Notes from the Mathematical Underground

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> Animals are fearful of anything new and unfamiliar; under less threatening conditions, however, they want to explore the new and become familiar with it.
> (Haenlein 1999)

> Thinking is a Dynamical System with the real world as Initial Condition
> Francesca Schremmer

Last Spring, while addressing the New York State Mathematical Association of Two-Year Colleges, I found myself asking "What is our purpose?" A nasty thing to do at the end of a banquet concluding a long conference day, I even amplified: ”We call ourselves educators but what do we really mean by education? We say we teach mathematics but what do we mean by mathematics?" I didn't, but should have also asked: why it is that "*unlike other sciences, mathematics, or at least some parts of it, is taught to all schoolchildren*" (Howson, Kahane et al. 1990) and, I might add, to all two-year college students.

Answers to these questions are of two types illustrated in an article comparing classroom activities in Japan and the U.S.. The first is illustrated by the answer given when asked to identify the main thing they wanted students to learn from a lesson: "*sixty one percent of U.S. teachers described "skills": They wanted the students to be able to perform a procedure, solve a particular kind of problem, and so on.*". (Stigler and Hiebert 1998)

Indeed, we claim "skills" to be necessary on *practical* grounds as in "To get a good job, get an education". However, you need only think of the number of well paid people, your physician or your attorney for example, who have told you: "Ah! you are in mathematics! *I* never went beyond calculus. I don't remember any of it and, anyway, I never use *any* math" to see that this necessity derives only from a societal decision that "math" be used as a screening device—say for admission to medical or law school. Yet, uncountably many "standards" take it to be the unarguable reason for our teaching of "basic skills".

But if, for instance, it is deemed that being able to obtain the result of basic arithmetic operations is an absolute must, why don't we just teach how to use an arithmetic calculator and leave it at that? In fact, before we know it, everyone will have a permanent, direct connection to the web. Ah but, my colleagues cry, it is the *acquisition* of these skills, it is the *learning* process, that are *really* important. It is the *conceptual* growth of our students that we have at heart, they wail. Upon which, they proceed, to take an ubiquitous example, to "factor quadratics".

Of course, they never specify over what set said quadratics are to be factored. While conceptually most regrettable, there is a very good reason for that as no one would want to specify "single-digit integers" as is the case in over 90% of all factoring exercises proposed in basic algebra texts. Why doesn't, for instance, x^2

–2 ever appear? Or, for that matter, $x^2 + 2$? Note, though, how religiously basic algebra textbooks, when they come to solving $x^2 + 2 = 0$ by the "quadratic formula", pronounce that it has no *real* solution. (By the way, has no one ever bothered to ask students what this qualifier means to them?) And isn't it telling enough that we must reverse-engineer these exercises? An issue that everybody dismisses without even, dare I say it, giving it a thought is that factoring not even being, under these circumstances, a *process* it is incompatible with the *conceptualization* of anything. But "factoring quadratics" blooms eternal.

Or, consider, if you will, the NASA-AMATYC-NSF collaborative project, ten years in the making, with at least twenty "team members" along twelve scientists/engineers. While it might be otherewise extremely worthwhile, I note that, in a flyer you may have received a few months ago, the qualifiers of the word "mathematics" in all its thirteen occurrences—mathematics *faculty* (3), mathematics *curriculum*, Mathematics *Reform*, Mathematics *Explorations* (3), mathematics *for use*, *pertinent* mathematics *courses*, mathematics *classroom materials*, mathematics *classes*, mathematics *classrooms*—hardly suggest that any attention was given to mathematics per se—and I don't mean necessarily with the bias of this column—but *exclusively* to data analysis[1]. This, presumably for "pedagogical" reasons that, other than invocation of "relevance" to the students, remain completely unexamined and, in fact, not even stated.

I find it deplorable in this respect that the recently adopted AMATYC Strategic Plan Goals and Objectives for 2000-2005 want to "*encourage research on teaching and learning*" (II D) but does not even mention any reflection on *contents*. Presumably, these are cast in stone and to suggest that our understanding of these contents could be improved would be insulting. In other words, the difficulties experienced by the students are assumed to result only from our inability to make the various "pieces of information" (aka recipes) more palatable. That these difficulties could result from our unwillingness to rely on our students' understanding of the logic that connects these pieces would appear to be unthinkable.

Contrast this with, to return to the above-mentioned article, the fact that "[Japanese] *teachers act as if mathematics is a set of relationships between concepts, facts and procedure*s. [They] *act as if mathematics is inherently interesting.* […] *They believe that students will be interested in exploring mathematics by developing new methods for solving problems.*" Shockingly unrealistic, isn't it?

I would like to elaborate on the Japanese standpoint not just because it's also mine but because, even from the "necessary skills" viewpoint, we could do a lot better by respecting the logic of the contents we teach. Or, if you prefer, if we cannot treat our students as intelligent beings for ethical reasons, then let it be at least on the basis of effectiveness.

