exchange
common denominator

Chapter 1

Fractions

We now deal with the case when the two collections consist of *different* kinds of objects.

1. In that case, we just *cannot* aggregate the two collections and, on the board, we can only write them as a *combination* and that is that!

For instance, given ***dollar, dollar*** and ***dime, dime, dime***, on the counter, we can only write the combination 2 **Dollars** & 3 **Dimes**. (Even if we were to "put together" ***dollar***, ***dollar*** and ***dime***, ***dime***, ***dime*** as ***dollar***, ***dollar***, ***dime***, ***dime***, ***dime***, we would not have a *collection*: what *denominator* would we use in the number-phrase?)

2. However, *if* we can **exchange** the objects in the two collections for collections of *identical* objects, *then* we have a **common denominator** and the combination can then be "turned" into an addition.

♠ For instance, if we can change ***quarter*** for ***nickel***, ***nickel***, ***nickel***, ***nickel***, ***nickel*** and ***dime*** for ***nickel***, ***nickel***, then we can change
quarter and ***dime***, ***dime***
for
nickel, ***nickel***, ***nickel***, ***nickel***, ***nickel*** and ***nickel***, ***nickel***, ***nickel***, ***nickel***,
that is for
nickel, ***nickel***, ***nickel***, ***nickel***, ***nickel***, ***nickel***, ***nickel***, ***nickel***, ***nickel***.

❖ On the board, we proceed exactly in the same manner as on the counter and we write:

If: 1 **Quarter** $= 5$ **Nickels** and 1 **Dime** $= 2$ **Nickels**,

then: 1 **Quarter** & 2 **Dime** $= 5$ **Nickels** $+ 2(2$ **Nickels**$)$

$= 9$ **Nickels**

However, while changing facilities will often be available, we should not expect that they always will.

1. We already saw above that the same ideas continue to apply even when the rate of exchange from one denominator to the next is not always the same or does not exist as for instance when we deal with goods in the British units. But, as we also saw already there, these ideas do not work as smoothly. We return to money because, even there,
Hundred-dollar-bill
Fifty-dollar-bill
Twenty-dollar-bill
Ten-dollar-bill
Five-dollar-bill
with ***Hundred-dollar-bill***, ***Fifty-dollar-bill***, ***Twenty-dollar-bill***, ***Ten-dollar-bill***, ***Five-dollar-bill***, ***One-dollar-bill***, we observe that 1 ***Hundred-dollar-bill*** exchanges for 2 ***Fifty-dollar-bill*** but that 1 **Fifty** does not exchange for any number of ***Twenty-dollar-bill***. However, both ***Fifty-dollar-bill*** and ***Twenty-dollar-bill*** exchange for any of the following common denominators: ***Ten-dollar-bill***, ***Five-dollar-bill***, ***One-dollar-bill***.
It is a mildly interesting exercise to learn how to add and subtract under such headings but we shall turn our attention to something more fruitful which is how to introduce ever-smaller units, that is going *downward*.
This is of course what we just did above, in Systematic Arithmetic, when we defined ***dime***, ***cent*** and ***mill***. For instance, starting with ***One-dollar-bill*** as origin, 1 ***quarter*** is defined as 1 of-which-4-will-exchange-for-1-***One-dollar-bill***, where the phrase of-which-4-will-exchange-for-1-***One-dollar-bill*** is the denominator , and, say, 3 ***quarter*** is read as 3 of-which-4-will-exchange-for-1-***One-dollar-bill***.
Unfortunately, such denominators are traditionally coded in a manner that, if historical, is not very evocative of what they represent. The best way to deal with this issue is to morph one into the other:
i. 3 ***quarter*** (In which ***quarters*** is the denominator.)
ii. 3 of-which-4-will-exchange-for-1-***One-dollar-bill*** (In which of-which-4-will-exchange-for-1-***One-dollar-bill*** is the denominator.)

iii. 3 4 ? 1-***One-dollar-bill*** (In which 4 ? 1-***One-dollar-bill*** is the denominator.)

iv. (in which is the denominator.)

v. (in which is the denominator.)

vi. ***One-dollar-bill*** (in which ***One-dollar-bill*** is the denominator.)

Note that, here, the horizontal line in iv, v, and vi is merely a separator called fraction bar. Note that in v and vi there ought to be a1 before ***One-dollar-bill*** but it goes without saying.

Unfortunately, two things happened that made the use of fractions a lot less easy than it should be. The first is that, as usual, the unit, here ***One-dollar-bill***, almost always goes without saying. The second is that the 4 in the above code is called the denominator while it is only part of the denominator.

2. However, from this we can now deal easily with the usual topics regarding fractions.

a. For some reason lost in history, a fraction whose numerator is more than the denominator is deemed to be improper and to some extent inadmissible. What one is supposed to do is to change a maximum of parts for units. For instance, given 7 of-which-4-will-exchange-for-1-***One-dollar-bill***, we can exchange 4 of-which-4-will-exchange-for-1-***One-dollar-bill*** for 1 ***One-dollar-bill*** after which we have the combination 1 ***One-dollar-bill*** & 3 of-which-4-will-exchange-for-1-***One-dollar-bill***. Similarly, given 23 of-which-4-will-exchange-for-1-***One-dollar-bill***, we want to exchange as many of them for ***One-dollar-bill***s. So we must find out how many ***One-dollar-bill***s we will be getting and how many parts will remain. In other words, we divide 23 by 4 to get the number of ***One-dollar-bill*** and the number of those of-which-4-will-exchange-for-1-***One-dollar-bill*** that will remain.

Since ***One-dollar-bill***s = 5 ***One-dollar-bill***s & 3 of-which-4-will-exchange-for-1 ***One-dollar-bill***, it is natural to exchange 5 ***One-dollar-bill***s for 20 of-which-4-will-exchange-for-1 ***One-dollar-bill*** after which they have 23 of-which-4-will-exchange-for-1-***One-dollar-bill***. And vice versa.

b. When the denominators are the same, they get ***One-dollar-bill*** + ***One-dollar-bill*** = ***One-dollar-bill*** in exactly the same manner as 7 ***nickel*** + 9 ***nickel*** = 16 ***nickel***.

c. When the denominators are not the same, they must change to a common denominator before they can add: ***One-dollar-bill*** & ***One-dollar-bill*** = ***One-dollar-bill*** + ***One-dollar-bill*** = ***One-dollar-bill*** in exactly the same manner as 7 ***dime*** & 3 ***quarter*** = 14 ***nickel*** + 15 ***nickel*** = 29 ***nickel***.

3. In the case of multiplication and division, the difficulty is, again, with the denominators but a picture readily shows that, for instance, = 24 1-inch x 1-inch & 12 1-inch x 1-of-which-5-will-exchange-for-1-inch & 6 1-of-which-2-will-exchange-for-1-inch x 1-inch & 3 1-of-which-2-will-exchange-for-1-inch x 1-of-which-5-will-exchange-for-1-inch, that is, more familiarly, = 24 + 12 + 6 + 3 = 27 + + .
Note. While one might want at this point to make the students realize that , it is of no use here and, by the time they will have to use it, that is when investigating functions, they will have forgotten. Anyhow, it will be much more satisfying to do with the notion of nearness, in conjunction with .