Mathematics For Learning

With Inflammatory Notes For The Education Of Educologists

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Math begins with arithmetic.

Arithmetic is said to have originated, some four or five thousands years ago, when Babylonian merchants were faced with the problem of **accounting** for more *goods* and *money* than they could handle personally. The solution was to **represent** the goods in the warehouse and the money in the safe by various scratches on clay tablets so that rich merchants could see the situation their business was in without the inconvenience of having to go to the warehouse and/or to open the safe.

As time went by and businesses grew more and more intricate, though, the scratching system had to become powerful enough to represent what needed to be accounted for. Eventually, the *scribes* (literally, scratchers) who had been doing the accounting had to invent **double-entry bookkeeping** to represent even more complicated business activities and thereafter became known as **accountants**.

We shall recreate, with a bit of poetic license, but *from a mathematical point of view*, the way the accounting system might have evolved[1] over the centuries to deal successively with:

1. Money on a counter,
2. Money changing hands across a counter,
3. Goods on a counter,
4. Goods changing hands across a counter,
5. Goods exchanged for money (that is buying and selling).

We shall thus pretend to work in the real world and, instead of scribing clay tablets, we will write on the **board procedures** to represent our pretended real world activities.[2]

[1] Of course, arithmetic didn't evolve in *that* order but what reason do Educologists have for clinging to the historical order? The order followed here seems to be rather logical and natural. In any case, the thinking underlying the historical stages was probably more or less as discussed here.

[2] Throughout, I shall make a typographical distinction between real world ***objects*** and the **names** that represent them on the board. The reader may or may not notice it and it hardly matters: If s/he doesn't, no damage is done. If s/he does, no damage is done either since the distinction will make sense. On the other hand, should the distinction *not* be made, this would certainly confuse the keener reader. And, if nothing else, the distinction might cue educologists as to what to do on the *counter* versus what to write on the *board*. But it is absolutely not a distinction to be enforced on the students as, in any case, it is a difficult one to maintain unfailingly. It is only here as something that might help to clarify things.

Chapter I. Counting

In which it will be seen that adopting a Model Theoretic viewpoint (= distinguishing objects *from their* **names***) leads us in the most natural manner to separating* quantity *from* quality *and thus to the parallel processing of combinations of powers of ten all the way up to 9,999.*

1. Accounting for Money

1. We begin with the problem of how to represent on the board *money* sitting on a **counter**. (Note by the way that banks used to be called *countinghouses* and that accountants are often referred to as "bean counters.")

   Starting with ***dollars*** and ***dimes*** on the counter, we use the words **Dollars** and **Dimes** as **denominators**[3] that is names/symbols/denominations to represent on the board what money sits on the counter. For instance, we represent ***dollar, dollar, dime*** on the counter by writing **Dollar, Dollar, Dime** on the board.

2. The first breakthrough in the development of arithmetic was the realization that objects of *different* kinds have to be accounted for *separately*. When dealing with objects all of the *same* kind, it will be convenient to refer to them collectively as a **collection**. For instance, we shall refer to ***dime, dime, dime, dime, dime, dime, dime*** on the counter as a collection of ***dimes***. On the other hand, ***dime, dime, nickel, nickel, nickel, nickel, nickel*** is *not* a collection. What would it be a collection of? We will deal with that matter right after we have dealt with collections.

   a. When dealing with a collection, we can represent it by *writing* on the board a **(counting-) number-phrase**.[4] We shall call this to **count** the collection. A number-phrase consists of:

   - a **numerator**[5] to indicate *how many* ***objects*** there are in the collection, which we do here by writing one **stroke** on the board for each ***object*** in the collection, and,
   - a single denominator to indicate *what kind* ***objects*** the collection is made of.

   For instance, instead of representing the collection ***dime, dime, dime, dime, dime, dime, dime*** by writing **Dime, Dime, Dime, Dime, Dime, Dime, Dime** on the board, we can count the collection by writing the number-phrase /////// **Dimes** on the board.

[3] An Educologist once warned me very, very sternly about "the possible confusion in learners' minds between this use of the term and its association with fractions in the usual meaning of the word"! More about this when we get to fractions.

[4] Words within parentheses are for the purists. After they have been properly introduced, they can safely go without saying as they can always be recovered from the context if need be.

[5] While "denominators" represent "objects," we cannot say that "numerators" represent "numbers" because we have not said what numbers *are* and we are *not* about to do so because, while objects can be *exhibited*, numbers have to be *defined*.

