. SIGN OF OUTPUTS

a. Given a function $NEGEVEX$ whose input-output rule is of the form
\[ x \xrightarrow{NEGEVEXP} NEGEVEXP(x) = (+4)x^{-\text{even}} \]
we want to find out the sign of the outputs for positive inputs. We have
\[ + \xrightarrow{NEGEVEXP} NEGEVEXP(+) = (+)(+)^{+\text{even}} = (+) \cdot (+) \cdot \ldots \cdot (+) \]
\[ = (+) = + \]
We see that, since an even number of + multiply to +, the sign of the output is equal to the sign of the coefficient divided by + and so the sign of the output is the same as the sign of the coefficient.

b. Given a function $NEGODEXP$ whose input-output rule is of the form
\[ x \xrightarrow{NEGODEXP} NEGODEXP(x) = (-4)x^{-\text{odd}} \]
we want to find out the sign of the outputs for positive inputs. We have
\[ + \xrightarrow{NEGODEXP} NEGODEXP(+) = (-)(+)^{-\text{odd}} = (-) \cdot (+) \cdot \ldots \cdot (+) \]
\[ = (-) = - \]
We see that, since an odd number of + multiply to +, the sign of the output is equal to the sign of the coefficient divided by + and so is the same as the sign of the coefficient.

So, we see that when the exponent is negative the parity of the exponent does not matter and we have that, for positive inputs, the sign of the output is the same as the sign of the coefficient.

3. Altogether, we see that neither the sign nor the parity of the exponent matters and we have that, for positive inputs, only the sign of the coefficient
matters since the sign of the output is always the same as the sign of the coefficient.

**THEOREM 2.** For positive inputs, the sign of the output of a power function is the same as the sign of the coefficient regardless of everything else.

**Sign of the outputs for negative inputs**

We consider here the following question:

**QUESTION 3.** Which features of the input-output rule control the sign of the output for a negative input.

1. We start with the case where the exponent is positive and we investigate separately the sub-case where the exponent is even and the sub-case where the exponent is odd.

   a. Given a function \( \text{POSEVEXP} \) whose input-output rule is of the form

\[
x \xrightarrow{\text{POSEVEXP}} \text{POSEVEXP}(x) = (-7)x^{+\text{even}}
\]

we want to find out the sign of the outputs for negative inputs. We have

\[
- \xrightarrow{\text{POSEVEXP}} \text{POSEVEXP}(-) = (-)(-)^{+\text{even}} = (-) \cdot (-) \cdot (-) \cdot \ldots \cdot (-)
\]

\[
= (-) \cdot (+) = -
\]

We see that, since an even number of \(-\) multiply to \(+\), the sign of the output is equal to the sign of the coefficient multiplied by \(+\) and so the sign of the output is the same as the sign of the coefficient.

b. Given a function \( \text{POSODEXP} \) whose input-output rule is of the form

\[
x \xrightarrow{\text{POSODEXP}} \text{POSODEXP}(x) = (-7)x^{+\text{odd}}
\]
we want to find out the sign of the outputs for negative inputs. We have
\[ + \xrightarrow{\text{POSODEXP}} \text{POSODEXP}(+) = (-)(-)^{\text{odd}} \]
\[ = (-) \cdot (-) \cdot (-) \cdot \ldots \cdot (-) \]
\[ \quad \text{odd number of } - \]
\[ = (-) \cdot (-) \]
\[ = + \]

We see that, since an odd number of \(-\) multiply to \(-\), the sign of the output is equal to the sign of the coefficient multiplied by \(-\) and so the sign of the output is the opposite of the sign of the coefficient.

So, we see that when the exponent is positive the parity of the exponent does matter and we have that, for negative inputs,
- when the exponent is even, the sign of the output is the same as the sign of the coefficient.
- when the exponent is odd, the sign of the output is the opposite of the sign of the coefficient.

2. We now continue with the case where the exponent is negative and we investigate again separately the sub-case where the exponent is even and the sub-case where the exponent is odd.

\textbf{a.} Given a function \textit{NEGEVEXP} whose input-output rule is of the form
\[ x \xrightarrow{\text{NEGEVEXP}} \text{NEGEVEXP}(x) = (+4)x^{-\text{even}} \]
we want to find out the sign of the outputs for negative inputs. We have
\[ + \xrightarrow{\text{NEGEVEXP}} \text{NEGEVEXP}(+) = (+)(-)^{\text{even}} \]
\[ = \frac{(+)}{(-) \cdot (-) \cdot \ldots \cdot (-)} \]
\[ \quad \text{even number of } - \]
\[ = \frac{(+)}{(+)} \]
\[ = + \]

We see that, since an even number of \(-\) multiply to \(+\), the sign of the output is equal to the sign of the coefficient divided by \(+\) and so the sign of the output is the same as the sign of the coefficient.

\textbf{b.} Given a function \textit{NEGODEXP} whose input-output rule is of the form
\[ x \xrightarrow{\text{NEGODEXP}} \text{NEGODEXP}(x) = (+4)x^{-\text{odd}} \]
we want to find out the sign of the outputs for negative inputs. We have
\[ \text{NEGODEXP}(-) = (+)(-)^{+\text{odd}} \]
\[ = (+)
\[ = (-) \cdot (-) \cdot \ldots \cdot (-) \text{ odd number of } - \]
\[ = (+) \]
\[ = (-) \]
\[ = - \]

We see that, since an odd number of $-s$ multiply to $-$, the sign of the output is equal to the sign of the coefficient divided by $-$ and so the sign of the output is the opposite of the sign of the coefficient.

So, we see that when the exponent is negative the parity of the exponent does matter and we have that, for negative inputs,
- when the exponent is even, the sign of the output is the same as the sign of the coefficient.
- when the exponent is odd, the sign of the output is the opposite of the sign of the coefficient.

3. Altogether then, for negative inputs, it is the parity of the exponent that controls the sign of the outputs and we have:

**THEOREM 3.** For negative inputs, the sign of the output of a power function relative to the coefficient depends on the parity of the exponent:
- When the exponent is even, the sign of the output is the same as the sign of the coefficient,
- When the exponent is odd, the sign of the output is the opposite of the sign of the coefficient.

### 10.3 Qualitative graphs

The information we are thus able to get at this point from the input-output rule of a power function is already enough to translate into what we shall call a **qualitative graph**, by which, for the time being, we shall mean the simplest, most regular, most natural graph that is consistent with the above type of information. After we shall have described another two features of a function, we shall make the concept somewhat more precise.

We look at two examples.
Given, for instance, a function \( \text{POSEVEXP} \) whose input-output rule is \( x \xrightarrow{\text{POSEVEXP}} \text{POSEVEXP}(x) = (-7)x^{+\text{even}} \)

\[ \begin{aligned}
1. \text{ We begin by looking for the local areas, that is the areas in which the graph is in } & \text{when the inputs are small and when the inputs are large. For that purpose, we shall need the size and the sign of the outputs and, while we shall essentially proceed as in the previous sections, we shall now do so in a more compact manner:} \\
\text{• When the inputs are } +\text{large}, \text{ we have} \\
+\text{large} \xrightarrow{\text{POSEVEXP}} \text{POSEVEXP}(+\text{large}) = (-1)(+\text{large})^{+\text{even}} \quad = (-1) \cdot (+\text{large}) \cdot (+\text{large}) \\
\text{even number of (+large)} \\
\text{= (} -1 \text{) } \cdot (+\text{large}) \\
\text{= } -\text{large}
\end{aligned} \]

From the graphic viewpoint, this means that the part of the graph for + large inputs is in the following local area:

- When the inputs are - large, we have

\[ \begin{aligned}
-\text{large} \xrightarrow{\text{POSEVEXP}} \text{POSEVEXP}(-\text{large}) = (-1)(-\text{large})^{+\text{even}} \\
\quad = (-1) \cdot (-\text{large}) \cdot (-\text{large}) \cdot (-\text{large}) \\
\text{even number of (-large)} \\
\quad = (-1) \cdot (+\text{large}) \\
\text{= } -\text{large}
\end{aligned} \]
From the graphic viewpoint, this means that the part of the graph for 
− large inputs is in the following local area:

- When the inputs are + small, we have

\[ \begin{align*}
+ \text{small} & \overset{POSEVEXP}{\longrightarrow} POSEVEXP(+ \text{small}) = (-1)(+ \text{small})^{\text{even}} \\
& = (-1) \cdot (+ \text{small}) \cdots (+ \text{small}) \\
& = (-1) \cdot (+ \text{small}) \\
& = - \text{small}
\end{align*} \]

From the graphic viewpoint, this means that the part of the graph for 
− small inputs is in the following local area:
• When the inputs are \(- small\), we have

\[
- small \xrightarrow{POSEVEXP} POSEVEXP(- small) = (-1)(- small)^{+even} \\
= (-1) \cdot (- small) \cdot \cdots \cdot (- small) \\
= (-1) \cdot (+ small) \\
= - small
\]

From the graphic viewpoint, this means that the part of the graph for \(- small\) inputs is in the following local area:
b. Altogether, then, the graph of the function \( \text{POSEVEXP} \) will be going through the following four local areas:

```
 as follows
```
c. Note that while the following graph, for instance,

might appear simpler, it really isn’t since straight lines and sharp angles are not something one often finds in nature. More about this later. The following graph too would not be very likely
because it gives the wrong impression about the outputs for sufficiently large inputs in that, for instance, the graph could be thought to continue as follows

which of course cannot be the case since \( \text{POSEVEXP} \) gives negative outputs for all large inputs.

Or, the graph might oscillate as in
but, up front, this graph looks awfully complicated for a function with as simple an input-output rule as \( POSEVEXP \).

2. Given, as another example, a function \( NEGODEXP \) whose input-output rule is

\[
x \xrightarrow{NEGODEXP} NEGODEXP(x) = (-4)x^{-\text{odd}}
\]

We want to find out the local areas when the inputs are small and when the inputs are large.

a. While we shall essentially proceed as before, we shall now do so in an even more compact manner:

- We start with positive inputs.
  - Since the sign of the outputs for positive inputs is always the same as the sign of the coefficient, the sign for positive inputs is \(-\).
  - Since the exponent is negative, the coefficient will be divided by the copies of the inputs and the outputs for + large inputs will be - small the outputs for + small inputs will be - large

From the graphic viewpoint, this means that the local areas for + large and + small inputs are
Looking at negative inputs, since the exponent is odd, the sign of the outputs for negative inputs will be the opposite of the sign of the coefficient and will thus be +.

