Chapter 2

Accounting for Money *On* the Counter (II).

In which it will be found that combinations, much cumbersome to work with, can be "coded" in a marvelously efficient way. The search being a bit convoluted, it will be conducted in careful stages, focussed on what is written on the board and illustrated of course with money, it being the only entity whose TEN *to 1 rate of change is likely to be familiar to U. S. students.*

Note. There is nothing sacred about TEN: it is simply how many fingers (digit is just a fancy word for finger) we have on our two hands and TEN is not the only number of digits that we could have used. Deep down in their innards, computers use TWO digits, 0 and 1, because any electronic device is either off or on. At a higher level, computers engineers may use EIGHT (0, 1, 2, 3, 4, 5, 6, 7) or SIXTEEN digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f.) The Babylonians used SIXTY digits, a historical remnant of which is the fact that there are SIXTY seconds to a minute and SIXTY minutes to an hour. In fact, all that we will do can be easily redone with *any* number of digits.

2.1 (*Decimal*) Headings

A somewhat natural idea is to write the denominators only once and then write just the numerators as needed. The problem is how to know which numerator goes with which denominator. What we do is to write the denominators into a **heading** such as

Clevelands	Franklins	Hamiltons	Washingtons	Dimes

with the numerators directly underneath the corresponding denominators. For instance, we will write the combination 3 **Franklins** & 1 **Washington** & 7 **Dimes** as follows:

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
	3		1	7

Thus, each *column* corresponds to a collection of a *different kind* of objects. While this may look like going from cumbersome to *very* cumbersome, we will now see how much easier it is to *work* with.

2.2 Adding Under A Heading

We saw earlier that the problem with *addition* was mostly that the aggregation of collections produces a collection larger than any of the collections we start with so that to count the aggregate collection we could be in a position where we may not have "large" enough denominators. We shall deal with that problem presently and here we shall deal with the bookkeeping that is involved and this is where adding under a heading makes things easy.

Suppose for instance that we wanted to add the three combinations,

8 **Hamiltons** & 7 **Dimes**
8 **Washingtons** & 2 **Dimes**
5 **Hamiltons** & 3 **Washingtons** & 4 **Dimes**

First, we write the three combinations under the heading:

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
		8		7
			8	2
		5	3	4

Adding up the **Dimes**

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
		8		7
			8	2
		5	3	4
				THIRTEEN

gives us THIRTEEN **Dimes** but we cannot *write* that. *If* we can *change* TEN of the THIRTEEN **Dimes** for 1 **Washington**, then the situation becomes[1]:

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
			1	
		8		7
			8	2
		5	3	4
				3

[1]No Educologist has ever bothered to explain what "carry over" is supposed to *evoke* in this context.

Adding up the **Washingtons**

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
			1	
		8		7
			8	2
		5	3	4
			TWELVE	3

gives us TWELVE **Washingtons** but we cannot *write* that. Again, *if* we can *change* TEN of the TWELVE **Washingtons** for 1 **Hamilton**, then the situation becomes:

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		1	*1*	
		8		7
			8	2
		5	3	4
			2	3

Finally, adding the **Hamiltons**

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		1	*1*	
		8		7
			8	2
		5	3	4
		FOURTEEN	2	3

gives us FOURTEEN **Hamiltons** but we cannot *write* that. Again, *if* we can *change* TEN of the FOURTEEN **Hamiltons** for 1 **Franklin**, then the situation becomes:

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
	1	*1*	*1*	
		8		7
			8	2
		5	3	4
	1	4	2	3

Thus, *if* we have *changing facilities*, we can then carry out the addition[2] and the result will be 1 **Franklin** & 4 **Hamiltons** & 2 **Washingtons** & 3 **Dimes**.

2.3 Subtracting Under A Heading.

While, in the case of addition, the problem was on the *board* in that it had to do with not having large enough denominators, in the case of subtraction by

[2]Might this be the origin of "carryover"?

remove
subtraction

contrast, we shall see that the problem is mostly with what we have on the *counter*.

1. Given money on the counter, we can **remove** some, or all, of it. As with addition, **subtraction** is a (board) procedure to get the resulting number-phrase.

a. When the collections are *small*, things are simple.

