Notes From The Mathematical Underground

Alain Schremmer.

The opinions expressed are those of the author, and should not be construed as representing the position of AMATYC, its officers, or anyone else.

Whenever I am about to begin working on a new installment of these Notes I worry about some of the above words: *The opinions expressed (…) should not be construed as representing the position of (…) anyone else*. But, this time, having read *The False Crisis in Science Education,* an article in Scientific American prompted by the fact that America's high school juniors have placed near last on the Third International Mathematics and Science Study (TIMSS) (Gibbs and Fox 1999), I feel vindicated even if it refers to high schools and deals with mathematics only secondarily. Nevertheless, even though it rightfully criticizes TIMSS as well as the sulphuric conclusions generally arrived at in its wake, I would fault the article itself for not really doing much more than *mentioning* the important issues.

One such I mentioned in my first column here (Fall 94), writing "We declare that mathematics is useful but what do we mean by that? It is often said that this country lacks mathematicians or, at least, that a good background in mathematics helps in getting a good job" and adding that I had my doubts. The article in Scientific American would seem to support those doubts:

> "U.S. Research Feels No Crisis. (…) At most, 13 percent of American workers require higher math for their jobs. (…) Many take it as axiomatic that the science and math skills of high school graduates are critical to the health of the U.S. economy. But several lines of evidence contradict that assumption. …"

Another interesting point in the article is that "*Textbooks, however, do seem to matter*. … *With few exceptions, American texts covered many more topics than the foreign books and covered them over and over*." I take this to mean that, as I have kept insisting, contents matter but the article does not elaborate. I wish the AMATYC would finally address the issue instead of indulging in "standards writing", as fashionable as the latter might be.

The article ends with another, most important issue:

> "There is plenty of time after high school for scientists-to-be to learn the minute facts of science," [William F. McComas of the Univesity of Southern California] says. What [people] need from the schools, [Paul DeHard] Hurd [of Stanford University] elaborates, are the *higher thinking skills* (My emphasis) "to distinguish evidence from propaganda, probability from certainty, rational beliefs from superstitions, data from assertions, science from folklore, theory from dogma." And opportunity from crisis."

This is something I have already advocated a couple of times and to which I will return but I really wish the Scientific American had elaborated.

In any case, all this raises the question of how to evaluate student performance or, more mundanely, how to arrive at minimally meaningful grades and this, in turn, brings us back to the manner in which we conceive mathematics education

If teaching means imparting skills and we want to check their acquisition, there is little doubt that multiple-choice tests are optimal[1]. In fact, multiple-choice tests are so identified with skills acquisition that, conversely, anything measured by multiple-choice test is usually held to be, almost by definition, non-conceptual. For instance, there has been a convoluted, if paradoxical, tendancy to dismiss the TIMSS results on the basis of most of the questions being multiple-choice.

But what if the goal should be mathematical literacy, or, better yet, the ability to investigate a given situation and compare it to, as well as distinguish it from, other a priori similar ones and then to report meaningfully on one's finding? In that case, open questions are generally held to be obvioulsy the only ones to make sense. Mathematicians however ought to distrust the obvious.

First, in phrasing multiple-choice questions aiming at checking *understanding*, it is important to be *oblique*. For instance, to check what students make of the discriminant of a quadratic function, we might ask "How many zero(s) does $f(x) = x^2 + x + 1$ have?" but we should also ask "How many times does $f(x) = x^2 + x + 1$ change sign?" as well as "Which of the following is *sufficient* to guaranteee that Discriminant $f = 0$? (a) f has a zero between +2 and +5, (b) f changes sign at +4, (c) f does *not* change sign at +4, … " or "Given $f(x) = x^2 + 3x + c$, which of the following will ensure that Discriminant $f < 0$? (a) $c = 1$, (b) $c > 5$, … ." Etc.

Even more important, while we focus a great deal on the acquisition of *operating skills*, we usually pay no attention to that of *concepts* and, if possible, even less to that of the corresponding *language*. A definition and a couple of examples definitely won't do and neither can we rely on the everyday meaning and connotations of terms such as negative, increasing, maximum, etc. And then there are the difficulties students have such as **with** finding the value of a function at 3 **which they confuse with** finding *where* a function has the value 3 **or with**: *Where* does (a given) f have such and such feature (e.g. positive, increasing, infinite, concave up, turning down, inflecting up[2], minimal, etc)? as, these being *inverse* problems, the desired features refer to *outputs* while the request itself is for *inputs*. Another important issue is that students find it difficult to conceptualize from graphics, e.g. to see a line in a coordinate system as determining a function and to see where it has a given feature. All of this has to be checked along with the contents.