From the graphic viewpoint, this means that the local areas for − large and − small inputs will be the opposite of those for + large and + small inputs.

b. Altogether, then, the graph of the function \( \text{NEGODEXP} \) will be going through the following four local areas:
c. Here again, for all we really know, the graph could be, for instance, wavering as in
but again, this graph looks awfully complicated for a function with as simple an input-output rule as \( \text{NEGODEXP} \).

In any case, we shall discuss and settle this kind of issue after we have described a couple of features that a function can have.

### 10.4 Slope

We now use

\[
x \xrightarrow{\text{POSEVEXP}} \text{POSEVEXP}(x) = (-7)x^{+\text{even}}
\]

to describe a feature which is not readily apparent from the input-output rule but is quite clear on its qualitative graph

1. From the graphic point of view, we use the qualitative graph
a. Take two positive inputs such as in the picture below, call them \textit{input} \#1 and \textit{input} \#2, and get their outputs, \textit{output} \#1 and \textit{output} \#2:

We observe that
- the \textbf{input change} (also known as \textbf{run}) from \textit{input} \#1 to \textit{input} \#2 is \textbf{positive}
- the \textbf{output change} (also known as \textbf{rise}) from \textit{output} \#1 to \textit{output} \#2 is \textbf{negative}

that is, the sign of the \textit{output change} is the \textbf{opposite} from the sign of the \textit{input change}. We shall say that the function $POSEVEXP$ is \textbf{sloping down} from \textit{input} \#1 to \textit{input} \#2.
b. Similarly, take two negative inputs such as in the picture below, call them input #3 and input #4, and get their outputs, output #3 and output #4:

\[
\begin{align*}
\text{input #1} = +3 & \quad \text{JANE}(+3) = (-7)(+3)^4 \\
& = (-7) \cdot (+3) \cdot (+3) \cdot (+3) \\
& = -567 = \text{output #1}
\end{align*}
\]

and

\[
\begin{align*}
\text{input #2} = +5 & \quad \text{JANE}(+5) = (-7)(+5)^4 \\
& = (-7) \cdot (+5) \cdot (+5) \cdot (+5) \\
& = -4375 = \text{output #2}
\end{align*}
\]
10.4. SLOPE

ii. We compute the changes.

\[
\text{input change} = \text{input \#2} - \text{input \#1} \\
= (+5) - (+3) \\
= (+5) + (-3) \\
= +2
\]

and

\[
\text{output change} = \text{output \#2} - \text{output \#1} \\
= (-4375) - (-567) \\
= (-4375) + (+567) \\
= -3808
\]

iii. We compute the slope

\[
slope = \frac{\text{output change}}{\text{input change}} \\
= \frac{-3808}{+2} \\
= -1904
\]

b. We pick input \#3 = -2 and input \#4 = -5,

i. We compute the outputs:

\[
\text{input \#3} = -2 \xrightarrow{JANE} JANE(-2) = (-7)(-2)^4 \\
= (-7) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) \\
= -112 = \text{output \#3}
\]

and

\[
\text{input \#4} = -6 \xrightarrow{JANE} JANE(-6) = (-7)(-5)^4 \\
= (-7) \cdot (-6) \cdot (-6) \cdot (-6) \cdot (-6) \\
= -9072 = \text{output \#4}
\]

ii. We compute the changes.

\[
\text{input change} = \text{input \#2} - \text{input \#1} \\
= (-6) - (-2) \\
= (-6) + (+2) \\
= -4
\]
and

\[\text{output change} = \text{output } \#2 - \text{output } \#1\]
\[= (-9072) - (-112)\]
\[= (-9072) + (+112)\]
\[= -8960\]

iii. We compute the slope

\[\text{slope} = \frac{\text{output change}}{\text{input change}}\]
\[= \frac{-8960}{-4}\]
\[= +2240\]

3. Comparing what we got from the qualitative graph with what we got from the input-output rule, we conclude that
- the graph is sloping up when the slope is positive
- the graph is sloping down when the slope is negative

10.5 Concavity

We now use

\[x \xrightarrow{\text{NEGODEXP}} \text{NEGODEXP}(x) = (-4)x^{+\text{odd}}\]

to describe another feature of a function which is also not readily apparent from the input-output rule but is quite clear on its qualitative graph.

1. From the graphic point of view, we use its qualitative graph
a. Take two positive inputs, call them input #1 and input #2, and get their graph-points, graph – point #1 and graph – point #2, as in the following picture.

We observe that the arc can be seen as part of the “cap” but cannot be seen as part of the “cup”.

When the arc can be seen as part of a “cap”, we shall say that the function is concave down.

So, the function NEGODEXP is concave down for all positive inputs.

b. Similarly, take two negative inputs, call them input #3 and input #4, and get their graph-points, graph – points #3 and graph – points #4, as in the following picture:
We observe that the arc can be seen as part of the “cup” but cannot be seen as part of the “cap”.

When the arc can be seen as part of a “cup”, we shall say that the function \( NEGODEXP \) is concave up. So, the function \( NEGODEXP \) is concave up for all negative inputs.

2. From the input-output rule point of view, we must take a specific instance of \( NEGODEXP \) in order to compute the outputs and their changes and we take

\[
x \xrightarrow{JILL} JILL(x) = (-4)x^{-3}
\]

a. We pick input \#1 = +3 and input \#2 = +5,

i. We compute the outputs:

\[
input \#1 = +3 \xrightarrow{JILL} JILL(+3) = (-4)(+3)^{-3} = \frac{-4}{(+3) \cdot (+3) \cdot (+3)} = \frac{-4}{+27} = -0.148 + (\ldots) = output \#1
\]
and

\[
\text{input } \#2 = +5 \xrightarrow{JILL} \text{JILL}(+5) = (-4)(+5)^{-3} \\
= -\frac{4}{5^3} \\
= -\frac{4}{125} \\
= -0.032 = \text{output } \#2
\]

ii. We compute the changes:

\[
\text{input change} = \text{input } \#2 - \text{input } \#1 \\
= (+5) - (+3) \\
= (+5) + (-3) \\
= +2
\]

and

\[
\text{output change} = \text{output } \#2 - \text{output } \#1 \\
= (-0.032) - (-0.148 + (\ldots)) \\
= (-0.032) + (+0.148 + (\ldots)) \\
= +0.116 + (\ldots)
\]

b. Now, we take again input \#2 = +5 and we pick input \#3 = +7

i. We compute output \#3

\[
\text{input } \#3 = +7 \xrightarrow{JILL} \text{JILL}(+7) = (-4)(+7)^{-3} \\
= -\frac{4}{7^3} \\
= -\frac{4}{343} \\
= -0.012 + (\ldots) = \text{output } \#3
\]

ii. We compute the changes.

\[
\text{input change} = \text{input } \#3 - \text{input } \#2 \\
= (+7) - (+5) \\
= (+7) + (-5) \\
= +2
\]
and

\[
output \ change = output \ #3 - output \ #2 \\
= (-0.012 + (\ldots)) - (-0.032) \\
= (-0.012) + (+0.032) \\
= +0.020 + (\ldots)
\]

3. Here we will use the qualitative graph to keep track of things,

\begin{itemize}
  \item[a.] We observe that, even though
    the second input change (from input +5 to input +7)
    is the same as
    the first input change (from input +3 to input +5)
    the corresponding output changes are not the same:
    the second output change (from input +5 to input +7), equal to +0.020,
    is algebra-smaller than
    the first output change (from input +3 to input +5), equal to +0.116 + (\ldots).

This fits the fact that the graph is concave down for all positive inputs.

  \item[b.] Since the second input change (from input +5 to input +7) is the same as the first input change (from input +3 to input +5), the fact that the second output change (from input +5 to input +7) is algebra-smaller than the first output change (from input +3 to input +5) means that the second slope (from the graph-point for input +5 to the graph-point for input +7) is algebra-smaller than the first slope (from the graph-point for input +3 to the graph-point for input +5).
\end{itemize}
So, the fact that the graph is concave down corresponds to the fact that the slope gets algebra-smaller when the inputs get algebra-larger.

10.6 Types of Power Functions

This is mostly a systematic putting-together of the information we have gathered so far about power functions.

1. There are two main types of power functions:
   1. The power functions whose exponent is positive
   2. The power functions whose exponent is negative

   a. When the exponent of a power function is positive, so that the coefficient is multiplied by the copies of the input, then regardless of everything else,
      1. the smaller the inputs are in size, then even smaller the outputs are in size, and the local areas are horizontal rectangles
      2. the larger the inputs are in size, then even larger the outputs are in size, and the local areas are vertical rectangles
CHAPTER 10. POWER FUNCTIONS

From a graphic viewpoint, this means that with *positive* exponent power functions,

- when the inputs are near 0, then the slope of the graph is nearly 0
- when the input are near ∞, then the slope of the graph is nearly ∞

**b.** When the *exponent* of a power function is *negative*, so that the coefficient is divided by the copies of the input, then regardless of everything else,

- the smaller the inputs are in size, then even the larger the outputs are in size, and the local areas are *vertical* rectangles
- the larger the inputs are in size, then even the smaller the outputs are in size, and the local areas are *horizontal* rectangles

From a graphic viewpoint, this means that with *negative* exponent power functions,

- when the inputs are near 0, then the slope of the graph is nearly ∞
- when the input are near ∞, then the slope of the graph is nearly 0
c. There are a number of very nice “symmetries” worth noting:

- The behaviors relative to size near 0 and near $\infty$ are **reciprocal**, that is the size of the outputs when the inputs are near 0 is the reciprocal of the size of the outputs when the inputs are near $\infty$: when either ones are *small*, the others are *large*.
- When the *sign* of the exponent changes, then, relative to size, the behavior near 0 and the behavior near $\infty$ are exchanged regardless of everything else.
- All power functions whose coefficient is +1 give output +1 for the input +1. For the input −1, the output is +1 or −1 depending on whether the parity of the exponent is *even* or *odd*.
- All power functions whose coefficient is −1 give output −1 for the input +1. For the input −1, the output is −1 or +1 depending on whether the parity of the exponent is *even* or *odd*. 
2. We look systematically at the local areas for each type of power functions.

