Mathematics For Learning

With Inflammatory Notes for the Mortification of Educologists and the Vindication of "Just Plain Folks"

Alain Schremmer

The opinions expressed are those of the author and should not be construed as representing the position of AMATYC, its officers, or anyone else.

[Editor's note: In the Spring 2004 issue of *The AMATYC Review*, Schremmer introduced his idea for an open-source serialized text: *Mathematics For Learning.* The Preface to the text appeared in the Spring 2004 issue with a new chapter in each subsequent issue of *The AMATYC Review*. This issue contains Chapter 6.]

6. Repeated Multiplications and Divisions

Given a *number-phrase* we investigate what is involved in **repeated** multiplications or *repeated* divisions by a given *numerator*[1]

A Problem With English

English can be confusing when we want to *indicate* "how many times" the operation is to be repeated.

EXAMPLE 1. When we tell someone

Divide 375 **Dollars** 3 times by 5

multiplication is not involved. It can also easily be confused with

Divide 375 **Dollars** *by* 3 times 5

A workaround would seem just to avoid using the word "by" but it is awkward and even misleading when we *say* it and downright dangerous when we *write* it.

EXAMPLE 2. To *say*

Multiply 7 **Dollars** by 2, 3 times

[1] Educologists will note our departure from the usual treatment but conflating *unary operators* and *binary operations* is not exactly helpful. Also, i. the *binary* aspect has been omitted for the sake of brevity, but ii. some of the points made here would be unnecessary had the treatment of multiplication that appeared earlier on been a full one.

can be correctly understood but requires one to make a pause after saying 2 as, otherwise, it will be understood to mean

Multiply 7 **Dollars** by 2 or 3.

To *write* it can be correctly understood but requires one to pay attention to the comma between the 2 and the 3 as otherwise it will be understood to mean

Multiply 7 **Dollars** by 23

Templates

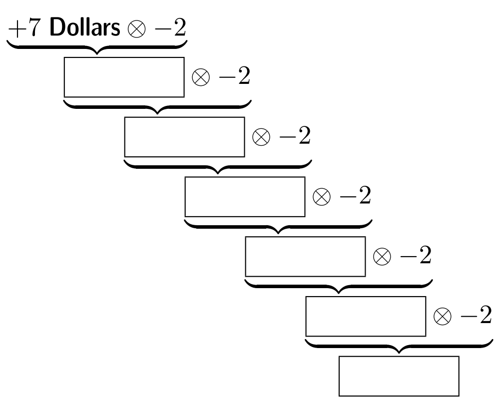
Perhaps surprisingly, writing specifying-phrases for *repeated* operations is not a simple matter.

1. Given a *number-phrase*, whose *numerator* we will refer to as the **coefficient**, and:

- given a *numerator*, called the **base**, by which the given number-phrase is to be repeatedly multiplied or repeatedly divided,
- given a *numerator*, called the **plain exponent**, to indicate how many multiplications or how many divisions we want done on the given *number-phrase*,

the simplest way to *specify* how many repeated multiplications or how many divisions we want done on the given *number-phrase* is to use a **staggered template**.

EXAMPLE 3. When we want the number-phrase $+7$ **Dollars** *multiplied* by 6 copies of -2, we write the following *staggered template*:



(1^{st} multiplication by -2)
(2^{nd} multiplication by -2)
(3^{rd} multiplication by -2)
(4^{th} multiplication by -2)
(5^{th} multiplication by -2)
(6^{th} multiplication by -2)
(Result of the repeated multiplications)

The staggered template specifies what is to be done at each stage and therefore what the result will be.

2. Quite often, though, we will not want to *get* the result but just be able to *use* or to *discuss* the repeated operations and, in that case, the use of *staggered* templates is cumbersome. So, what we will do is to let the boxes "go without saying" which will allow us to write an **in-line template**.

EXAMPLE 4. We can

- *Declare up front* that the *in-line* template is in **Dollars** and then write:

$$-208 \odiv -2 \odiv -2 \odiv -2 \odiv -2$$

- Write the in-line template for the numerators *within square brackets* and then write the denominator **Dollars**

$$\left[-208 \odiv -2 \odiv -2 \odiv -2 \odiv -2\right] \textbf{Dollars}$$

The Order of Operations

The use of *in-line* templates for repeated operations, though, poses a problem: how do we know for sure in what order the recipient of an *in-line* template is going to do the operations?