♠ For instance, given ***dollar, dollar, dollar, dollar, dollar***, ***dime, dime, dime***, if we remove, say, ***dollar, dime, dime***, we are then left with ***dollar, dollar, dollar, dollar, dime***.

❖ The (board) procedure consists in just "erasing" the names of the objects we are removing:

[5 **Dollars** & 4 **Dimes**] − [1 **Dollar** & 3 **Dimes**] = **Dollar**, **Dollar**, **Dollar**, **Dollar**, **Dollar**,
Dime, **Dime**, **Dime**, **Dime**,
from which we must erase
Dollar,
Dime, **Dime**, **Dime**
= **Dollar**, **Dollar**, **Dollar**, **Dollar**, ~~**Dollar**~~,
Dime, ~~**Dime**~~, ~~**Dime**~~, ~~**Dime**~~,
= **Dollar**, **Dollar**, **Dollar**, **Dollar**,
Dime
= 4 **Dollars** & 1 **Dime**

b. When the collections are larger and are represented by (*counting-*) *number-phrases*, we want a (board) procedure to get the (counting-) number-phrases for the resulting collection in terms of the (*counting-*) *number-phrases* for the given collections.

♠ Suppose for instance that, from SEVENTY-EIGHT ***dollars*** in the form of SEVEN ***ten-dollar-bills*** and EIGHT ***one-dollar-bills***, we wanted to *remove* TWENTY-FIVE ***dollars***, in the form of TWO ***ten-dollar-bills*** and FIVE ***one-dollar-bills***.

What we do in the real world is in fact exactly represented by what we write on the board, so we skip what we do on the counter.

❖ On the board, we can represent the actual money by, say, the (counting-) number-phrases 7.8 **Hamiltons** and 25 **Washingtons**. What we choose as unit (denominator) does not matter since the first thing we do is to place the (counting-) number-phrases under a heading:

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
		7 2	8 5	

Then of course we get: placeholder

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		7	8	
		2	5	
		5	3	

that is 5 **Hamiltons** & 3 **Washingtons**.
Which we can write as 0.053 **Clevelands**, 0.53 **Franklins**, 5.3 **Hamiltons**, 53. **Washingtons** or 530. **Dimes**.

c. Most of the time, though, we need to *change*.

♠ Suppose for instance that, instead of SEVENTY-EIGHT ***dollars***, we had only SEVENTY-THREE ***dollars*** in the form of SEVEN ***ten-dollar-bills*** and THREE ***one-dollar-bills*** from which to *remove* TWENTY-FIVE ***dollars***, in the form of TWO ***ten-dollar-bills*** and FIVE ***one-dollar-bills***. While, obviously, we still have enough money to remove TWENTY-FIVE ***dollars***, the problem is that we don't have enough ***one-dollar-bills***.
Since, here again, what we do in the real world is in fact exactly represented by what we write on the board, we now move on to that:
❖ On the board, this means that, from 7.3 **Hamiltons** we want to *subtract* 25 **Washingtons**:

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		7	3	
		2	5	

We immediately run into a problem even though we obviously have plenty enough money to pay out 25 **Washingtons**. The problem is that 3 **Washingtons** is not enough from which to pay out 5 **Washingtons**. However, if we can change 1 **Hamiltons** for TEN **Washingtons**, then the situation becomes[3]

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		~~7~~ 6	THIRTEEN	
		2	5	
		4	8	

From which we get the result 4 **Hamiltons** & 8 **Washingtons**.
Which we can write as 0.048 **Clevelands**, 0.48 **Franklins**, 4.8 **Hamiltons**, 48. **Washingtons** or 480. **Dimes**.

2. Until now, we have been using 0 only as a **placeholder** when writing number-phrases. It is with *subtraction* that 0 becomes absolutely necessary as a *numerator* because we want to be able to write the result of, say, 3 **Franklins** – 3 **Franklins** as 0 **Franklin**.

[3]Educologists will certainly have no trouble explaining why they advocate the term "borrowing" even though it is both completely unfounded and perfectly non-explanatory.

place-notation
(decimal-counting) number-phrase
pick
unit denominator

3. Finally, as long as we are dealing with money on the counter, writing stuff such as 3 **Washingtons** − 5 **Washingtons** makes no sense whatsoever because, when there are only THREE ***ten-dollar-bills*** on the counter,we cannot remove FIVE ***one-dollar-bills***!