1. • When the inputs are + large, we have
   • When the inputs are − large, we have
   • When the inputs are + small, we have
   • When the inputs are − small, we have
2. • When the inputs are + large, we have
   • When the inputs are − large, we have
   • When the inputs are + small, we have
   • When the inputs are − small, we have
3. • When the inputs are + large, we have
   • When the inputs are − large, we have
   • When the inputs are + small, we have
   • When the inputs are − small, we have
4. • When the inputs are + large, we have
   • When the inputs are − large, we have
   • When the inputs are + small, we have
   • When the inputs are − small, we have

Most of the time, we shall not concern ourselves with specific inputs but only with inputs having certain features.

For instance, we take the power function whose input-output rule is as an example and we will investigate the SIZE and SIGN of the outputs that returns when we use inputs of a given SIGN and SIZE. We could of course look at SIGN and SIZE separately,

1. We use small positive inputs: + small (−1)(+small) −3 = = −large
   where the small rectangle is where the correspondent plot-points will have to be.
2. We use small negative inputs: − small (−1)(− small) −3 = = + large

Figure 10.1: +∞ is left of ∞ and −∞ is right of ∞.
3. Altogether, this gives us the local box when $x$ is near 0 which is made up of the two separate pieces shaded outside the window. The graph when $x$ is near 0 (when $x$ is small) will be in this local box. We code this information as follows:
- SIGN height near 0 = (+, −)
- SIGN height near 0 = (+, −)

Note. The code left of the comma refers to the feature of the output when the input is left of the center-point which, here, is 0 and the code right of the comma refers to the feature of the output when the input is right of the center-point.

4. We take large inputs: $+\text{ large } (-1)(+\text{ large}) -3 = -\text{ small } -\text{ large } (-1)(-\text{ large}) 3 = +\text{ small}$

The graphic representation is the local box when $x$ is near made up of the two pieces shaded outside the window:

The graph when $x$ is near (when $x$ is large) will be in this local box. We code this information as follows:
- SIGN height near $\pm \infty$ = (−, +)
- SIZE height near $\pm \infty$ = (small, small)

Note. The code left of the comma refers to the feature of the output when the input is left of the center-point which, here, is and the code right of the comma refers to the feature of the output when the input is right of the center-point. Be careful though: even though large positive inputs (+ large) are right of 0, this is irrelevant because 0 is the center-point for small inputs. The center-point for large inputs is and large positive inputs are left of .

5. Altogether, the plot points for small inputs and for large inputs will be in the two local boxes:

### 10.7 Sign of Slope

- SIGN slope near 0 = ( , ) (The center point is 0 and $0_+$ is the right half of the neighborhood of 0 and $0_-$ is the left half of the neighborhood of 0.)
- SIGN slope near $\infty$.) = ( , ) (The center point is $\infty$ so that $+\infty$.) is left of $\infty$.) and $\infty$.) is right of $\infty$.)

### 10.8 Sign of Concavity
10.9 Local Features

Figure 10.2: Finite Part of Graph.
Figure 10.3: Graph near infinity.
Figure 10.4: Local Concavity-sign.
Figure 10.5: Local Features.
Figure 10.6: Local Height-sign.
Figure 10.7: Local Slope-sign.
10.10 From Local to Global

From just the local graph near either 0 or $\infty$, we can get the qualitative global graph. From the local graph from one side of either 0 or $\infty$, we can get the qualitative global graph if we know the parity.

10.11 Global Graphs

10.12 From Rule to Graph

From the features of the Input-Output rule to the features of the (qualitative global) graph.

1. a. We compute the outputs, keeping in mind that when we multiply an even number of copies of a negative number the result is a negative number. We take small positive inputs: $+ \text{small} (1)(+\text{small}) +\text{even} = (1) = (1) (+ \text{small}) = \text{small}$ We take small negative inputs: $\text{small} (1)(-\text{small}) +\text{even} = (1) = (1) (+ \text{small}) = \text{small}$ We take large positive inputs: $+ \text{large} (1)(+\text{large}) +\text{even} = (1) = (1) (+ \text{large}) = \text{large}$ We take large negative inputs: $\text{large} (1)(-\text{large}) +\text{even} = (1) = (1) (+ \text{large}) = \text{large}$

b. In graphic terms, this gives us the following local boxes Figure 1.

c. We code the results as follows SIGN height near 0 = ( , ) SIZE height near 0 = (small , small) SIGN height near $\infty$ = ( , ) SIZE height near $\infty$ = (large , large)

2. a. However, looking closer at the above computations, we see that the outputs are not just small (or large), but they are smaller (or larger) than the inputs that they come from.

b. In graphic terms, the local boxes are not squares but rectangles in a manner that corresponds to the relative sizes of the inputs and outputs as in Figure 2.

3. a. Now we don’t just look at one (undisclosed) input at a time but at inputs that change and see how the corresponding outputs change.

b. Graphically this translates to the following run arrows, rise arrows and slope segments:

c. We code this as follows SIGN slope near 0 = ( , ) SIGN slope near $\infty$ = ( , ) (Remember that the center point is .) 0+ is right of 0
10.13. FROM GRAPH TO RULE

a. With small/large inputs, the size of the outputs determines the sign of the power. b. With positive inputs, the sign of the outputs determines the sign of the coefficient. c. With negative inputs, the sign of the outputs determines the parity of the power.

10.13 From Graph to Rule

From the features of the (qualitative global) graph to the features of the Input-Output rule. Conversely, we ask what feature of the graph entails what feature of the input-output rule. For instance, to say of a power function \( f \) that small large means that \( \text{SIGN power} = \) and to say that + means that \( \text{SIGN coefficient} = \) and then to say that means that \( \text{PARITY power} = \) even. Information about slope or concavity is a bit more difficult to utilize as, in practice, it requires that we already know the local graphs of the power functions and that we consider the two possibilities for \( \text{SIGN power} \) and check if they yield the same conclusion. Eventually however, we observe that, very fortunately, THE LOCAL GRAPHS OF A POWER FUNCTION BOTH NEAR 0 AND NEAR ARE DETERMINED BY JUST ITS LOCAL BOX NEAR EITHER 0 OR and that, therefore, so are its type of input-output rule and its global qualitative graph:

10.14 Comparing degrees

The higher the degree, the boxier the graph.

10.15 Comparing types

10.16 Exceptional Power Functions
Chapter 11

Polynomial Functions

11.1 Introduction

Starting with this chapter and until the end of PART TWO: FUNCTIONS SPECIFIED DIRECTLY, for each type of functions, our goal will be:

- To find general statements about the qualitative features of this type of functions, given by an input-output rule.
- To find a global qualitative graph for this type of functions (to serve as a basis for interpolating-extrapolating plots into actual global quantitative graphs):

  - To find the conditions that a plot of a function of this type must meet in order to be interpolated-extrapolated into a global quantitative graph.

  After we have dealt with constant functions, our approach will be based on the fact that, while, no matter how many plot points we have, they can never be interpolated-extrapolated into a global quantitative graph, local graphs can be, at least to a “certain” extent. In particular, in the presence of “certain” information, local graphs near “certain” input points can be interpolated-extrapolated a long way. Thus, for each given kind of functions, we shall proceed as follows:

  - We shall find out how to get the local input-output rule near a given input point,
  - We shall see how to construct from the local input-output rule the local graph near a given input point,
We shall determine near which “certain” input points we need to find a local graph, how far these local graphs can be interpolated-extrapolated and what information is needed to decide that.

We shall see how to join smoothly the local graphs near these “certain” points into an essential global qualitative graph.

In other words, other than with the constant functions, our general strategy will be the long way in

11.2 Operations On Functions

Before we can start on this program, though, we need to develop a couple of tools.

Indeed, since, earlier on, we defined polynomials as combinations of powers, and since polynomial functions will be functions whose output is a polynomial, it should not be surprising that we first need operations on functions to construct new functions out of given functions so as to look upon polynomial functions as constructed from power functions, that is as linear combinations of power functions.

How we look at these operations will depend on whether we look at them from the input-output rule viewpoint or from the graphic viewpoint.

To keep things simple from the graphic point of view, we will be looking here at functions specified by plots rather than by graphs.

11.2.1 Dilations Of Functions

The dilation of a given function $f$ with a given dilation factor $a$, where $a$ is a signed number, is a new function whose name is $[a \circ f]$ and whose outputs are the outputs of the given function $f$ multiplied by the given number $a$. 
11.2. OPERATIONS

1. For instance, say some function called $QG_{34}$ is given by the table

<table>
<thead>
<tr>
<th>$x$</th>
<th>-6</th>
<th>-5</th>
<th>-3</th>
<th>-2</th>
<th>0</th>
<th>+1</th>
<th>+2</th>
<th>+3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$QG_{34}(x)$</td>
<td>-4</td>
<td>+4</td>
<td>+5</td>
<td>+5</td>
<td>-1</td>
<td>0</td>
<td>-5</td>
<td>+5</td>
</tr>
</tbody>
</table>

and that we want to look at $[-3 \odot QG_{34}]$, the dilation of the function $QG_{34}$ by a factor of $-3$.

Then, for instance, given the input $-2$, we get its output under the dilated function $[-3 \odot QG_{34}]$ as follows

$$x \overset{[-3 \odot QG_{34}]}{\longrightarrow} [-3 \odot QG_{34}] (x) = (-3) \cdot QG_{34}(-2)$$

$$= (-3) \cdot (+5)$$

$$= -15$$

In that manner, we can obtain the table for the dilated function $[-3 \odot QG_{34}]$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>-6</th>
<th>-5</th>
<th>-3</th>
<th>-2</th>
<th>0</th>
<th>+1</th>
<th>+2</th>
<th>+3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-3 \odot QG_{34}] (x)$</td>
<td>+12</td>
<td>-12</td>
<td>-15</td>
<td>-15</td>
<td>+3</td>
<td>0</td>
<td>+15</td>
<td>-15</td>
</tr>
</tbody>
</table>

2. From the graphic viewpoint and while, when we computed the outputs, the “rule of signs for multiplication” took automatically care of negative dilation factors as well as positive dilation factors, here we will have to look at the two cases separately.

a. We thus begin by explaining what we mean by a plain stretching. For example, given some function called $ZW_{009}$ whose plot is

![Plot of Lavinia](image)

say we want the plot of $[(+3) \odot ZW_{009}]$ or $[(-3) \odot ZW_{009}]$.