Let me thus use "graphing" rational functions as my example, if only because it is usually held to be one of the things it is almost hopeless to teach, and has, by now, all but disappeared from calculus textbooks even though there does not seem to have ever been a study of why this ought to be so. In fact, this disappearance is quite deplorable because graphing rational functions is an obvious place in which to develop "*the higher thinking skills*". The ideal test on rational functions indeed consists of the single sentence "Investigate the function $f(x) = \ldots$" followed by, taking the once fashionable "Writing across the Curriculum" theme seriously, the demand for a narrative report. But then, if a narrative such as the one I once gave here (Spring 97) should certainly rate an A, what of the less than perfect paper? There are several issues here.

First is that, at least initially, students brought up in show and tell and drill have no idea of what's required of them and, in particular, have a very hard time making the difference between what they had to write during their investigation and the report they would have to write—in (mathematical) English—to support their conclusion, i.e. the (global[3] qualitative) graph. This, of course, is not surprising given the kind of "math" exam they took before and, in fact, the extent to which a grade in mathematics can depend on an activity which far transcends mathematics is indeed

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[1] Strangely enough though, it is often those instructors doing strictly nothing but show and tell and drill who are most adamantly opposed to multiple-choice tests. But *that* is another story.

[2] I apologize for this made-up term but I find it useful.

[3] There is no such thing of course. Here, a **global** graph is the smallest graph that can be safely extrapolated.

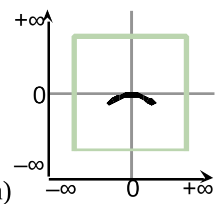
an issue. The third issue is the basis on which to rate, hopefully in a consistent manner, such reports. For instance, should one go easy on the occasional computational mistake? But what if it led the student to give a completely incorrect description of the function? Should one not, in that case, hold it against the student that s/he did not spot it, that s/he did not realize that things did not fit, did not "compute"? A fourth isssue is that—while I hope I did not myself commit outright crimes in this respect—essays leave much too much room for unconscious prejudice and/or favoritism[4]. And finally, a fifth issue is that essays do not lend themselves easily to the type of large scale information gathering necessary to establish the superiority of this or that approach in mathematics education or to validate/invalidate this or that department or to evaluate faculty.

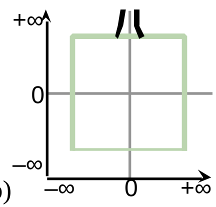
Apart from these issues, there is the preliminary matter of the knowledge and understanding *necessary* to just graph rational functions. Since I have argued (Spring 97) that it is the reliance on the sign of the derivatives to get the variation and the concavity that is at the heart of the problem while the systematic use of local (Laurent) polynomial approximations works very well for students with even only a background in *basic algebra*, it is the context I will assume here.

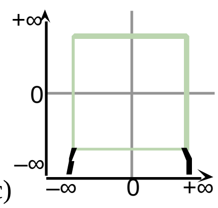
It is tempting to begin by verifying the students' proficiency in the algebra of polynomials as most students do *not* come that proficient in it. But, as soon as they see, say, that not being able to evaluate the expression $b^2 - 4ac$ also bars them from finding the zero points of a quadratic or the turning points of a cubic, etc, most students will, somehow, quickly learn what they need. *Ab initio* questions on algebra thus tend to be psychologically counterproductive.

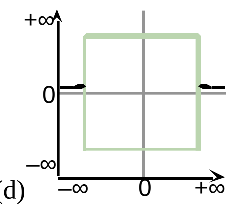
The first thing to check is complete familiarity with power functions, with positive as well as negative exponents, near 0 and near ∞. But, while we indeed should ask questions such as:

1. Which of the following is the graph of the function $f(x) = -x^{-2}$ near infinity?

(a) 

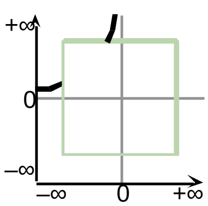
(b) 

(c) 

(d) 

(e) None of the preceeding[5].