1. When the operation being repeated is *multiplication*, it turns out that the order in which the operations are done does *not* matter

EXAMPLE 5. Given the in-line template in **Dollars**

$$17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

the recipient might choose to compute it as

$$\underbrace{2 \times 2}$$

$$\underbrace{4 \times 2}$$

$$\underbrace{8 \times 2}$$

$$\underbrace{2 \quad \times \quad 16}$$

$$\underbrace{2 \quad \times \quad 32}$$

$$\underbrace{17 \quad \times \quad 64}$$

$$\boxed{1088}$$

etc but, it does not matter as the result will always be 1088.

However, proving *in general* that the *order* in which the *multiplications* are done does *not* matter takes some work because, as the number of copies gets large, the number of ways in which the multiplications could be done gets even larger and yet, to be able to make a *general statement*, we would have to make sure that *all* of these ways have been accounted for. So, for the sake of time, in the case of repeated *multiplications*, we will take the following *general statement* for granted:

THEOREM 1: The order in which the *multiplications* are done does *not* matter.

2. In the case of repeated *division*, though, the order usually makes a *huge* difference.

EXAMPLE 6. Given the in-line template in **Dollars**

$$448 \div 2 \div 2 \div 2 \div 2 \div 2 \div 2$$

which, when done from left to right, gives 7 as the result, the recipient might choose to compute it as

$$\underbrace{2 \div 2}$$

$$\underbrace{1 \div 2}$$

$$\underbrace{0.5 \div 2}$$

$$\underbrace{2 \quad \div \quad 0.25}$$

$$\underbrace{2 \quad \div \quad 8}$$

$$\underbrace{448 \quad \div \quad 0.25}$$

$$\boxed{1796}$$

etc.

Thus, in the case of repeated *divisions* it is crucial to agree on the order in which to do them and so, in the absence of any instructions to that effect, we will use

DEFAULT RULE # 1: The order in which the *divisions* are to be done is *from left to right.*

The Way to Power

Eventually, we will devise a very powerful language to deal both with repeated multiplications and repeated divisions but, before we can do that, we need to clear the way.

1. While, as we have seen, 1 does tend to "go without saying", what we can do when the *coefficient* in a repeated operation is 1 depends on whether the operation being repeated is *multiplication* or *division.*

(a) When it is *multiplication* that is being repeated, we can let the coefficient 1 go without saying. However, the number of multiplications is then one less than the number of copies.

(b) When it is *division* that is being repeated, we *must* write the coefficient 1 as, if we did not, we would be getting a different result[2].

EXAMPLE 7. Given the in-line template in **Dollars**

$$1 \div 2 \div 2 \div 2 \div 2 \div 2$$

the 1 cannot go without saying because, while the *given* in-line template computes to $\frac{1}{32}$, if we don't write the coefficient 1, we get an in-line template with coefficient 2 to be divided by 4 copies of 2:

$$2 \div 2 \div 2 \div 2 \div 2$$

which computes to $\frac{1}{8}$.

2. *Repeated divisions* are related to *repeated multiplications.* Indeed, instead of dividing a coefficient by a number of copies of the *base*, we can i. multiply 1 repeatedly by the number of copies of the base or ii. divide the coefficient by the *result* of the repeated multiplication.

The advantage of the second way of computing in-line templates involving repeated *divisions* is that while we now have one more *operation* than we had *divisions*, the first multiplication, multiplying the coefficient 1 by the first copy of the base, is no work and, as we saw above, need in fact not even be written so that the number of operations *requiring work* is the same in both cases. But now all operations except one are *multiplications* which are a lot less work than *divisions.*

However, here again, proving *in general* that the results are always the same takes some work so that, for the sake of saving time, we will take for granted that:

THEOREM 2: A repeated division is the same as a single division of the coefficient by the result of 1 multiplied repeatedly by the same number of copies of the base[3].

3. In order to *specify* the second way of computing, we can write either a **bracket in-line template** or a **fraction-like template**.

[2]This is more obvious with the use of fraction bars instead of $\div$.
[3]This is where the binary aspect becomes really useful.