Imagine the plot of $ZW_{009}$ printed of a sheet of stretchable, transparent rubber lying on top of the window but attached to the 0-output level line with a copy of the plot printed on the window itself with white circles just underneath the black circles so that they are not visible.
positive vertical stretching. Now grab the rubber sheet on its top and bottom outer edges and stretch it vertically with a stretch factor equal to $\text{SIZE}(+3) = \text{SIZE}(-3) = 3$, that is equal to the size of the dilation factor:

The black circles show the plot of $[(3) \odot ZW_{009}]$ which is the plain vertical stretching of $ZW_{009}$ with the coefficient 3.

We can now see (at least some of) the white circles of the copy of the plot of $ZW_{009}$ and the white circles that we do not see are the plot points for the inputs $-4$ and $+2$ because the output $ZW_{009}(-4) = -4 \cdot 0 = 0$ and the output $ZW_{009}(+2) = +2 \cdot 0 = 0$ and since the rubber sheet is attached to the 0 output level line, the plot points for $[(3) \odot ZW_{009}](-4)$ and $[(3) \odot ZW_{009}](+2)$ stay right on top of them.

b. Since, when $a$ is positive, we identify $a$ with $\text{SIZE} a$, the positive vertical stretching $[(+3) \odot ZW_{009}]$ is the same as the plain vertical stretching $[(3) \odot ZW_{009}]$.

c. Since, when $a$ is negative, we identify $a$ with $-\text{SIZE} a$, we obtain the plot of $[(-3) \odot ZW_{009}]$ as follows
We start as before and do a vertical stretch with a stretch factor equal to \(\text{size}(-3) = 3\), that is again equal to the size of the dilation factor.

In which the white circles show the plot of \(ZW_{009}\) and the grey circles show the plot of \([(+3) \odot ZW_{009}]\).

But, because the stretch factor is negative, we need an extra step to get the plot of \([(-3) \odot ZW_{009}]\) from the plot of the plain stretch, \([3 \odot ZW_{009}]\), namely a vertical flip:

By a **negative stretching**, we shall thus mean a plain vertical stretching followed by a **vertical flip**.

**d.** Depending on whether the dilation factor is larger than 1 or smaller than 1, we shall speak of an **expansion** or a **contraction**.

For instance, given the plot of \(XQ_{111}\)
the plain stretching $[(2) \odot XQ_{111}]$ is an expansion

while the plain stretching $[(\frac{1}{2}) \odot XQ_{111}]$ is a contraction.

3. From the input-output rule viewpoint, if the output of a function $f$ is given by some given expression involving the input

$$x \xrightarrow{f} f(x) = [\text{some expression involving } x]$$
then the output of the function \([a \circ f]\), its dilation by a factor \(a\), is given by \(a\) multiplied by the same given expression involving the input \(x\):

\[ x \xrightarrow{[a \circ f]} [a \circ f](x) = a \cdot \text{same expression involving } x \]

Thus, for instance, the (-3) dilation of the power function whose input-output rule is

\[ x \xrightarrow{f} f(x) = (-1)x^{-5} \]

will be the function whose input-output rule is

\[ x \xrightarrow{[(-3) \circ f]} [(-3) \circ f](x) = (-3) \cdot [(-1)x^{-5}] \]

\[ = (+3)x^{-5} \]

4. In all of the above, we used the symbol \(\circ\) and the brackets \([\ ]\) for the sake of clarity but from now on, in the course of our investigations, we will almost conform to the general usage:

- We shall reuse the symbol \(\cdot\) instead of using the symbol \(\circ\),
- We shall use brackets only when writing outputs.

Thus, later on for example,

<table>
<thead>
<tr>
<th>we will write</th>
<th>rather than</th>
<th>or the usual</th>
<th>but write the outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3 \cdot f)</td>
<td>([3 \circ f])</td>
<td>(3f)</td>
<td>([3 \cdot f]) (-4)</td>
</tr>
<tr>
<td>(m \cdot L)</td>
<td>([m \circ L])</td>
<td>(mL)</td>
<td>([m \cdot L]) (+51)</td>
</tr>
<tr>
<td>((+2) \cdot A)</td>
<td>([(+2) \circ A])</td>
<td>(+2A)</td>
<td>([(+2) \cdot A]) (\frac{5}{2})</td>
</tr>
<tr>
<td>(a \cdot \text{COS})</td>
<td>([a \circ \text{COS}])</td>
<td>(a \text{COS})</td>
<td>([a \cdot \text{COS}]) ((-\frac{2\pi}{3}))</td>
</tr>
<tr>
<td>(\sqrt{5} \cdot \text{EXP})</td>
<td>([\sqrt{5} \circ \text{EXP}])</td>
<td>(\sqrt{5}\text{EXP})</td>
<td>([\sqrt{5} \cdot \text{EXP}]) ((x))</td>
</tr>
</tbody>
</table>

11.2.2 Addition Of Functions

The addition of two functions \(f\) and \(g\) is a new function whose name is \(f \oplus g\) and whose output is that of the addition of the two given functions.

Actually, here we will use the symbol \(\oplus\) for the sake of clarity but, later on, when we use addition of functions on actual functions, we will conform to the usual practice of reusing the symbol +.

Thus, later on,

<table>
<thead>
<tr>
<th>we will write (\text{COS} + \text{SIN})</th>
<th>rather than (\text{COS} \oplus \text{SIN})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{COS} + \text{SIN})</td>
<td>(\text{COS} \oplus \text{SIN})</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
constant function
constant term

Etc.
we will write \( \cos + \sin \) rather than .

11.2.3 Linear Combinations Of Functions

Thus, later on,

\[
\begin{array}{c|c}
\text{we will write} & \text{rather than} \\
3\cos - 5\sin & (+3) \odot \cos \oplus (-5) \odot \sin \\
? & ? \\
? & ?
\end{array}
\]

11.3 Constant Functions

Functions whose \textit{input-output rule} is of the form:

\[
x \xrightarrow{C_a} C_a(x) = ax^0
\]

where \( a \) stands for a given \textit{signed} number are called \textbf{constant functions}. \( C_a \) will be the name of the constant function \textit{specified} by the signed number \( a \). The given signed number \( a \) is called the \textbf{constant term} of \( C_a \).

For example, \( C_{-3} \) is the \textit{constant function} whose input-output rule is

\[
x \xrightarrow{C_{-3}} C_{-3}(x) = (-3)x^0
\]

that is the function whose \textit{output} is equal to \(-3\) multiplied by \textit{zero} copy of the \textit{input}.

Then, when given an input, say \(+4\), we write

\[
+4 \xrightarrow{C} C(x)|_{x=+4} = (-3)x^0|_{x=+4} = (-3)(+4)^0 = -3
\]

\textit{Note.} However, unless there is need for the above complete way of writing things, as there will be when, for instance, we have to deal with more than one \textit{constant function} at a time, we shall usually write a simpler form such as

\[
x \xrightarrow{C} C(x) = a
\]
In particular, the power \( x^0 \) in the term \( ax^0 \) will often “go without saying”. For instance, instead of writing, as in the above example,

\[
x \xrightarrow{C_{-3}} C_{-3}(x) = (-3)x^0
\]

we shall usually write

\[
x \xrightarrow{C} C(x) = -3
\]

### 11.3.1 Graphing Constant Functions

We can proceed in either one of two ways.

1. We can deal with a constant function in exactly the same manner as we dealt with the unit functions, saying that since \( ax^0 = a \) the input-output rule of a constant function boils down to

\[
x \xrightarrow{C_a} C_a(x) = a
\]

the input does not appear in the output so that, no matter what the input \( x \) is, the output of the constant function \( C_a \) will remain equal to the number \( a \). As a result, the global quantitative graph of the constant function \( C_a \) will be the output level line for the output \( a \).

For instance, given the constant function \( C_{+4} \), its input-output rule is

\[
x \xrightarrow{C_{+4}} C_{+4}(x) = (+4)x^0 = +4
\]

and its global quantitative graph is

![Diagram of Global Quantitative Graph of \( C_{+4} \)]

The disadvantage of getting the graph this way is that it depends too much on the fact that the function is a constant function so that it will not carry over to other functions.
2. The other way is based on the similarity of constant functions with unit functions.

   a. First, we observe that constant functions are very similar to the two unit functions

   \[ x \xrightarrow{\text{UNIT}_+} \text{UNIT}_+(x) = (+1)x^0 = +1 \]

   and

   \[ x \xrightarrow{\text{UNIT}_-} \text{UNIT}_-(x) = (-1)x^0 = -1 \]

   namely the two “exceptional power functions” whose input-output rule involve zero copy of the input.

   b. To take advantage of this similarity, we need to use the dilation of a function.

   For instance, the (+4) dilation of the unit function \( \text{UNIT}_+ \) is the function whose name is \( (+4)\text{UNIT}_+ \) and whose input-output rule is

   \[ x \xrightarrow{(+4)\text{UNIT}_+} (+4)\text{UNIT}_+(x) = (+4) \cdot (+1)x^0 = (+4)x^0 \]

   c. Then we can look at a constant function as just a dilation of either one of the functions \( \text{UNIT}_+ \) or \( \text{UNIT}_- \).

   For instance, the constant function \( C_{+4} \) whose input-output rule is

   \[ x \xrightarrow{C_{+4}} C_{+4}(x) = (+4)x^0 = +4 \]

   is the same as the (+4) dilation of the function \( \text{UNIT}_+ \):

   \[ x \xrightarrow{(+4)\text{UNIT}_+} (+4)\text{UNIT}_+(x) = (+4) \cdot [(+1)x^0] = (+4)x^0 = +4 \]

   but observe that it is also the same as the (-4) dilation of the function \( \text{UNIT}_- \):

   \[ x \xrightarrow{(-4)\text{UNIT}_-} (-4)\text{UNIT}_-(x) = (-4) \cdot [(-1)x^0] = (+4)x^0 = +4 \]
d. From this point of view, we could get the graph of a constant function \( C_a \) from the graph of either unit function by a vertical stretch with stretch factor \( a \). However, this would require that we define negative vertical stretching as well as positive vertical stretching and it will be simpler just to define plain vertical stretching, with the stretch factor an unsigned number, as follows:

Then, to get the graph of \( C_a \),

- when \( a \) is positive, look at the stretched graph of \( UNIT_+ \),
- when \( a \) is negative, look at the stretched graph of \( UNIT_- \).

