Notes from the Mathematical Underground

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While this column purports to criticize the mathematics we teach as a matter of course, my friends keep reminding me that being critical doesn't go over too well in academic circles. As a result, rather than to criticize, I shall often prefer to appear "constructive" by offering alternatives. The danger however, as the same friends keep warning me with glee, is that these notes could well turn into some Gospel According To Schremmer. So, once again, let me remind the reader that this column wants to be a "forum" where to argue about what's taken to be given.

In the meantime, let me start with how we choose textbooks. As any publisher will tell you, we just open the book at three or four places to see how the author handles certain pet items of ours and, if s/he does it our way, we like the book! At this point, I find it very difficult not to indulge in heavy sarcasm. What this says though is that we take the contents of any given course for granted, if possibly up to a permutation and with a few twists and variations allowed here and there. In this, by the way, our view of mathematics is not overwhelmingly unlike that of non-mathematicians: a certain number of problems need to be covered and learning mathematics means learning the corresponding recipes. Appropriately wrapped, of course, in "theory". But just look at the exercises and/or, even better, at the exams: As has often been observed, each and every question corresponds to a key on a calculator!

However, even if I were to agree that a course consists only of teaching how to handle a given number of problems, I would nevertheless argue that the recipes that we use are not the only possible ones and that, in fact, they are not the most efficient ones. When I do so though, my interlocutors seem to fall prey to the acute discomfort usually associated with having to deal with the citizens released a few years ago from State Hospitals into the Community, i.e. onto city sidewalks. So, instead, I will sketch an alternative: To use an analogy, and if we compare the usual set of recipes to a CISC processor, what I will advocate here corresponds to a RISC processor. In other words, instead of having a complex set of instructions, essentially one for each problem, I will propose here a reduced set of instructions which can be used over and over again, in various assemblies, to deal with a large set of problems.

To take again basic arithmetic as my example, start with the idea that numbers should never be used without units: Never say 3 but 3 which we define of course as with the important proviso that 1 is the same as —*one* apple is the same as *an* apple. (How many times have you had to say that $x = 1x$?) Then, be sure to call 2 **apples** +3 **oranges** anything but a *sum*. It is bad enough that we usually cannot avoid using the symbol +. I will use the term *combination*. Thus, since in a combination + cannot be computed, + is not to be read as "plus" but as "and". In contradistinction, the *sum* 2 **apples** +3 **apples** computes to 5 **apples**. The main question then is: When can we turn a combination into a sum? The answer is that, *if* we can exchange, for example, 1 **apple** for 25 **strawberries** and 1 **orange** for 12 **strawberries**, *then* we can compute 2 **apples** + 3 **oranges** = 2 • 25 **strawberries** + 3 • 12 **strawberries** = 50 **strawberries** + 36 **strawberries** = 86 **strawberries**! This notion of *exchange*, which we already encountered in the case of the monetary combinations I discussed last Fall, namely such as **5 Franklin**, **2 Hamiltons**, **7**

Washingtons, is indeed an ubiquitous one.

Next, recall that the top of a fraction *numerates* the unit that its bottom *denominates* so that we should think of 2/3 🍎 as 2 🍎/3, that is as 2 **third-of-an-apple**. Naturally, we can exchange 🍎/3 for 2 🍎/6 so that, while 2/5 🍎 +3/7 🍎, that is 2 🍎/5+ 3🍎/7 , is a combination and cannot be computed, it can be *exchanged* for the sum 14 🍎/5•7 + 15 🍎/7•5 which computes to 29🍎/35 . Of course, sooner or later, we must actually cut the apple up and make the connection with division: 2 🍎/3 is what each one of 3 persons gets when dividing up 2🍎 even if this does not always work, as when dividing eggs.

We then need to introduce multiplication but certainly not as "repeated addition" because how would we then define • ? Note that when we *add* two numbers, we think of them as *lengths* and we visualize the addition as

3 *length* units	5 *length* units	8 *length* units
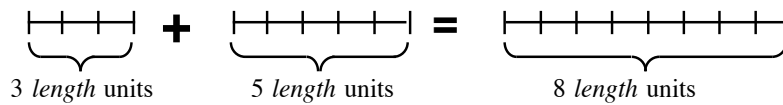
So, when we *multiply* the same two numbers, we should visualize the product as the cardinal of a cartesian product:
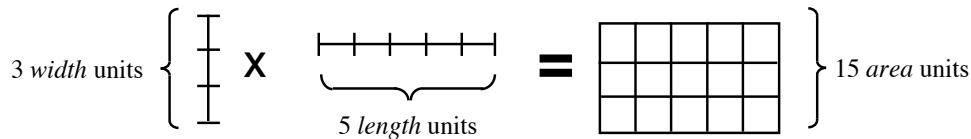
(Shades of matrix multiplication!) But then,
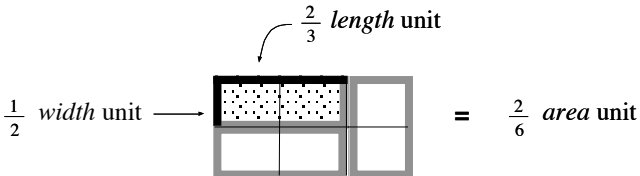
Eventually of course, multiplication by whole numbers must be identified with additive powers.