2.4 (*Decimal-Counting*) Number-Phrases

1. While headings are convenient when we want to *work* with *several* combinations, they are much too cumbersome when all we want is just to *write one* combination. So we will now develop yet another way, called **place-notation**, which will be convenient *both* to write a single combination and to add several combinations.

a. Suppose that, for a while, we wanted to deal with combinations such as 7 **Franklins** & 2 **Hamiltons** & 4 **Washingtons**.
Since the *denominators* remain fixed for the duration and can thus "go without saying", a natural idea would be just to give the *numerators* 7, 2, 4. The problem, though, is that someone being given these numerators wouldn't know under which denominator to write each numerator and thus wouldn't be able to reconstruct the combination.
For instance, s/he wouldn't know if the numerators 7, 2, 4 should be placed under the heading this way

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
7	2	4		

or that way

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		7	2	4

or *any* other way.
So, we cannot give someone just the *numerators*. Along with the numerators, we must also give some information as to *how* the numerators are to be placed under the heading.

b. To that purpose, we shall now introduce a new type of (counting) number-phrase that we shall call a **(decimal-counting) number-phrase**. Here is how we shall proceed.
First, we "**pick**" one of the denominators and call it our **unit (denominator)**. By "pick", we mean that our unit (denominator) can be *any* of the denominators and that the choice is entirely *ours*. The price for this freedom, though, is that we will have to indicate up front what our choice of unit (denominator) will be[4].
Thus, to write a (decimal-counting) number-phrase,

- we write the *numerators* in the order that they appear in the combination,
- we write the *unit* (denominator)

[4]This is somewhat similar to what is called a *declaration* in software engineering.

- we mark with a **decimal pointer** which of the numerators is to be placed under the unit (denominator).

decimal pointer
(decimal) numerator
decimal point

All the individual numerators, together with the decimal pointer, will be collectively referred to as **(decimal) numerator**. Thus, just like a (counting) number-phrase consisted of a numerator and a denominator, a (*decimal*-counting) number-phrase consists of a (decimal) numerator and a unit denominator.
Note. To see how things work, we will use for a while $\leftharpoondown$ as our decimal pointer. But, after this section, we shall conform and just use a dot called a **decimal point**. (However, this use of a dot as decimal pointer is not universal and many languages use a comma instead of a dot.)
The unfortunate thing about decimal *points*, as opposed to decimal *pointers*, is that we will have to *remember* that the decimal point *points to its left*, the way the decimal pointer $\leftharpoondown$ does.
Example 1. After we have taken **Hamiltons** as our unit, we can rewrite the combination 7 **Franklins** & 2 **Hamiltons** & 4 **Washingtons** as the number-phrase $72_{\leftharpoondown}4$ **Hamiltons**.
Then, anyone given the (decimal-counting) number-phrase $72_{\leftharpoondown}4$ **Hamiltons** would be able to recover the combination because,

- since 2 is being pointed at, 2 goes under **Hamiltons** since it is the unit (denominator),
- since 7 is left of 2, 7 goes under **Franklins**, the denominator left of **Hamiltons**, and,
- since 4 is right of 2, 4 goes under **Washingtons**, the denominator right of **Hamiltons**.

that is,

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
	7	2	4	

which is indeed 7 **Franklins** & 2 **Hamiltons** & 4 **Washingtons**.

c. There is however a problem with combinations such as 5 **Franklins** & 3 **Hamiltons** & 8 **Dimes** which, under a heading, looks like

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
	5	3		8

Say we take again **Hamiltons** as our unit (denominator). The problem is that what is immediately to the right of **Hamiltons** is **Washingtons** rather than **Dimes**. So, we cannot write $53_{\leftharpoondown}8$ **Hamiltons** because *that* (decimal-counting) number-phrase would be read as meaning

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
	5	3	8	

that is 5 **Franklins** & 3 **Hamiltons** & 8 **Washingtons**.