2. Of which of the following is this a local graph of?



(a) $f(x) = -x^{+\text{even}}$ near 0.

(b) $f(x) = -x^{+\text{even}}$ near ∞.

(c) $f(x) = -x^{+\text{odd}}$ near 0.

(d) $f(x) = -x^{+\text{odd}}$ near ∞.

(e) None of the Preceeding.

we should also force students to pause and consider:

3. Which of the following is *necessarily* true of power functions of the type $f(x) = +x^{-\text{even}}$?

(a) The power of $f(x)$ is positive. (b) The power of $f(x)$ is negative. (c) The coefficient of $f(x)$ is positive. (d) The coefficient of $f(x)$ is positive. (e) None of the Preceeding.

4. Given that the (output of the) *power* function $f(x)$ is positive whether x is positive or negative, which of the following is *necessarily* true?

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[4] How would you react to a student telling you that lim sin$h$/h is 1 because sin$h$= h + (…) so that sin$h$/h = 1 + (…)? Some of my colleaugues cry "Anathema!".

[5] The choice "None of the preceeding" opens things a bit if used as correct choice as often as the other choices. Which is not always easy to achieve.

(a) The power of $f(x)$ is positive. (b) The power of $f(x)$ is negative. (c) The power of $f(x)$ is even. (d) The power of $f(x)$ is odd. (e) None of the preceeding.

And we should also check their knowledge of the technical vocabulary:

5. Find if, near 0 (resp. ∞), power functions of the form $f(x) = -x^{+\text{odd}}$

(a) Remain positive (b) Remain negative (c) Change from positive to negative. (d) Change from negative to positive. (e) None of the preceeding.

or

(a) Are increasing (b) Are decreasing (c) Turn down (= change from increasing to decreasing). (d) Turn up (= change from decreasing to increasing. (e) None of the preceeding.

or

(a) Are concave up (b) Are concave down (c) Are inflecting down (= change from concave up to concave down). (d) Are inflecting up (= change from concave down to concave up. (e) None of the preceeding.

or, from the *optimization* viewpoint,

(a) Reach a maximum. (b) Reach a minimum. (c) Are monotonic. (d) Have a saddle. (e) None of the preceeding.

and

6. Given $f(x) = -x^{+\text{odd}}$, 0 (resp ∞) is

(a) An even zero. (b) An odd zero. (c) An even pole. (d) An odd pole. (e) None of the preceeding.

but also

7. Given that the (output of the) *power* function $f(x)$ is small when x is large, which of the following is *necessarily* true?

(a) $f(x)$ is small when x is small. (b) $f(x)$ is large when x is small. (c) The power of $f(x)$ is even. (d) The power of $f(x)$ is odd. (e) None of the preceeding.

and

8. Which of the following is *necessarily* true of $f(x) = -x^{+\text{odd}}$?

(a) When $x = 0^+$, then $f(x) = 0^+$. (b) When $x = 0^+$, then $f(x) = 0^-$. (c) When $x = 0^+$, then $f(x) = +\infty$. (d) When $x = 0^+$, then $f(x) = -\infty$. (e) None of the preceeding.

The next necessary item is the ability to *expand* expressions of the form $(x_0+h)^n$ and there are two sides to this. The first is the binomial aspect, but here again, there usually isn't much trouble with, say,

9. Which of the following is *not* a term in the expansion of $(5 + h)^4$

(a) $+625$ (b) $+500h$ (c) $+600h^2$ (d) $+20h^3$ (e) None of the preceeding.

The second aspect is the orders of magnitude that are involved and *that* is more subtle:

10. Which of the following is the *second most important term* in the expansion of $(5 + h)^4$

(a) $+625$ (b) $+500h$ (c) $+600h^2$ (d) $+20h^3$ (e) None of the preceeding.

Then there is what term in a (Laurent) polynomial approximation gives a desired *feature*.

11. Given $f(+3+h) = -5h^{-2} +4h^{-1} -3 +7h -2h^4 + (\ldots)$, which *term* gives the variation of f near +3?

(a) -3 (b) $+7h$ (c) $-5h^2$ (d) $+4h^{-1}$ (e) None of the preceeding.

