+ (...)

Chapter 1

Real Numbers And Their Graphic Representation

Engineers like to say that the only *real* Real Numbers are the Decimal Numbers .

1.1 Decimal Approximations

Since, other than counting-numbers that were defined *directly*, all other numbers were specified *indirectly*, that is as solutions of equations that we can usually solve only approximately, we will use real numbers only as code to designate collectively all the many Decimal Numbers that are approximate solutions of the equation.

After whatever decimal number we use, though, we shall have to write **+ (...)**. read "plus a little bit too small to matter in the current situation" because the equality will not be exact . We look at a couple of examples.

Example 1. We view $\frac{11}{7}$ as a *specifying-phrase*, namely the solution of the equation $7x = 11$, that is $\frac{11}{7}$ stands for whatever decimal number 7 copies of which will *add* to 11 within the approximation required by the situation. So, we might use any one of the following.

- $\frac{11}{7} = 1 + (...)$
  because
  $7 \cdot \big[1 + (...)\big] = 7 \cdot 1 + 7 \cdot (...) = 7 + (...)$
- $\frac{11}{7} = 1.5 + (...)$
  because
  $7 \cdot \big[1.5 + (...)\big] = 7 \cdot 1.5 + 7 \cdot (...) = 10.5 + (...)$
- $\frac{11}{7} = 1.57 + (...)$

because
$7 \cdot \left[1.57 + (...)\right] = 7 \cdot 1.57 + 7 \cdot (...) = 10.99 + (...)$

- $\frac{11}{7} = 1.571 + (...)$
  because
  $7 \cdot \left[1.571 + (...)\right] = 7 \cdot 1.571 + 7 \cdot (...) = 10.997 + (...)$
- $\frac{11}{7} = 1.5714 + (...)$
  because
  $7 \cdot \left[1.5714 + (...)\right] = 7 \cdot 1.5714 + 7 \cdot (...) = 10.99998 + (...)$

Thus, in practice, depending on the circumstances, we might replace $\frac{11}{7}$ by any one of the following:
$1 + (...)$
$1.5 + (...)$
$1.57 + (...)$
$1.571 + (...)$
$1.5714 + (...)$

Example 2. Similarly, we view $\sqrt[3]{13}$ as a *specifying-phrase* standing for whatever *decimal number* is a solution of the equation $x^3 = 13$ that is, $\sqrt[3]{13}$ stands for whatever decimal number 3 copies of which will *multiply* to 13 within the approximation required by the situation. So, we might use any one of the following.

- $\sqrt[3]{13} = 2 + (...)$
  because
  $\left[2 + (...)\right]^3 = 8 + (...)$
- $\sqrt[3]{13} = 2.3 + (...)$
  because
  $\left[2.3 + (...)\right]^3 = 12.167 + (...)$
- $\sqrt[3]{13} = 2.4 + (...)$
  because
  $\left[2.4 + (...)\right]^3 = 13.824 + (...)$
- $\sqrt[3]{13} = 2.35 + (...)$
  because
  $\left[2.35 + (...)\right]^3 = 12.977875 + (...)$
- $\sqrt[3]{13} = 2.351 + (...)$
  because
  $\left[2.351 + (...)\right]^3 = 12.994449551 + (...)$

Thus, in practice, depending on the circumstances, we might replace $\sqrt[3]{13}$ by any one of the following:
$2 + (...)$,
$2.4 + (...)$,

Sign
Size
same-size
algebra-smaller
larger-in-size
algebra-larger
smaller-in-size

2.35 + (...),
2.351 + (...),
2.3513 + (...),

And, as we shall see presently, when in need to record how small (...) is, we will use a power of h instead.

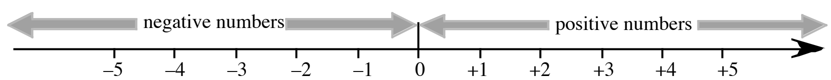
1.2 Sign and Size of a Number

Given a type of function, in order to find out under what conditions we can "join smoothly" a plot into a graph, we shall investigate the kind of outputs this type of functions returns for various kinds of inputs.

Thus, before we can start on our program, we need to look at numbers from a new point of view.

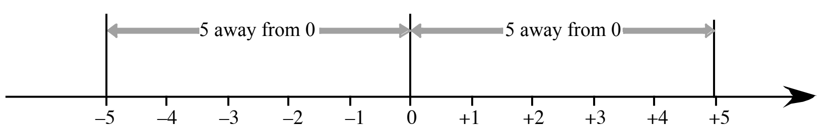
1. The **Sign** of a number is the side of 0 that the number is on the ruler. For instance,

- Sign of -7 is *negative* because -7 is *left of* 0 on the ruler.
- Sign of $+3$ is *positive* because $+3$ is *right of* 0 on the ruler.



Sign *means* Which Side of 0.