Note that, even though a number-phrase is much more economical a way to represent on the board a collection of ***objects*** on the counter than writing one denominator for each ***object***, there is ***no loss of information*** as, essentially, we are merely *separating* **quantity** from **quality**. This is actually a very powerful idea that we shall use again and again.[6]

While it is tempting to refer to ***dime, dime, dime, dime, dime, dime, dime*** as a collection of /////// ***dimes***, we shall resist the temptation because /////// are marks on the board while ***dimes*** are ***objects*** on the counter and we don't want to mix marks and objects.)

Another point about language is that we shall often speak of **the count** of a collection to refer to the *result* of counting the collection. We will thus use "count" as shorthand for "(counting-) number phrase" even though it could be argued that there is a difference between what the two mean.

b. We now deal with the question of what to do when we have ***objects*** of *different* kinds. For instance, say we have ***dime, dime, nickel, nickel, nickel, nickel, nickel***, on the counter. Of course, we could write ////// **Coins** but then we would be losing information and, for instance, wouldn't know how much *money* we have. So what we will do is to count the ***dimes*** and the ***nickels*** *separately* and write // **Dimes** & //// **Nickels**, where & means "and," and we will call this a **combination** of **Dimes** and **Nickels**. Combinations too will turn out to be a very powerful idea and one that will come up again and, in fact, combinations are the subject of a whole part of mathematics called LINEAR ALGEBRA.[7]

3. The second breakthrough in the development of arithmetic occurred when Indian scribes introduced as numerators the **digits** 2, 3, ..., 9 to be used instead of //, ///, ..., /////////. Because Arabs brought these to Europe, the whole world now writes, say, 3 **Dimes** instead of /// **Dimes**. While this may not seem to be an earth-shattering improvement, we will see presently all that this made possible.

a. Note that, at this point we can count only up to 9 **Dimes**. We shall see in a short while what to do after that.

b. In the meantime, we also introduce, if only for the sake of consistency, the digit 1 just as we use in English the phrase "one dime" as well as the phrase "a dime." The numerator 1 however often "goes without saying." For instance, when we write **2 Dollars & Dime** we understand of course that **Dollars** is the denominator in a number-phrase whose numerator is 2 but we have to *remember* that, here, **Dime** is not a denominator but stands for a *number-phrase* whose numerator is 1 and whose denominator is **Dime** so that **2 Dollars & Dime** stands for **2 Dollars & 1 Dime**. This is often expressed as a **default rule**: "when no numerator is given, the numerator 1 is intended and goes without saying."

[6]Of course, it is precisely at this stage that we decide to "abstract" *quality* better to concentrate on *quantity*! Educologists should note that it is also where we begin to "lose" a great many people.

[7]Unfortunately, in Linear Algebra too, the *denominators* that correspond to the *dimensions* of the space in which we are to operate "go without saying" and therefore may or may not be "understood" which is probably why the dual space remains unreachable for such a long time. See NOTMU, Fall 2003.

The trouble is that this default rule is often abbreviated as "when there is no numerator, the numerator is 1" which can be confusing because, when there is no numerator, it is tempting to think that there is no ***object*** either! To be on the safe side, we will avoid letting the numerator 1 go without saying.

c. Finally, we note that we have not yet introduced the digit 0 and this is because we have had no need for it yet. Historically, the digit 0 was in fact a much later invention. More about 0 later!

4. Another thing that we shall often have to deal with is the fact that collections can come wrapped together into **bundles**. For instance, ***dimes*** can come in rolls wrapped in paper.

   ♠ For instance, we may have on the counter[8] a ***bundle of three dimes***.

   ❖ We shall represent this on the board[9] by writing the denominator (3 **Dimes**) in which the parentheses represent the wrapping.

   Then of course we can count bundles.

   ♠ For instance, we may have on the counter ***bundle of three dimes, bundle of three dimes***.

   ❖ We represent this on the board by writing 2(**3 Dimes**) in which 2 is the numerator and (**3 Dimes**) is the denominator.

   We now look at what happens when we **unwrap** the bundles.

   ♠ On the counter, unwrapping ***bundle of three dimes, bundle of three dimes*** gives us ***dime, dime, dime, dime, dime, dime***.