For instance, we get the global quantitative graph of the constant function \( C_{+4} \), that is the function whose input-output rule is

\[
x \mapsto_{C_{+4}} C_{+4}(x) = (+4)x^0 = +4
\]

by a vertical stretch of the function \( UNIT_+ \) with stretch factor (4):

and we get the global quantitative graph of the constant function \( C_{-4} \), that is the function whose input-output rule is

\[
x \mapsto_{C_{-4}} C_{-4}(x) = (-4)x^0 = -4
\]

by a vertical stretch of the function \( UNIT_- \) again with stretch factor (4):
11.3.2 Operating On Constant Functions

1. Just like we looked upon constant functions as dilations of unit functions, we can dilate constant functions and this gives us constant functions. However, there is no reason here to limit ourselves to positive dilation factors.

   a. When the dilation factor is positive, things are just the same as above.

   For instance, if we dilate the constant function $C_{+2}$ by a factor of $(+3)$, we get the constant function $[(+2)C_{+2}]$:

   $$x \xrightarrow{[(+3)C_{+2}]} [(+3)C_{+2}] (x) = (+3) \cdot [(+2)x^0]$$
   $$= (+6)x^0$$
   $$= +6$$

   which is the same as the constant function $C_{+6}$

   $$x \xrightarrow{C_{+6}} C_{+6}(x) = (+6)x^0$$
   $$= +6$$

   From the graphic point of view, starting from the graphs of $C_{+2}$

The advantage of getting the graph this way is that it uses dilation and stretching, a tool that we will have to use with all other polynomial functions.
we get the graph of \([+(2)C_{+2}]\) by a vertical \((+3)\) stretch of the graph of \(C_{+2}\)

The resulting graph outputs +6 for any input and thus is the graph of \(C_{+6}\).

b. When the dilation factor is negative, the graphic aspect is a bit more complicated.
For instance, if we dilate the constant function \(C_{+2}\) by a factor of \((-3)\), we get the constant function \([(-3)C_{+2}]\):

\[
x \xrightarrow{(-3)C_{+2}} (-3)C_{+2}(x) = (-3) \cdot (+(2)x^0) = (-6)x^0 = -6
\]

which is the same as the constant function \(C_{-6}\)

\[
x \xrightarrow{C_{+6}} C_{+6}(x) = (-6)x^0 = -6
\]

From the graphic point of view, though, starting from the graphs of \(C_{+2}\)
we get the graph of $C_{+6}$ by a vertical (+3) stretch of the graph of $C_{+2}$ followed by a vertical flip:

The resulting graph outputs +6 for any input and thus is the graph of $C_{+6}$.

Observe that we could have done the flip first and then a (+3) stretch:
2. We will need to add constant functions.
For example, given the two constant functions, $C_2$ and $C_5$,

\[ x \rightarrow C_2(x) = +2 \]

and

\[ x \rightarrow C_5(x) = +5 \]

we can add them as follows

\[ x \rightarrow [C_2 \oplus C_5](x) = C_2(x) + C_5(x) \]

\[ = [+2] + [+5] \]

\[ = +7 \]

which is the constant function $C_7$:

From the graphic point of view, starting from the graphs of $C_2$ and $C_5$
we get the graph of $C_{+7}$ by stacking the graph of $C_{+5}$ on top of the graph of $C_{+2}$.

The resulting graph outputs +7 for any input and thus is the graph of $C_{+7}$.

11.4 Piecewise Constant Functions
11.5 Linear Functions

Functions whose *input-output rule* is of the form

\[ x \xrightarrow{L_a} L_a(x) = ax^{+1} \]

where \(a\) stands for a given *signed* number, are called *linear functions*. \(L_a\) will be the *name* of the linear function *specified* by the signed number \(a\). The given signed number \(a\) is called the *linear coefficient* of \(L_a\). For example, \(L_{-3}\) is the linear function whose input-output rule is

\[ x \xrightarrow{L_{-3}} L_{-3}(x) = (-3)x^{+1} \]

that is the function whose *output* is equal to \(-3\) multiplied by one copy of the input.

*Note*. Unless there is need for the above complete way of writing things, as there will be when, for instance, we have to deal with more than one linear function at a time, we shall usually write a simpler form such as

\[ x \xrightarrow{L} L(x) = ax \]

In particular, the *exponent* \(+1\) in the term \(ax^{+1}\) will often “go without saying”.

For instance, instead of writing, as in the above example,

\[ x \xrightarrow{L_{-3}} L_{-3}(x) = (-3)x^{+1} \]

we shall usually write

\[ x \xrightarrow{L} L(x) = -3x \]

Then, when given an input, say \(+4\), we write

\[ +4 \xrightarrow{L} L(x)|_{\text{when } x=+4} = -3x|_{\text{when } x=+4} \]

\[ = -3(+4) \]

\[ = -12 \]
11.5.1 Graphing Linear Functions

As opposed to the case of constant functions, we cannot graph linear functions directly from the input-output rule and there is thus only one way to proceed.

1. First, we observe that linear functions are very similar to the identity function

\[
x \xrightarrow{ID} ID(x) = ( +1 ) x^1 = +x
\]

and to the opposite function

\[
x \xrightarrow{OPP} OPP(x) = ( -1 ) x^1 = -x
\]

namely the two “exceptional power functions” whose input-output rule involve one copy of the input. In fact, we can look at a linear function as just a dilation of either one of the functions ID or OPP.

For instance, the linear function \( L_{-3} \) whose input-output rule is

\[
x \xrightarrow{L_{-3}} L_{-3}(x) = (-3) x^1 = -3x
\]

is the same as the function which is the \((-3)\) dilation of the function ID:

\[
x \xrightarrow{(-3)ID} (+3) ID(x) = (-3) \cdot \left[ (+1) x^0 \right] = (-3) x^1 = -3x
\]

but observe that it is also the same as the function which is the \((+3)\) dilation of the function OPP:

\[
x \xrightarrow{(+3)OPP} (+3) OPP(x) = (+3) \cdot \left[ (-1) x^1 \right] = (-3) x^1 = -3x
\]

2. From this point of view, we could get the graph of a linear function \( L_a \) from either the graph of the function ID or the graph of the function OPP by a vertical stretch with stretch factor \( a \). However, here as with constant functions, we shall use a plain vertical stretch with the stretch factor equal to \( \text{SIZE}(a) \), that is equal to the size of the dilation factor and,
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• when \( a \) is positive, stretch the function \( ID \),
• when \( a \) is negative, stretch the function \( OPP \).

For example,

a. We get the global quantitative graph of the linear function \( L_{+\frac{1}{2}} \), that is the function whose input-output rule is

\[
x \overset{L_{+\frac{1}{2}}}{\longrightarrow} L_{+\frac{1}{2}}(x) = \left(\frac{1}{2}\right)x^1
\]

\[
= +\frac{1}{2}x
\]

by a vertical stretch with stretch factor \( \left(\frac{1}{2}\right) \) of the function \( ID \):

b. We get the global quantitative graph of the linear function \( L_{-\frac{1}{2}} \), that is the function whose input-output rule is

\[
x \overset{L_{-\frac{1}{2}}}{\longrightarrow} L_{-\frac{1}{2}}(x) = \left(-\frac{1}{2}\right)x^1
\]

\[
= -\frac{1}{2}x
\]

by a vertical stretch again with stretch factor \( \left(\frac{1}{2}\right) \) but of the function \( OPP \):
### 11.5.2 Operating On Linear Functions

We can add and dilate linear functions just the way we dilated and added constant functions in Section 11.3. While essentially the same, stretching and stacking graphs will be just a little bit more difficult to see.

1. When we dilate by a positive factor,

For instance, given the linear function $L_{\frac{1}{2}}$ we can dilate it by a factor of $(+3)$ as follows:

\[
x \xrightarrow{\left[(+3)L_{\frac{1}{2}}\right]} \left[(+3)L_{\frac{1}{2}}\right](x) = (+3) \cdot \left(\frac{1}{2}\right)x^1
= \left[(+3) \cdot \left(\frac{1}{2}\right)\right]x^1
= \left(\frac{3}{2}\right)x
\]

From the graphic point of view, we get the graph of $\left[(+3)L_{\frac{1}{2}}\right]$ by a vertical $(+3)$ stretch of $L_{\frac{1}{2}}$. 
2. When we dilate by a negative factor, we must include a vertical flip. For instance, given the linear function \( L_{-\frac{1}{2}} \) we can dilate it by a factor of \((-3)\) as follows:

\[
x \rightarrow \left[ (-3)L_{-\frac{1}{2}} \right] \rightarrow \left[ (-3)L_{-\frac{1}{2}} \right] (x) = (-3) \cdot \left( -\frac{1}{2} \right) x^1
\]

\[
= \left[ (-3) \cdot \left( -\frac{1}{2} \right) \right] x^1
\]

\[
= \left( \frac{3}{2} \right) x
\]

From the graphic point of view, we get the graph of \( \left[ (-3)L_{-\frac{1}{2}} \right] \) by a vertical \((+3)\) stretch of \( L_{-\frac{1}{2}} \) followed by a vertical flip:

Observe that we could just as well have gotten the graph of \( \left[ (-3)L_{-\frac{1}{2}} \right] \) by a vertical flip of \( L_{-\frac{1}{2}} \) followed by a \((3)\) stretch:
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11.5.3 Linearity

Finally, we must mention an extremely important double feature which, among the functions in this text, linear functions will be the only ones to have.

- Part of this feature is that if we input two numbers, say +5 and +2, into a linear function, say $L_{+3}$, and add the outputs:

  \[
  +5 \rightarrow L_{+3} \rightarrow +15 \quad +2 \rightarrow L_{+3} \rightarrow +6
  \]

  \[
  +15 + 6 = +21
  \]

  we would have gotten the same result by first adding the original numbers +5 and +2 and then inputting the result in the linear function:

  \[
  +5 + 2 = +7 \rightarrow L_{+3} \rightarrow +21
  \]

- The other part of this feature is that, if we input a number, say +5, into a linear function, say $L_{+3}$, and multiply the output by some number, say +2, we would have gotten the same result by multiplying the original number +5 by the number +2 and inputting the result into the linear function $L_{+3}$:

  \[
  +5 \rightarrow L_{+3} \rightarrow +15 \rightarrow +10 \rightarrow L_{+3} \rightarrow +30
  \]

  \[
  +15 \times 2 = +30
  \]

The reader should check that constant functions do not have that feature.
Unsurprisingly, this double feature is called \textit{linearity} but, surprisingly enough, there is a whole part of mathematics, called \textsc{Linear Mathematics}, dealing with the far-reaching consequences of this apparently simple feature. In this text, though, we shall not make much use of it.
11.6 Affine Functions

Functions whose input-output rule is of the form

\[ x \rightarrow_{A_{a,b}} A_{a,b}(x) = ax^1 + bx^0 \]

where \( a \) and \( b \) stand for given signed numbers, are called affine functions. \( A_{a,b} \) will be the name of the affine function specified by the two signed numbers \( a \) and \( b \). The given signed number \( a \) is called the linear coefficient of \( A_{a,b} \) and the given signed number \( b \) is called the constant term of \( A_{a,b} \). For instance, \( A_{-3,+5} \) is the affine function whose input-output rule is

\[ x \rightarrow_{A_{-3,+5}} A_{-3,+5}(x) = (-3)x^1 + (+5)x^0 \]

that is of the function whose output is equal to \(-3\) multiplied by one copy of \( x \) plus \(+5\) multiplied by zero copy of \( x \).