0 In order to get 5 **Franklins** & 3 **Hamiltons** & 8 **Dimes** we must indicate that there are no **Washingtons**, This is where we introduce the new digit **0** and we now write 53$_{\leftarrow}$08 **Hamiltons** as *that* gives

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
	5	3	0	8

which is 5 **Franklins** & 3 **Hamiltons** & 0 **Washingtons** & 8 **Dimes** and indeed the same as 5 **Franklins** & 3 **Hamiltons** & 8 **Dimes**.

2. There are two default rules. (We now use ".", the decimal *point*, instead of "$_{\leftarrow}$", the decimal *pointer*.)

• When the decimal point is to the right of the numerator, as in 7204. **Dimes**, it is customary not to write it and just to write 7204 **Dimes** and the corresponding default rule is:

WHEN THERE IS NO DECIMAL POINT, IT GOES WITHOUT SAYING THAT THE DECIMAL POINT IS TO THE RIGHT OF THE (DECIMAL) NUMERATOR.

• When there is no doubt as to what the unit (denominator) is, say **Hamiltons**, it is customary not to write it and, for instance, just to write 72.04 instead of 72.04 **Hamiltons**. The corresponding default rule is:

WHEN THERE IS NO DENOMINATOR, IT GOES WITHOUT SAYING THAT THE DENOMINATOR IS THE UNIT (DENOMINATOR).

This way of putting it, though, is *very dangerous* because it depends on us *remembering* what the unit is[5]. Unfortunately, it is done all the time.

Note. It is usual to write, for instance, .56 Hamiltons rather than 0.56 Hamiltons. The energy saving is not worth it and *we* shall *not* do so in this book.

3. (Decimal-counting) number-phrases have several advantages:

a. As we already saw, *we* can pick any denominator *we* want to be our unit (denominator). For instance, a rich person might pick **Clevelands** as her/his unit (denominator) while the rest of us would probably pick **Washingtons** as our unit (denominator).

b. Not only does place-notation allow us to pick any denominator we want to be our unit (denominator) but, by placing the number-phrase back under the heading, we can easily change the denominator we want to use as our unit (denominator)—and to adjust the decimal point accordingly. For instance, placing 72.04 **Hamiltons** back under the heading,

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
	7	2	0	4

we see that the (decimal-counting) number-phrase 72.04 **Hamiltons** could just as well be rewritten as any of the following (decimal-counting) number-phrases:

[5]This is perhaps one more place to remind Educologists that memory is the weakest part of the mind, that it is the first to go and that, so far, humans can be defined as thinking entities, that is entities amenable to logic.

0.7204 **Clevelands**
7.204 **Franklins**
720.4 **Washingtons**
7204. **Dimes**

Any of the above means the same as 7 **Franklins** & 2 **Hamiltons** & 4 **Dimes**.

c. Another advantage is that if, for whatever reason, we needed to have "smaller" or "larger" (decimal) numerators, it would be easy to do. We would change the digit being pointed and change correspondingly the denominator to be used as unit. We would do that by placing the number-phrase back under a heading.

d. Finally, there is another, intriguing, advantage. We saw earlier that with our heading we could now count all the way up to 9 **Clevelands** & 9 **Franklins** & 9 **Hamiltons** & 9 **Washingtons** & 9 **Dimes**. What if we wanted to add 1 **Hamilton**? Let us do it under a heading:

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
9	9	9 1	9	9

We have no trouble reaching the following stage (in the presence of changing facilities):

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
1	*1*			
9	9	9 1	9	9
TEN	0	0	9	9

At this point, though, the trouble is that we do not have any denominator beyond **Clevelands** so that we cannot change the TEN **Clevelands** and we cannot write the result either as a combination or as a (decimal-counting) number-phrase. On the other hand, if we pick any of the existing denominators, say **Franklins**, then we can write the number-phrase 100.099 **Franklins**.

Note. To some extent this would be cheating because we still do no have a denominator under which to place the leftmost 1 but, other than that, everything looks fine and we could even say that we wrote the decimal number phrase *pending the creation* of that new denominator!!! In case you should worry, though, we will develop in the next chapter several ways to get *automatically* as many denominators as we need.

To be continued with, much to the contentmen of Physicists, Chemists, Biologists, and other, assorted Scientists, a detailed investigation of the various ways in which insisting on being very *systematic leads to being able to write very, very large numbers fairly painlessly.*