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1 But I must admit to being awed by the apparition of

$$POS = \sum_{n=0}^{s} P(n) = \sum_{n=0}^{s} \frac{(\lambda T)^n * e^{-\lambda T}}{n!}$$

in the *first* Laboratory Technical Activity (LTA), suggested for Foundations and College Algebra courses. (Page 1-8)

I hold that our show and tell and drill approach is precisely what created the situation that we have to cope with—and that not only in remedial courses. Indeed, imagine yourself in a room full of high voltage equipment unfamiliar to you and being given a sequence of tasks such as: If this light over there is blue, then press that green button yonder and if that other light there is yellow, then pull this lever here but only halfway unless this dial here is past the quarter mark in which case, if …, etc, etc for a couple of pages. Keep in mind that you will be zapped to death at your third error. I think that you will agree that you are dead meat and that you would have had a much better chance, and might even have made it, had you been made to understand the logic organizing the equipment.

Similarly, what we do in what is delicately called "developmental math" adds up to a very painful death. We start, perhaps because we know students *can* count, with fractions introduced with examples such as *the fraction of failing students in a class is 1 out of 4*. But then what should they do when they know perfectly well that, with 1 point out of 4 on the first quiz and 2 points out of 4 on the second, they now have 3 points out of 8? Just as in elementary school, by drilling them to death, we may get them to write $\frac{1}{4}+\frac{2}{4}=\frac{3}{4}$ on the Exam, sometimes even on the Final but with the same long term results as in … elementary school. How can we even dare talk about "developmental math" when we forestall the very notion of *development* by requiring the same kind of drilling for each new "topic"?

It isn't as if there were no truly developmental alternative. Consider, for instance, the following *development* which, ironically, is a lot less "abstract" than the usual presentation in that it acknowledges the ubiquity of *denominators*.

(1) Start with *counting* numbers but, since that's what we encounter in real life, always deal with *denominated* counting numbers—i.e. with numbers of *objects* as in = 3 in which is clearly the *denominator* and 3 the *numerator*. And do note that = 1—an apple is one apple. Then, 1 + 2 = + = =3. (Contrast this with what you do with the disembodied 1 + 2 = 3.) We can count that way as long as we have digits.

(2) The big ideas are: a) to *aggregate* objects and count and aggregate the aggregates. For instance, it is traditional to aggregate eggs by the DOZEN. But since we have only ten digits, it is more efficient to aggregate by the TEN—and count up to 99. Then we can aggregate aggregates—and count up to 999. Etc. And then, b) to introduce the concept of objects *equivalent* to aggregates as in the sequence: mile, furlong, rod, yard, feet, inch, but the difficulty here is that the exchange rate is not constant in contrast with the familiar sequence **Cleveland**, **Franklin**, **Washington**, **dime**, **penny**, **mill**, in which each object is *equivalent* to (an aggregate of) TEN of the next one. The processes of "carrying over" and "borrowing", if not the terms themselves, then make perfect sense as explicited in my first column here (Fall 94). And if we choose, say, **Washington** as *principal* object, call it now **Dollar**, we can use as denominators: **Kilo-Dollar**, **Hecto-Dollar**, **Deka-Dollar**, **Dollar**, **deci-Dollar**, **centi-Dollar**, **milli-Dollar**.

(3) Now think of a Frenchman fresh off the boat who knows what a **Dollar** is and who can only count on his fingers and who encounters a **quarter**. Of course, *without even knowing what a quarter is,* he can tell you that 1 **quarter** + 2 **quarters** = 3 **quarters**. Great isn't it? Still, he would like to know what a **quarter** is but

telling him that it is equal to 100 pennies will not help since he doesn't know what a penny is and to tell him that it is a *fourth* of a dollar won't do either since all he can do is counting and, anyhow, he knows better than cut up a paper bill into four pieces like a pie. But if you tell him that 4 **quarters** are *equivalent* to 1 **Dollar**, he will be in business. If we introduce $\frac{\textbf{Dollar}}{\textbf{4}}$ as a much better name for **quarter**, then the preceding becomes[2]: $1\ \frac{\textbf{Dollar}}{\textbf{4}} + 2\ \frac{\textbf{Dollar}}{\textbf{4}} = 3\ \frac{\textbf{Dollar}}{\textbf{4}}$.

(4) We can then shift to the notation $\frac{1}{\mathbf{4}}$ **Dollar** + $\frac{2}{\mathbf{4}}$ **Dollar** = $\frac{3}{\mathbf{4}}$ **Dollar** but we must keep in mind that **4** is part of what *denominates* the numerators 1, 2, and 3.

What about fractions with different denominators? Indeed, the above development is incomplete and we need to introduce the notion of (linear) *combination*:

(1') In 1 **apple** + 2 **bananas**, the "+" does not denote an addition but a combination. Perhaps we should use "&" as in one **apple** and two **bananas**.