Note. Unless there is need for the above complete way of writing things, as there will be when, for instance, we have to deal with more than one affine function at a time, we shall write a simpler form such as

\[ x \rightarrow_{A} A(x) = ax + b \]

In particular, the exponent \( +1 \) in the term \( ax^1 \) and the whole power \( x^0 \) in the term \( bx^0 \) will often “go without saying”.

For instance, instead of writing, as in the above example,

\[ x \rightarrow_{A_{-3,+5}} A_{-3,+5}(x) = (-3)x^1 + (+5)x^0 \]

we shall usually write

\[ x \rightarrow_{A} A(x) = -3x + 5 \]

Then, when given an input, say +4, we write

\[ +4 \rightarrow_{A} A(x)|_{\text{when } x=+4} = -3x + 5|_{\text{when } x=+4} = -3(+4) + 5 = -7 \]

\(^1\)Educologists will rightfully observe that we are not being quite consistent: \( ax^1 = ax \) is the linear term and \( bx^0 = b \) is the constant term of which \( a \) and \( b \) are the coefficients.
11.6.1 Graphing Affine Functions

In order to graph affine functions we need both operations.

1. First we observe that the input-output rule shows that affine functions can be looked upon as combinations of the two different types of “exceptional” power functions, that is of the power functions with exponent 0 and with exponent +1, namely:

- the two unit functions
  
  \[ x \xrightarrow{UNIT_+} UNIT_+(x) = +1 \]

  and

  \[ x \xrightarrow{UNIT_-} UNIT_-(x) = -1 \]

- the identity function

  \[ x \xrightarrow{ID} ID(x) = +x \]

- and the opposite function

  \[ x \xrightarrow{OPP} OPP(x) = -x \]

For example, the affine function

\[ x \xrightarrow{A} A(x) = -3x + 5 \]

can be looked upon as a combination of a dilation of

\[ x \xrightarrow{OPP} OPP(x) = -x \]

and a dilation of

\[ x \xrightarrow{UNIT_+} UNIT_-(x) = +1 \]

We shall call such a combination of dilations a linear combination and thus we shall say that affine functions are linear combinations of the exceptional power functions.

Observe, though, that there are four ways in which we can look at a given affine function as linear combination of exceptional power functions.
So, just the way we introduced in Section 11.3 stretching as the graphical representation of dilating and stacking as the graphical representation of adding, we now need to introduce the graphical representation of combination.

As it happens, we can stack two functions of different type almost as easily as

To get the graph of an affine function, we shall see it as a linear combination of exceptional power functions, graph these exceptional power functions individually, stretch them and stack the results.

### 11.6.2 Interpolating From A Local Graph Near Infinity

In what follows, we shall investigate the function \( x \xrightarrow{A} A(x) = +\frac{1}{2}x + 17 \).

Our approach is based on the fact that, when \( x \) is near \( \infty \), \( +\frac{1}{2}x \) is larger in size than +17. So, we first look for the local graph of the affine function \( x \xrightarrow{A} A(x) = +\frac{1}{2}x + 17 \) when \( x \) is near \( \infty \), that is for large inputs.

1. We begin by graphing the linear function \( x \xrightarrow{L} L(x) = +\frac{1}{2}x \) as dilation of the identity function \( x \xrightarrow{ID} ID(x) = +x \) with dilation factor equal to \( \frac{1}{2} \).

2. The graph of the approximate linear function \( x \xrightarrow{L} L_{\ldots}(x) = +\frac{1}{2}x + (\ldots) \) is a band with the graph of \( L \) as median.
3. Since, when \( x \) is large, +17 is small compared to \( x \), the graph of \( x \rightarrow A(x) = +\frac{1}{2}x + 17 \) is somewhere in that band that acts therefore as a local frame.

In order to get a qualitative global graph, we must join the two pieces of the graph near \( \infty \) in some way and the question is what information about the way we need to do so. For that, we shall need some General Statements about the graph inside the window and so we shall return to the issue after we have developed the latter in Section 11.6.4.

In the meantime, we shall draw what we shall call the essential global graph, that is the simplest graph that fits the information we have.

### 11.6.3 Extrapolating A Local Graph Near A Finite Point

We now look at the output of an affine function for inputs that are restricted to being near a given input.

1. Given, for instance, the function \( x \rightarrow A(x) = 3x + 17 \), and given an input, say \( x_0 = -5 \), we localize the input-output rule at \( x_0 = -5 \) as
follows:

\[-5 + h \rightarrow^A f_{-5}(h) = +3x + 17\bigg|_{x=-5}\]

\[= +3(-5 + h) + 17\]

\[= -15 + 3h + 17\]

and, collecting terms of the same order of magnitude,

\[= +2 + 3h\]

We will call

\[-5 + h \rightarrow^A f_{-5}(h) = +2 + 3h\]

the local input-output rule when \(x\) is near \(x_0\).

**Note.** The terms in the output of the local input-output rule are listed in order of diminishing order of magnitude.

**2.** We now look at the graphic representation of each of these terms by taking them into successive consideration:

**a.** We consider the effect of the term +2 in the output if it were alone:

\[-5 + h \rightarrow^A +2\]

*Graphically*, this says that the output point for any input near \(-5\) is at the point +2 on the output ruler so that the plot points are on the piece of horizontal straight line

\[\text{Output Ruler} \quad \text{Input Ruler} \quad -5 \quad +2 \quad \text{Window}\]

**b.** We consider the effect of adding some fuzz to +2 in the output:

\[-5 + h \rightarrow^A +2 + (...)\]
Graphically, this says that the output points for inputs that are near $-5$ will be near the point $+2$ on the output ruler, rather than exactly at the point $+2$, and that the plot points are now going to be, rather than exactly on the horizontal line, somewhere in the grey area which is the graphic fuzz that corresponds to the fuzz term (...) that we added in the output of the local input-output rule:

\[ -5 + h \xrightarrow{A} +2 + 3h \]

Graphically, this says that the plot points are not just anywhere in the grey area but on the piece of slanted line:

where the piece of slanted line is the graph of the power function

\[ h \xrightarrow{p} p(h) = (+1)h^{+1} \]

translated near $-5$ and added on top of $+2$. because, since this is all qualitative, we ignore the size of the dilation factor, 3, but keep its sign, +.
11.6. AFFINE FUNCTIONS

11.6.4 General Statements

1. We localize

\[ x \xrightarrow{A} A(x) = ax + b \]

near a yet unspecified point \( x_0 \):

\[ x_0 + h \xrightarrow{A} f_{x_0}(h) = ax + b \bigg|_{x=x_0+h} = a(x_0 + h) + b = ax_0 + ah + b \]

and, collecting terms of the same order of magnitude, we get

\[ = [ax_0 + b] + [a]h \]

2. Observe that the order of magnitude of the term \([ax_0 + b]\) is 0 because \([ax_0 + b]\) is “of the order of \(10^0 = 1\)”, which is to say that it is “in the ones”, while the order of magnitude of the term \([a]h\) is \(-1\) because \([a]h\) is “of the order of \(10^{-1} = \frac{1}{10} = 0.1\)” which is to say that it is “in the tenths”. This is the reason why we write \([a]h\) in second position after \([ax_0 + b]\).

3. This gives us the following two general statements:

- The height of the local graph, which is given by the coefficient of \(h^0\), that is \([ax_0 + b]\), depends on \(x_0\) and therefore will be different “everywhere”, that is for all inputs.

- The slope of the local graph, which is given by the coefficient of \(h^{-1}\), that is \([a]\), is independent of \(x_0\) and therefore will be the same everywhere, that is for all inputs.

11.6.5 Qualitative Global Graph

On the basis of the above general statements, we can now obtain a qualitative global graph in either one of two ways:

1. We can start from the local graph near infinity of Section 11.6.2:
200  

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finite graph

the two pieces of which we join with a straight line, the finite graph.

2. We can start from local graphs near finite points as follows:

a. We construct local graphs near, say, three different points, $x_4$, $x_2$, $x_3$. On the above basis, they would look something like this:

b. However, this is not possible because that would mean that inputs such as $x_4$ would have two outputs:
whereas the input-output rule will pair the input $x_4$ with only one output, namely $+3x_4 + 17$.

c. As a result, the local graphs must all line up and so the finite graph must be a straight line:

3. Of course, the finite graph must line up with the local graph near $\infty$ and, either way we get the general statement:

*The global graph of an affine function is a straight line.*

### 11.6.6 Global Problems

- Where is the output equal to/smaller than/larger than a given number? algebraically/in size.
- Where is the output smallest/largest)? Algebraically/In size? Absolutely/In a given interval?
- **Boundary Value Problem**: What is the input-output rule of an affine function given its output for two different inputs.
- **Initial Value Problem**: What is the input-output rule of an affine function given its slope and given its output for a given input.
11.7  Piecewise Affine Functions

11.8  Quadratic Functions

11.9  Cubic Functions

11.10 Quartic Functions And Above
Chapter 12

Rational Functions

*Transition inputs* (inputs that are just outside the window. needed for the cases where slope and/or concavity are lost and need to be recovered by division.)

Important for image interpretation because they are the first functions to have steep gradients near finite inputs. See Meyer?
12.1 Local Investigations
12.1. LOCAL INVESTIGATIONS

12.1.1 Local investigation of the rational function

\[ x \xrightarrow{f} f(x) = \frac{x}{x^2 - 1} \]

when \( x \) is near 0.

\[
x \text{ near } 0 \xrightarrow{f} f(x)|_{\text{when } x=0+h=h} = \frac{x}{x^2 - 1}|_{\text{when } x=h} = \frac{h}{h^2 - 1}
\]

which we must rewrite as

\[
= \frac{h}{-1 + h^2}
\]

because \( h^2 \) is smaller in size than \(-1\)

At this point we have a choice:

- We can compute blindly
- We can think ahead about what is going to happen and take advantage of it so as to do as little work as possible.