EXAMPLE 8. We can write an in-line template in **Dollars** as

$$+448 \div \left[+1 \ \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2\right]$$

or as

$$+448 \div \left[-2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2\right]$$

or as

$$\frac{+448}{+1 \ \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2}$$

or as

$$\frac{+448}{-2 \otimes -2 \otimes -2 \otimes -2 \otimes -2 \otimes -2}$$

Power Language

We are now ready to introduce a way of writing specifying-phrases that will work both for *repeated multiplications* and for *repeated divisions.*

1. The idea is to *write* a **monomial specifying-phrase** in which we just write:

- the *coefficient*
- the symbol × as a **separator**
- the **power** which consists of the *base* with a **signed exponent**
    - whose *size* indicates the number of copies of the base to be used,
    - whose *sign* indicates whether the coefficient should be *multiplied* or *divided* by the copies.

We then *read* monomial specifying-phrases as

"Coefficient *multiplied/divided* by number of *copies* of the base"

EXAMPLE 9. Given the in-line *template*

$$448 \div \left[2 \times 2 \times 2 \times 2 \times 2 \times 2\right]$$

we *write* the monomial specifying-phrase,

$$448 \times 2^{-6}$$

which we *read* as

448 divided by 6 copies of 2

2. As it happens, though, there is *no* procedure for identifying *monomial specifying-phrases* other than the procedures corresponding to *staggered templates.*

This is in sharp contrast with the case of *repeated additions* for which there is a much shorter procedure for getting the result of repeated additions that is based on *multiplication* and with the case of *repeated subtractions* for which there is a much shorter procedure for getting the result based on *division*[4].

Multiplying Monomial Specifying-Phrases

When we multiply a first monomial specifying-phrase by a second monomial specifying-phrase with the same base, the result turns out to be a monomial specifying-phrase with the **common base**. We can get the result either one of two ways[5].

EXAMPLE 10. We can

- replace each *monomial specifying-phrase* by the corresponding *in-line* template, change the order of the multiplications and write the resulting monomial specifying-phrase:

$$\begin{aligned}\left[17 \times 2^{+5}\right] \times \left[11 \times 2^{-2}\right] &= \frac{17 \times 2 \times 2 \times 2 \times 2 \times 2}{1} \times \frac{11}{2 \times 2} \\ &= \frac{17 \times 11 \times \not{2} \times \not{2} \times 2 \times 2 \times 2}{\not{2} \times \not{2}} \\ &= 17 \times 11\big] \times 2^{+(5-2)} \\ &= 187 \times 2^{+3}\end{aligned}$$

- multiply the coefficients and "oplus" the signed exponents:

$$\begin{aligned}\left[17 \times 2^{+5}\right] \times \left[11 \times 2^{-2}\right] &= \left[17 \times 11\right] \times 2^{+5 \oplus -2} \\ &= 187 \times 2^{+3}\end{aligned}$$

Dividing Monomial Specifying-Phrases

When we divide a first monomial specifying-phrase by a second monomial specifying-phrase with the same base, the result turns out to be a monomial specifying-phrase with the *common base*. We can get the result either one of two ways.

EXAMPLE 11. We can

- replace each *monomial specifying-phrase* by the corresponding *in-line* template using fraction bars, "invert and multiply", change the order of the multiplications, cancel and write

[4] Educologists will justly regret that space limitations prevented here a sytematic development of the parallel between *additive* powers and *multiplicative* powers.

[5] Educologists of course let students "experience" how much work is saved by having them do it both ways for a while.

the resulting monomial specifying-phrase:

$$
\begin{aligned}
\left[17 \times 2^{+7}\right] \div \left[11 \times 2^{+3}\right] &= \frac{17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{1} \div \frac{11 \times 2 \times 2 \times 2}{1} \\
&= \frac{17 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{1} \times \frac{1}{11 \times 2 \times 2 \times 2} \\
&= \frac{17}{11} \times \frac{\not{2} \times \not{2} \times \not{2} \times 2 \times 2 \times 2 \times 2}{\not{2} \times \not{2} \times \not{2}} \\
&= \frac{17}{11} \times 2^{+(7-3)} \\
&= \frac{17}{11} \times 2^{+4}
\end{aligned}
$$

- divide the coefficients and "ominus" the signed exponents:

$$
\begin{aligned}
\left[17 \times 2^{+7}\right] \div \left[11 \times 2^{+3}\right] &= \left[17 \div 11\right] \times 2^{+7 \ominus +3} \\
&= \frac{17}{11} \times 2^{+7 \oplus -3} \\
&= \frac{17}{11} \times 2^{+4}
\end{aligned}
$$