We now turn to multiplicative powers. By ax^n, we mean a stacks of n copies of x with the natural proviso that $ax^1 = ax$. That $ax^0 = a$ is not yet natural. Of course, as with $1🍎 = 🍎$, we have $x^n = 1x^n$. Clearly, $2x^4 + 3x^4 = 5x^4$:

while $2x^4 +3x^6$ is just a combination:

In arithmetic, the identification of additive powers with multiplication gives that $5 \cdot 3^4$ must be read as 5 *multiplied* by 4 copies of 3 that is as $5 \cdot 3 \cdot 3 \cdot 3 \cdot 3$. If nothing else, this prevents students having to compute 3^4 from multiplying 3 by 4. Now, $5 \cdot 3^0$ is equal to 5 since it means 5 multiplied by 0 copy of 3. Similarly, we define ax^{-n} as a/x^n, that is as *a* divided by *n* copies of *x*. This time however, we cannot omit the 1 in $1x^{-1} = 1/x^1$. The key to manipulating powers then resides in the fact that we can denote division with a negative exponent as well as with a fraction bar. Either way, we use the instruction "instead of dividing, multiply by the reciprocal" (as in "instead of subtracting, add the opposite").

And that's it! We are now equipped to justify just about any computation in Arithmetic and/or Basic Algebra. (Excluding "factoring by inspection" of course! But that's quite another story.) For instance, we get in this manner Laurent polynomials as well as the exponential notation for decimal numbers. By the way, the fact that evaluating the polynomial $5x^2 + 2x + 7 + 4x^{-1} + 3x^{-2}$ at 10 yields the number 527.43 points directly at the very heart of the Lagrangian differential calculus.

Observe that most of these "instructions" are in reality semantic definitions in that they dictate how to interpret certain terms and that, under these interpretations, the usual rules and recipes become quite natural, by which I mean that they come from the students' own non-school experience. In other words, we are operating here in a *model-theoretic* mode. Of course, RISC computations tend to be longer than CISC ones but even this can be seen as an advantage since it forces the students to concentrate on the computational strategy to be followed rather than on worrying whether they are using the right recipe. I would suggest that the reader make a "reduced set of instructions" in the subject of her/his interest and see what s/he can get out of it and I would be most interested in discussing here such attempts.

While none of the above has any originality whatsoever, this is certainly *not* how so-called elementary textbooks present these matters and I hope that it also illustrates how the way we cut up courses, e.g. separate arithmetic from basic algebra, affects the way we think about their contents. More generally, I would like to argue in some future column that the main problem students have with mathematics, and what drives them to memorization and, eventually, to distraction, is that the way things are presented in the conventional curriculum makes it completely impossible for them to see the broader picture, the overall architecture according to which these things hang together.

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In my first column, I had promised to discuss books you are unlikely to get from your friendly Publisher Representative and, indeed, if *Publish or Perish*, the publisher of Susan Bassein's *An Infinite Series Approach to Calculus*, is well known to geometers, it is not likely to have reps. But, because of space limitations, I can only hope here to entice someone to review it for the amatyc *Review* and I will limit myself to a quotation from the author's preface to the Instructor:

> In the usual development of the derivative functions defined by infinite series, the operation of finding the limit of the difference quotient $f(a+h) - f(a) / h$ as $h \to 0$ is *unnecessary*: the correct answer for the derivative of all polynomials, rational and algebraic functions (and, in fact, all real analytic functions, that is functions that can be expressed as convergent power series) can be found by performing the algebra necessary to cancel the h in the denominator and then setting $h = 0$ because the difference quotient of such functions has only a removable discontinuity at $h = 0$.

This is very close to Howard Levi's approach in *Polynomials, Power Series and Calculus* which I had mentioned and which was published in 1968 by *Van Nostrand*, now defunct but whose catalogue was picked up by *Springer Verlag*, and which might therefore reappear someday. The two treatments however could not be more different. Finally, a caveat: The approach by series

expansion is not to be confused, repeat, not to be confused, with Lagrange's approach which is by way of *asymptotic* expansions, a version of which we propounded in this *Review* a few years back.