We look at each choice separately:

- When we compute blindly, we must divide:

\[
\begin{array}{c}
\frac{-1 + 0h + h^2}{+h} \\
\underline{\overbrace{\overbrace{-1 + 0h + h^2}^{+h}}}
\end{array}
\]

Observe that we wrote in 0 to make sure things will remain lined up later on.

1. We divide the first into the first:

\[
\begin{array}{c}
\frac{-1 + 0h + h^2}{+h} \\
\underline{\overbrace{\overbrace{-1 + 0h + h^2}^{+h}}}
\end{array}
\]

This gives us

\[
x \text{ near } 0 \xrightarrow{f} f(x)|_{\text{when } x=h} = \frac{h}{-1 + h^2} = -h + (\ldots)
\]

but, since \(-h\) is one of the exceptional power functions, it does not have all three features:
– If we want the sign of the output or the sign of the slope, the power function \(-h\) will give it to us.
– If we want the sign of the concavity, it is inside (...) and we must push the division to dig it out.

2. We multiply

\[
\begin{array}{c}
-1 + 0h + h^2 \\
\hline
-1 + 0h + h^2 \\
\end{array}
\]

\[
\begin{array}{c}
h \\
h \\
\end{array}
\]

\[
\begin{array}{c}
+1 + 0h^2 + h^3 \\
-1 + 0h^2 - h^3 \\
\end{array}
\]

3. We subtract

\[
\begin{array}{c}
-1 + 0h + h^2 \\
\hline
-1 + 0h + h^2 \\
\end{array}
\]

\[
\begin{array}{c}
h \\
h \\
\end{array}
\]

\[
\begin{array}{c}
+1 + 0h^2 + h^3 \\
-1 + 0h^2 - h^3 \\
\end{array}
\]

4. We divide the first into the first

\[
\begin{array}{c}
-1 + 0h + h^2 \\
\hline
-1 + 0h + h^2 \\
\end{array}
\]

\[
\begin{array}{c}
h \\
h \\
\end{array}
\]

\[
\begin{array}{c}
+1 + 0h^2 + h^3 \\
-1 + 0h^2 - h^3 \\
\end{array}
\]

This gives us

\[
x \text{ near } 0 \xrightarrow{f} f(x)_{\text{when } x=h} = \frac{h}{-1 + h^2} = -h + h^3 + (...)\]

and the power function \(+h^3\) gives us the sign of the concavity.

• When we think ahead, we see that dividing \(-1+ (...)\) (instead of \(-1+h^2\)) into \(+h\) will give us \(-h + (...)\) which will give us the sign of the output and the sign of the slope but not the sign of the concavity.

If we need the sign of the concavity then we must divide as above.
12.1. LOCAL INVESTIGATIONS

12.1.2 Local investigation of the rational function

\[ x \xrightarrow{f} f(x) = \frac{x^3 - 8}{x - 1} \]

when \( x \) is near +2.

\[
\begin{align*}
\left. f(x) \right|_{x=+2+h} &= \frac{x^3 - 8}{x - 1} \bigg|_{x=+2+h} \\
&= \frac{(+2 + h)^3 - 8}{+2 + h - 1} \\
&= \frac{(+2 + h)^3 - 8}{+1 + h}
\end{align*}
\]

At this point we have a choice:

- We can compute blindly
- We can think ahead about what is going to happen and take advantage of it so as to do as little work as possible.

In either case, though, we need to know the addition formula:

\[(x_0 + h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3\]

- When we compute blindly, we write:

\[
\begin{align*}
\frac{(+2 + h)^3 - 8}{+1 + h} &= \frac{+8 + 3(+2)^2h + 3(+2)h^2 + h^3 - 8}{+1 + h} \\
&= \frac{+3(+2)^2h + 3(+2)h^2 + h^3}{+1 + h} \\
&= \frac{+12h + 6h^2 + h^3}{+1 + h}
\end{align*}
\]

and we must divide:

1. We divide the first into the first:

\[
\begin{array}{cccc}
+1 & +h & & \\
+12 & h & +6 & +h^3 & \\
\end{array}
\]

This gives us:

\[
\left. f(x) \right|_{x=+2+h} = \frac{+12h + 6h^2 + h^3}{+1 + h} = +12h + (...)
\]
but, since this is one of the exceptiona power functions, it does not have all three features:

– If we want the sign of the output or the sign of the slope, the power function $+12h$ will give it to us.
– If we want the sign of the concavity, it is inside (...) and we must push the division to dig it out.

2. We multiply

\[
\begin{array}{c|c|c|c}
+1 + h & +12h & +6h^2 & +h^3 \\
+12h & +12h & +12h^2 & +6h^2 & +h^3 \\
\end{array}
\]

3. We subtract

\[
\begin{array}{c|c|c|c}
+1 + h & +12h & +6h^2 & +h^3 \\
+12h & +12h & +12h^2 & +12h^2 & +6h^2 & +h^3 \\
\end{array}
\]

4. We divide the first into the first

\[
\begin{array}{c|c|c|c}
+1 + h & +12h & -6h^2 & +h^3 \\
+12h & +12h & +6h^2 & +h^3 \\
\end{array}
\]

This gives us

\[
x \text{ near } 2 \quad f \longrightarrow f(x) \bigg|_{x=+2} = \frac{+12h + 6h^2 + h^3}{+1 + h} = +12h - 6h^2 + (...)\]

and the power function $-6h^2$ gives us the sign of the concavity.

• When we think ahead about what $(+2 + h)^3$ is going to be equal to, we see that it is going to start with $(+2)^3 = +8$ so that, because of the $-8$ at the end, the numerator will really start with the next term in the expansion of $(+2 + h)^3$, that is with $+12h$ so that we will have:

\[
\frac{(+2 + h)^3 - 8}{+1 + h} = \frac{+12h + (...)}{+1 + (...) = +12h + (...)}
\]

Here again, since this is one of the exceptional power functions, it does not have all three features:

– If we want the sign of the output or the sign of the slope, the power function $+12h$ will give it to us.
12.1. LOCAL INVESTIGATIONS

– If we want the sign of the concavity, it is inside (...) and we must push the division to dig it out. Since the next term in the expansion of 
\((+2 + h)^3\) is \(+6h^2\), we write

\[
\frac{(+2 + h)^3 - 8}{+1 + h} = \frac{+12h + 6h^2 + (...) }{+1 + h}
\]

where we included the \(+h\) to balance our including the \(+6h^2\). Note
that there is no (...) after that because the \(+h\) is all there was in (...). So, we must divide

\[
\begin{array}{c}
+1 + h \\
\end{array}
\]

\[
\begin{array}{c}
+12h & +6h^2 & + (...) \\
\end{array}
\]

instead of

\[
\begin{array}{c}
+1 + h \\
\end{array}
\]

\[
\begin{array}{c}
+12h & +6h^2 & + h^3 \\
\end{array}
\]

as we did above.
So, in this case, the economy is not a big one.
12.1.3 Local investigation of the rational function

\[ f(x) = \frac{x - 1}{x^3 - 8} \]

when \( x \) is near +2.

\[ x \nearrow +2 \quad f(x) \bigg|_{x = +2 + h} = \frac{x - 1}{x^3 - 8} \bigg|_{x = +2 + h} = \frac{+2 + h - 1}{(+2 + h)^3 - 8} = \frac{+1 + h}{(+2 + h)^3 - 8} \]

At this point we have a choice:
- We can compute blindly
- We can think ahead about what is going to happen and take advantage of it so as to do as little work as possible.

In either case, though, we need to know the addition formula:

\[(x_0 + h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3\]

- When we compute blindly, we write:

\[
\begin{align*}
\frac{+1 + h}{(+2 + h)^3 - 8} &= \frac{+1 + h}{+8 + 3(+2)^2h + 3(+2)h^2 + h^3 - 8} \\
&= \frac{+1 + h}{+12h + 6h^2 + h^3}
\end{align*}
\]

and we must divide:

\[+12h + 6h^2 + h^3 \bigg) +1 \quad +h \]

1. We divide the first into the first:

\[+12h + 6h^2 + h^3 \bigg) +1 \quad +h \quad +\frac{1}{12}h^{-1} \]

2. We don't need to go any further since the first term we got, \(+\frac{1}{12}h^{-1}\), has slope.
When we think ahead about what \((+2 + h)^3\) is going to be equal to, we see that it is going to start with \((+2)^3 = +8\) so that, because of the \(-8\) at the end, the denominator will really start with the next term in the expansion of \((+2 + h)^3\), that is with \(+12h\) so that we will have:

\[
\frac{+1 + h}{(+2 + h)^3 - 8} = \frac{+1 + (...) + 12h}{+12h + (...) + 12h^{-1} + (...)}
\]

which should be compared with the division above.

So we have

\[
x \text{ near } +2 \xrightarrow{f} \left. f(x) \right|_{x=+2+h} = \frac{+1 + h}{(+2 + h)^3 - 8} = \frac{1}{12} h^{-1} + (\ldots)
\]

and, since this is a negative power function, it has all three features.
12.2 Global Investigations
12.2. GLOBAL INVESTIGATIONS

12.2.1 Investigate the rational function \( \text{Rat}_2 \) whose input-output rule is

\[
x \xrightarrow{\text{Rat}_2} \text{Rat}_2(x) = \frac{x - 1}{x^3}
\]

1. We look at inputs that are near \( \infty \), that is at large inputs (most of them are since the screen is only a very, very small part of the universe).

a. From the input-output rule point of view, we compute:

\[
x \text{ near } \infty \xrightarrow{\text{Rat}_2} \text{Rat}_2(x) \bigg|_{\text{when } x \text{ is near } \infty} = \frac{x - 1}{x^3} \bigg|_{\text{when } x \text{ is near } \infty} = x + (...) = \frac{1}{x^2} + (...) = (+1) \cdot x^{-2} + (...)
\]

In other words, when \( x \) is near \( \infty \), the output of the rational function \( \text{Rat}_2 \) is approximately the same as the output of the power function whose input-output rule is

\[
x \xrightarrow{\text{Power}} \text{Power}(x) = (+1) \cdot x^{-2}
\]

b. From the graphic point of view, the local graph of the power function when \( x \) is near \( \infty \) is

![Graph of Power Function](image)

Then, the fact that \( \text{Rat}(x) \bigg|_{\text{when } x \text{ is near } \infty} = (+1) \cdot x^{-2} + (...) \) means that, when \( x \) is near \( \infty \), the local graph of \( \text{Rat}_2 \) is in a strip whose median is the graph of the power function:
(The width of the strip corresponds to the (...) in the input-output rule.)

c. Looking at the features of the graph when \( x \) is near \( \infty \)

we see that:
- the output does not switch sign.
- the slope switches sign,
- the concavity does not switch sign,

2. We now ask if, when \( x \) is finite, the output \( \text{Rat}_2(x) \) must also be finite.

   a. From the graphic point of view, we are asking if the graph of \( \text{Rat}_2 \) for finite input is completely within the screen.