(2') 1 **Kilo-Dollar** + 2 **Hecto-Dollars** + 3 **Deka-Dollars** is thus a combination which, by the way, we can write variously as 1.23 **Kilo-Dollar**, 12.3 **Hecto-Dollar**, 12300. **deci-Dollars**, etc where the decimal point … points at the digit corresponding to the denominator so that we now have both the (decimal) number system and the metric system.

(3') Similarly, 1 **quarter** + 2 **dimes**, also known as $1\ \frac{\textbf{Dollar}}{\textbf{4}} + 2\ \frac{\textbf{Dollar}}{\textbf{10}}$ or as $\frac{1}{\mathbf{4}}$ **Dollar** + $\frac{2}{\mathbf{10}}$ **Dollar** is also a combination.

(4') Combinations can be *exchanged* for additions: Say we have 1 **apple** + 2 **bananas** and that 1 **apple** is worth 7 **strawberries** while 1 **banana** is worth 3 **strawberries**. Then, 1 **apple** + 2 **bananas** is worth 13 **strawberries**.

(5) Similarly, 1 **quarter** + 2 **dimes** is worth 9 **nickels** and "addition of fractions with different denominators" has lost its awesomeness. You might even say that it makes street sense.

It is often said of technicians that what most of them ever need of mathematics is to be able to use a few formulas. Possibly so, but it seems to me, if they use decimals a lot more than fractions, it is because, while *decimals* and *fractions* differ only by how systematic the denominators are, decimals allow for simple algorithms and *easy* approximation. Say you wanted to know how different $\frac{13}{7}$ and $\frac{9}{5}$ are. You could "teach" that, since the "cross products" 13•5 and 7•9 are pretty close, the fractions themselves must be pretty close (I would just compute the difference) but, either way, this does not obviously say how close the fractions are. On the other hand, division gives, say, 1.85 + (…) and 1.80. So, even from the "necessary skills" viewpoint, we could borrow a leaf from the technicians' book very much

[2] Be careful though that this notation can lead to confusion with "mixed numbers".

very much in the manner that mathematics is, once again, drawing much of its significant inspiration from engineering and physics.

Further down the line, say we wanted to investigate $f(x) = \frac{x-1}{x+1}$ near infinity. Division gives $f(x)$ exactly equal to a "mixed number", $f(x) = 1 - \frac{2}{x+1}$, or approximately equal to a "decimal number", $f(x) = 1 - 2x^{-1} + (\ldots)$. And, here again, the latter tells us immediately how close the approximation is.

Admittedly, the above development is sketchy but the reader should readily get the idea which is that, once a *conceptual framework* has been established, much finds a natural place, much comes for free. Indeed, if only because we deal with "definite numbers", "unknown numbers", and "numerical functions" in separate courses, we deprive our students of the immense power of isomorphism and logic. Polynomials, for instance, are combinations of powers of x just as decimal numbers are combinations of powers of TEN. Thus, $x^4 + 2x^4$ is naturally equal to $3x^4$ whereas I routinely see[3] $2x^8$, $3x^8$, $2x^5$. At the very least, you can counter the question "what do you do with the x^4?" with another question (🍎 = ?🍎) which is good pedagogical practice.

Of course, students need to be given whatever time might be necessary to get used to such a "developmental approach". That they already wasted a lot of time on "show and tell and drill" is unfortunate but makes it even more imperative that they be allowed to puzzle, to consider, to wonder… Indeed, they too "*are fearful of anything new and unfamiliar; under less threatening conditions, however, they want to explore the new and become familiar with it.*"

And, last but absolutely not least, there is another, broader side to "developmental education". As I already had the occasion to quote in an earlier column (Spring 96), but is important enough to bear repetition ad infinitum,

"*How do we bring about the cognitive health required by democratic government? A basic requirement is to cultivate in the populace a respect for intellectual values, an intolerance of intellectual vices or shortcomings.* [...]. *The forces of cretinisation are, and have always been, the biggest threat to the success of democracy as a way of allocating political power: this is the fundamental conceptual truth, as well as a lamentable fact of history.*"

However, "*people do not really like the truth; they feel coerced by reason, bullied by fact. In a certain sense, this is not irrational, since a commitment to believe only what is true implies a willingness to detach your beliefs from your desires.* [...]. *Truth limits your freedom, in a way, because it reduces your belief-options; it is quite capable of forcing your mind to go against its natural inclination. This, I suspect, is the root psychological cause of the relativistic view of truth, for that view gives me license to believe whatever it pleases me to believe.* [...]. *One of the central aims of education, as a preparation for political democracy, should be to enable people to get on better terms with reason – to learn to live with the truth.*"(McGinn 1990)

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3 I know, I know, this *never* happens with *your* students.

And *that,* in *any* case, is impossible to achieve by "show and tell and drill".

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