   ❖ On the board, we write naturally

$$
\begin{aligned}
2(\textbf{3 Dimes}) &= (\textbf{3 Dimes}), (\textbf{3 Dimes}) \\
&= \textbf{Dime}, \textbf{Dime}, \textbf{Dime}, \textbf{Dime}, \textbf{Dime}, \textbf{Dime} \\
&= \textbf{6 Dimes}
\end{aligned}
$$

Now that we have set the stage for counting, we will see how to count beyond 9 but how we will do this, and in what direction this will take us, will of course depend on what we will want to account for, on the kind of activities we want to represent and, in particular, on what **operation** we will want to perform.

2. Addition Leads to Large Collections

When we have two collections of objects, there are two cases:

- The two collections consist of *identical* kinds of objects

[8] I will use the symbol ♠ to signal that we are in the **semantic** mode, that is, working *in the real world*.

[9] I will use the symbol ❖ to signal that we are in the **syntactic** mode, that is, writing *on the board*.

- The two collections consist of *different* kinds of objects

1. When the two collections consist of *identical* kinds of objects, we can **collect** the objects in the two collections into a single *collection* and the **addition** of the two counts will be the (board) procedure that will give us the count of the resulting collection.

   a. First, to see the essence of addition, we deal with numerators that are "small" enough that we can still count the resulting collection(s) with the digits we have.

   ♠ For instance, if we have ***dime, dime*** and ***dime, dime, dime***, we can collect them all and get ***dime, dime, dime, dime, dime***.

   ❖ The (board) representation of this is:

$$\begin{aligned}\textsf{\textbf{2 Dimes}} + \textsf{\textbf{3 Dimes}} &= \textsf{\textbf{Dime}},\ \textsf{\textbf{Dime}} + \textsf{\textbf{Dime}},\ \textsf{\textbf{Dime}},\ \textsf{\textbf{Dime}}\\ &= \textsf{\textbf{Dime}},\ \textsf{\textbf{Dime}},\ \textsf{\textbf{Dime}},\ \textsf{\textbf{Dime}},\ \textsf{\textbf{Dime}}\\ &= \textsf{\textbf{5 Dimes}}\end{aligned}$$

   When we are dealing with identical bundles, we can write,[10] for instance:

$$\begin{aligned}4(\textsf{\textbf{2 Dimes}}) &= \textsf{\textbf{2 Dimes}} + \textsf{\textbf{2 Dimes}} + \textsf{\textbf{2 Dimes}} + \textsf{\textbf{2 Dimes}}\\ &= \textsf{\textbf{8 Dimes}}\end{aligned}$$

   b. Before we move on to "large" numerators, we note that *combinations* can also be added and that the result is still a combination. For instance, [**2 Dollars & 3 Nickels**] + [**4 Dollars & 7 Dimes**] = **Dollar, Dollar, Nickel, Nickel, Nickel, Dollar, Dollar, Dollar, Dollar, Dime, Dime, Dime, Dime, Dime, Dime, Dime** = **6 Dollars & 3 Nickels & 7 Dimes**.[11]

   c. We now turn to the case when the numerators that are "large" in the sense that we cannot count the resulting collection with the digits we have.[12] Indeed, the effect of collecting is rapidly to create collections too large for us to have a *numerator*

   For the sake of convenience in the discussion, but only for that, we *will* allow ourselves to use the words TEN, ELEVEN, TWELVE, etc., but only with the firm understanding that we cannot *write* them as *digits* on the board.[13]

   There is of course nothing to prevent us from collecting the objects in the two collections and the problem is only how we will *represent* the result on the board.

[10] Educologists will realize that the operation that we are dealing with here is definitely *not* multiplication but an entirely different concept, namely **additive power**.

[11] Undoubtedly, Educologist will have recognized that this is simply addition in a linear space.

[12] Even with computers, this is still an issue.

[13] Just as I distinguish typographically between real world ***objects*** and the **denominators** that represent them on the board, I shall make a typographical distinction between the *numbers* TWO, THREE, etc. of objects on the counter and the *numerators*, 2, 3 etc that are used to represent them on the board.

♠ For instance, collecting ***dime, dime, dime, dime, dime*** and ***dime, dime, dime, dime, dime, dime, dime*** gives us ***dime, dime, dime, dime, dime, dime, dime, dime, dime, dime, dime, dime***.

❖ But, as we have no digit to represent on the board TWELVE ***dimes*** on the counter, we can write only

$$\begin{aligned} 5 \textsf{ Dimes} + 7 \textsf{ Dimes} &= \textsf{Dime, Dime, Dime, Dime, Dime, Dime,} \\ &\quad \textsf{Dime, Dime, Dime, Dime, Dime, Dime} \\ &= \textbf{?} \textsf{ Dimes} \end{aligned}$$

What we do in real life of course is to **change** TEN ***dimes*** for a ***dollar***.