Another way to ask the same question is the following:

Is there a finite input, call it \( x_{\infty\text{-output}} \), such that the local frame is
(if the answer to the one is “yes”, the answer to the other is “no” and if the answer to the one is “no”, then the answer to the other is “yes”.)

b. From the input-output rule point of view, we are asking if there is a finite input, call it \( x_{\infty\text{-output}} \), whose output is \( \infty \)? But since we do not know how to compute with \( \infty \) itself, we ask:
In English: Is there a finite input, call it \( x_{\infty\text{-output}} \), such that when \( x \) is near \( x_{\infty\text{-output}} \), then \( \text{Rat}_2(x) \) is near \( \infty \)?
In Algebra: Is there a finite input, call it \( x_{\infty\text{-output}} \), such that when \( x = x_{\infty\text{-output}} + h, \text{Rat}_2(x) \) is large?
In other words, we want

\[
\text{Rat}_2(x)|_{x = x_{\infty\text{-output}} + h}
\]
to be large

i. So, first we localize near a temporarily unspecified \( x_0 \)

\[
x = x_0 + h \rightarrow \text{Rat}_2 \rightarrow \text{Rat}_2(x)|_{x = x_0 + h} = \frac{x - 1}{x^3} |_{x = x_0 + h} = \frac{[x_0 + h] - 1}{[x_0 + h]^3} = \frac{[x_0 - 1] + h}{x_0^3 + 3x_0^2h + 3x_0h^2 + h^3} = \frac{[x_0 - 1] + (...)}{x_0^3 + (...)}
\]

Note. Observe that since

\[
\text{Rat}_2(x)|_{x = x_0} = \frac{x_0 - 1}{x_0^3}
\]
the last line of the previous computation could have been obtained easily.

ii. Now since the output \( \text{Rat}_2(x) \big|_{x=x_0} \) is a fraction, there are two ways that it could be large:

- if its numerator, \([x_0 - 1] + (\ldots)\), were large. But \((\ldots)\) is small and \([x_0 - 1]\) cannot be large since \(x_0\) is finite. So the numerator cannot be large.
- if its denominator, \(x_0^3 + (\ldots)\), were small. But, while \((\ldots)\) is indeed small, \(x_0^3\) is not. So for the denominator to be small, it must reduce to \((\ldots)\) and so \(x_0^3\) must be equal to 0, that is \(x_0 = 0\)

Altogether then, the only neighborhood in which the output can be large is when \(x\) is near 0

3. We investigate the behavior of the function \( \text{Rat}_2 \) when \(x\) is near \(x_\infty\)-output, that is when \(x\) is near 0.

a. From the input-output rule point of view, we compute

\[
x = 0 + h = h \xrightarrow{\text{Rat}_2} \text{Rat}_2(x) \big|_{x=h} = \frac{x - 1}{x^3} \big|_{x=h}
\]

\[
= \frac{-1 + h}{h^3}
\]

\[
= \frac{-1 + (\ldots)}{h^3}
\]

\[
= \frac{-1}{h^3} + (\ldots)
\]

\[
= -h^{-3} + (\ldots)
\]

In other words, when \(x\) is near 0, the output of the rational function \( \text{Rat}_2 \) is approximately the same as the output of the power function whose input-output rule is

\[
x \xrightarrow{\text{Power}} \text{Power}(x) = (-1) \cdot x^{-3}
\]

b. From the graphic point of view, the local graph near 0 of the power function is
Then, the fact that $Rat_2(x)|_{x=h} = -h^{-3} + (...)$ means that, when $x$ is near 0, the local graph of $Rat_2$ is in a strip whose median is the graph of the power function.

c. Looking at the features of the graph when $x$ is near 0

we see that:
essential graph

- the output switches sign.
- the slope does not switch sign,
- the concavity switches sign,

4. Altogether, we have that the graph outside the screen is in the shaded strips

When we join smoothly these local graphs, we obtain the essential graph, that is the simplest possible qualitative global graph for the function $Rat_2$.

In most textbooks, this would be simply called “the” graph and shown as
which is quite misleading since, as an anthropocentric concession, even the
screen above was already disproportionately large and a graph more evoca-
tive of the size of the screen relative to the universe, would be:

In fact, an even more realistic picture would be:
5. Looking at the features that we already found, we can get a certain amount of additional, mostly qualitative, information:
In other words, we can see that there has to be at least:
- one finite input near which the concavity switches sign, namely the 0-concavity input $x_{0\text{-concavity}}$
- one finite input near which the slope switches sign, namely the 0-slope input $x_{0\text{-slope}}$
- two finite inputs near which the output switches sign, namely the 0-output input $x_{0\text{-output}}$ and the $\infty$-output input $x_{\infty}$.

We said “at least” because these features, which we shall call the **essential features** of the function $\text{Rat}_2$, are those that are forced by what happens beyond the screen just like, for instance, tides on earth are caused by the moon.

But, on the basis of our investigation, we cannot tell if the actual graph might not be, for instance,

![Diagram](image.png)

In other words, the graph may have **fluctuations**, that is min-max pairs. But these would not be caused by what happens beyond the screen and they would just be like, for instance, waves on the sea.

6. We would now like to **locate** the input intervals, if any, for which one of the three features has a **given sign**. This is quite independent from most of the previous investigation since this will involve only the input-output rule and solving an **inequation** which, as usual (See Section 1.3), means: (i) looking for those inputs where the output is either 0 or $\infty$ because, for a rational function, these are the only inputs where the output could switch sign. (ii) testing each of the intervals to find the inputs where the output actually **does** switch sign.

Unfortunately, here we will only be able to locate the inputs whose output has a given sign because we are not in a position to locate $x_{0\text{-slope}}$ and $x_{0\text{-concavity}}$ since we do not have an **input-slope rule** and an **input-concavity rule**. (These will be derived in Differential Calculus.)
For example, in what follows, we will want to locate the input intervals, if any, for which the output is positive.

**a.** We look for the inputs where the output could switch sign, that is the inputs where the output is either 0 or \( \infty \).

**i.** We already located \( x_{\infty\text{-output}} \), namely we found above that \( x_{\infty\text{-output}} = 0 \).

**ii.** We locate \( x_{0\text{-output}} \) exactly in the same manner: As already computed above,

\[
x = x_0 + h \xrightarrow{\text{Rat}_2} \text{Rat}_2(x) \big|_{\text{when } x = x_0 + h} = \frac{x_0 - 1 + (...)}{x_0^3 + (...)}
\]

Now, since the output \( \text{Rat}_2(x) \big|_{\text{when } x = x_0} \) is a fraction, there are two ways that it could be small:

- if its numerator, \([x_0 - 1] + (...)\), were small.
  - But, while (...) is indeed small, \([x_0 - 1]\) is not. So, for the numerator to be small, it must reduce to (...) and so \([x_0 - 1]\) must be equal to 0, that is \( x_0 = +1 \)
- if its denominator, \( x_0^3 \), were large.
  - But \( x_0^3 \) cannot be large since \( x_0 \) is finite. So the denominator cannot be large.

Altogether, the only possibility is \( x_{0\text{-output}} = +1 \)

**b.** We now test each one of the intervals determined by \( x_{\infty\text{-output}} = 0 \) and \( x_{0\text{-output}} = +1 \)

**i.** To test the interval from \(+1\) to \(+\infty\), we can use any number larger than \(+1\). We use a *large* positive number:

\[
+\text{large} \xrightarrow{\text{Rat}_2} \text{Rat}_2(x) \big|_{\text{when } x = +\text{large}} = \frac{x - 1}{x^3} \big|_{\text{when } x = +\text{large}} = \frac{+\text{large} + (...)}{(+\text{large})^3}
\]

\[
= \frac{+1}{(+\text{large})^2} + (...) = +\text{small} + (...)
\]

So the output will be positive in the whole interval from \(+1\) to \(+\infty\). Since this is what is being required, we *highlight* the interval:
ii. To test the interval from $-\infty$ to 0, we can use any number smaller than 0. We use a large negative number:

\[
-\text{large} \rightarrow \text{Rat}_2 \rightarrow \text{Rat}_2(x)\bigg|_{\text{when } x = -\text{large}} = \frac{x - 1}{x^3} \bigg|_{\text{when } x = -\text{large}} = -\text{large} + (...) = \frac{(-\text{large})^3}{(-\text{large})^3} = +1 + (...) = +\text{small} + (...)
\]

So the output will be positive in the whole interval from +1 to $+\infty$. Since this is what is being required, we highlight the interval:

![Input Ruler]

\[
-\infty \quad 0 \quad +1 \quad +\infty
\]

\[
\text{x}_0\text{-output} \quad \text{x}_0\text{-output}
\]

iii. To test the interval from 0 to +1, we can use any number between 0 and +1. Say we use +0.5:

\[
+0.5 \rightarrow \text{Rat}_2 \rightarrow \text{Rat}_2(x)\bigg|_{\text{when } x = +0.5} = \frac{x - 1}{x^3} \bigg|_{\text{when } x = +0.5} = +0.5 - 1 = \frac{(+0.5)^3}{(+0.5)^3} = -0.5 = +0.125 = -0.25
\]

So the output will be negative in the whole interval from 0 to +1. Since this is not what is being required, we blank-out the interval:

![Input Ruler]

\[
-\infty \quad 0 \quad +1 \quad +\infty
\]

\[
\text{x}_0\text{-output} \quad \text{x}_0\text{-output}
\]
Chapter 13

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