12. Which of the following *approximations* of $f(+3+h)$ gives the concavity of f near +3?

(a) $-4 + (\ldots)$ (b) $-1 +7h + (\ldots)$ (c) $-4 +4h^3 -2h^4 + (\ldots)$ (d) $-2h + (\ldots)$ (e) None of the preceeding.

At this point, the students must be able to *obtain* the necessary (*local*) expansions from the (*global*) rule that gives $f(x)$ in terms of x. This involves dividing in ascending powers of h for expansions near x_0 as well as in descending powers of x for expansions near ∞. To determine a local feature, they must know to *stop* the division with the first term that gives the desired feature.

13. Find if, near +3, the function $f(x) = \dfrac{x-3}{x^2-9}$ … (same choices as in **5.**)

Locating the change-points for a given feature is much more difficult in that it requires setting the coefficient of the appropriate power of h in the expansion of $f(x_0+h)$ (or the corresponding derivative) equal to 0 and solving for x_0 what often turns out to be a complicated equation. Fortunately, this is not needed to get the *essential* graph, that is the part of the graph in a window that is *forced* by the part outside the window, i.e. when either x or $f(x)$ is large.

Students however must be able to decide *where* to expand. Even though the answer is "near infinity and near the poles", this is invariably where they have the most trouble. In particular, finding the poles involves finding the zero(s) of the denominator but also expanding there and realizing that, when the expansion does not to start with a negative power, the point is not a pole but a removable singularity and therefore not of *essential* interest. Thus,

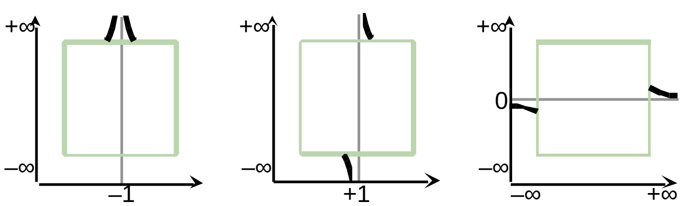
14. Given $f(x) = \dfrac{(x+2)^2}{x-2}$, $f(+2+h)$ is approximately equal to[6] …

15. When x is near ∞, the *graph* of the function $f(x) = \dfrac{x^3}{x-1}$ is approximately …

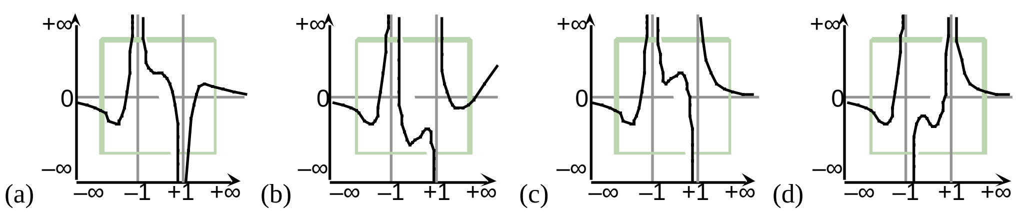
16. When x is near +3, the function $f(x) = \dfrac{x-3}{x^2-9}$ has: (a) an even zero. (b) an odd pole. …

Finally, can the students put these local graphs together into a (qualitative) global graph?

17. Given the following *local* graphs of a rational function f,



which of the following *could* be its *global* graph:



If they do, we can then ask about the *essential* global features of the function: *essential* zero points, *essential* turning points, *essential* inflection points?

Space does not allow for more but it ought to be clear by now how multiple choice questions can be used to check what needs to have have been understood *before* graphs can be attempted. Of

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[6] The notations $f(x)\big|_{x=+2+h}$ and $\left.\dfrac{(x+2)^2}{x-2}\right|_{x=+2+h}$ are, in fact, well liked by the students, probably because they hint at what to do next.

course, it is conceivable, if not likely, that, no mattter how many such questions a student is able to answer correctly, s/he may remain unable to graph a rational function on her/his own. What is likely though is that s/he would remain unable to write the narrative report to support it. Another problem is that such testing narrowly depends on the expository approach and I shall address these issues at some later time.

References

Gibbs, W. W. and D. Fox (1999). The False Crisis in Science Education. Scientific American. **281:** 87-93.

Comments, criticisms, rebuttals are most welcome and should be sent to:

Alain Schremmer
Mathematics Department
Community College of Philadelphia,
1700 Spring Garden Street
Philadelphia, PA 19130

Or e-mail SchremmerA @fast.net