♠ For instance, given

dime, ***dime***, ***dime***, ***dime***, ***dime*** and ***dime dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***,

we collect them

dime, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***,

and then we change TEN ***dimes*** for a ***dollar*** and we now have

dollar, ***dime***, ***dime***.

❖ The (board) representation of this is:

$$\begin{aligned} 5 \textsf{ Dimes} + 7 \textsf{ Dimes} &= \textsf{Dime, Dime, Dime, Dime, Dime, Dime,} \\ &\quad \textsf{Dime, Dime, Dime, Dime, Dime, Dime} \\ &= (\textsf{Dime, Dime, Dime, Dime, Dime, Dime,} \\ &\quad \textsf{Dime, Dime, Dime, Dime}), \textsf{ Dime, Dime} \\ &= 1 \textsf{ Dollar} \ \& \ 2 \textsf{ Dimes}. \end{aligned}$$

With only a slight abuse of language we can write this in a more friendly manner:

$$\begin{aligned} 5 \textsf{ Dimes} + 7 \textsf{ Dimes} &= \text{TEN } \textsf{Dimes, Dime, Dime,} \\ &= 1 \textsf{ Dollar} \ \& \ 2 \textsf{ Dimes}. \end{aligned}$$

Of course, the problem will be that we will have to keep introducing *ever-larger* denominators. For the sake of consistency, we will agree to stick to bills that change at a TEN to ONE **exchange rate** and thus to disregard ***five-dollar-bills***, ***twenty-dollar-bills*** and ***fifty-dollar-bills***. (However, we will deal with these later on.) To avoid getting mixed up between the actual money on the counter and the denominators on the board, we will write **Hamiltons**, **Franklins**, **Clevelands** as denominators for ***ten-dollar-bills***, ***hundred-dollar-bills*** and ***thousand-dollar-bills*** and also, since consistency is a virtue we will want to practice, we will write **Washingtons** instead of **Dollars**. But we will keep writing **Dimes** as denominator for ***dimes*** for lack of a better word.

♠ For instance, when we have to collect EIGHT ***dollars*** and FIVE ***dimes*** with SIX ***dollars*** and THREE ***dimes***, we will change TEN of the resulting FOURTEEN ***dollars*** on the counter for a ***ten-dollar-bill***.

❖ The (board) representation of this is:

[8 **Washingtons** & 5 **Dimes**] + [6 **Washingtons** & 3 **Dimes**]
= FOURTEEN **Washingtons** & 8 **Dimes**
= 1 **Hamilton** & 4 **Washingtons** & 3 **Dimes**

2. When two collections consist of *different* kinds of objects, we can only write a combination and that is that. For instance, given ***dollar, dollar*** and ***dime, dime, dime***, on the counter, we can only write the combination 2 **Dollars** & 3 **Dimes**. Even if we were to collect ***dollar, dollar*** and ***dime, dime, dime*** as ***dollar, dollar, dime, dime, dime***, we would not have a collection and no denominator to represent it.

   On the other hand, given a combination of two collections with different objects, *if* we can change the objects in the two collections for collections of *identical* objects, then we have a **common denominator** and the combination can then be turned into an addition.

   ♠ For instance, if we can change ***apple*** for ***nut, nut, nut*** and ***banana*** for ***nut, nut***, then we can change ***apple*** and ***banana, banana*** for ***nut, nut, nut*** and ***nut, nut, nut, nut***, that is for ***nut, nut, nut, nut, nut, nut, nut***.

   ❖ On the board, we proceed exactly in the same manner as on the counter and we write:

   If: 1 **apple** = 3 **nuts** and 1 **banana** = 2 **nuts**,
   then: 1 **apple** & 2 **bananas** = 3 **nuts** + 2 **nuts** + 2 **nuts**
   = 7 **nuts**

   But, while changing facilities will often be available, we should not expect that they always will.

   As a result of all this, we can now "count" all the way up to 9 **Clevelands** & 9 **Franklins** & 9 **Hamiltons** & 9 **Washingtons** & 9 **Dimes**. In order to continue, we would have to create even "larger" denominators. Instead, we shall now develop another, more systematic approach.

To be continued with a clever way of writing numbers and an explanation of what the mysteriously ill-named "carry over" really is. Much attention will continue to be lavished on the metric *aspect of U.S. money.*