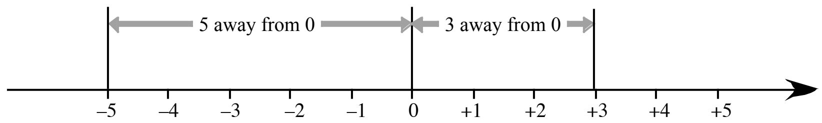
The **Size** of a number is *how far from* 0 the number is on the ruler. For instance, -5 and $+5$ are the **same-size**, namely 5, because they are both 5 away from 0:



Size *means* How Far From 0.

2. While -5 is *algebra-smaller* than $+3$ (see Section **??**), -5 is **larger-in-size** than $+3$ because -5 is *further away from* 0 than $+3$: -5 is 5 away from 0 while $+3$ is only 3 away from 0.
Similarly, while $+3$ is *algebra-larger* than -5, (see Section **??**), $+3$ is **smaller-in-size** than -5 because $+3$ is *closer from* 0 than -5: $+3$ is 3 away from 0 while -5 is 5 away from 0.

small-in-size
algebra-small
h
large-in-size
algebra-large
algebra-between



Larger/Smaller In Size *means* Farther from/Closer to 0.

3. We will have to distinguish:

- Numbers that are **small-in-size**, as opposed to numbers that are *algebra-small*. We shall use the letter h to stand for inputs that are *small-in-size*.
- Numbers that are **large-in-size**, as opposed to numbers that are *algebra-large*. Unfortunately, there is no generally accepted letter to stand for inputs that are *large-in-size*.

To an extent, *small-in-size* and *large-in-size* are relative concepts. For instance, whether a *gain* or a *loss*, ten-thousand dollars is a small sum of money for people like Bill Gates or George W. Bush while to "the rest of us" ten-thousand dollars is large enough. However, the words have one definite meaning that is the same for everybody. For instance, nobody likes losing a large sum of money. Bill Gates or George W. Bush wouldn't like to lose a million dollars just the way "the rest of us" wouldn't like to lose ten-thousand dollars. It is just the *cutoff point* that varies from people to people.

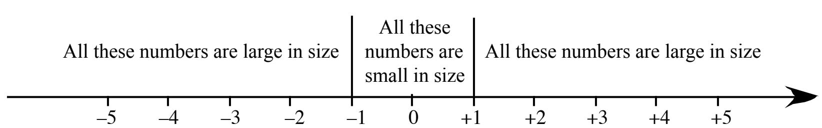
4. Still, for our purpose, it will be necessary to make the concepts a bit more precise. We observe that:

- Any number of copies of an original whose Size is *equal* to 1, and that is only either the number -1 or the number $+1$, multiply to a result which is the exact *same-size* as the original.

On the other hand,

- Any number of copies of an original whose Size is *larger* than 1, that is numbers either *algebra-smaller* than -1 or *algebra-larger* than $+1$, multiply to a result which is *larger-in-size* than the original.
- Any number of copies of an original whose Size is *smaller* than 1, that is numbers that are **algebra-between** -1 and $+1$, multiply to a result which is *smaller-in-size* than the original.

We thus have:

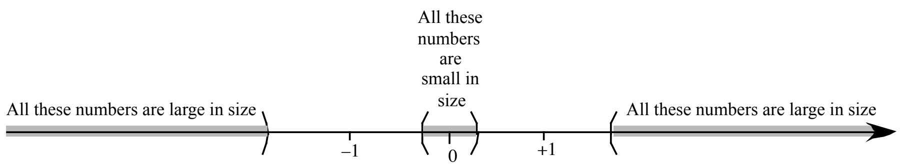


Small In Size and Large In Size (Strictly).

For instance, we could treat -1.1 as a *large-in-size* number because, say, three copies of -1.1 multiply to -1.331 which is *larger-in-size* than the original, -1.1.
Similarly, we could treat that $+0.2$ as a *small-in-size* number because, say, three copies of $+0.2$ multiply to $+0.008$ which is *smaller-in-size* that the original, $+0.2$.
Strictly speaking, this is the only thing we will need to know about a number to decide whether to treat it as *large-in-size* or *small-in-size*.

in-the-ones
order of magnitude
in-the-hundreds
by ... order(s) of magnitude
in-the-tenths

5. To make things easier, though, we will stay way away from -1 and $+1$ as in



Small In Size and Large In Size (Safely).

where the parentheses indicate that the endpoints are "open to change", according to the circumstances.

6. In fact, practically, we will "think" of the endpoints as being -10, -0.1, $+0.1$, $+10$ and this for the following reason.
Consider -7 and $+2$. They are both **in-the-ones** and therefore of the same **order of magnitude**. Multiplying, say, three copies of each shows that they are both *large-in-size*.
However,

- $+8$, the third power of $+2$, is *in-the-ones* and therefore of the *same* order of magnitude as the original.
- -343, the third power of -7, is **in-the-hundreds** and therefore of a *larger* order of magnitude than the original.

So, there is a *qualitative* difference between -7 and $+2$: While both are *large-in-size*, their third powers are *not* of the same order of magnitude. We shall say that while -7 and $+2$ are both *large-in-size*, they are so *by different orders of magnitudes*.

Similarly, consider -0.7 and -0.2. They are both **in-the-tenths** and therefore of the same *order of magnitude*. Multiplying, say, two copies of each shows that they are both *small-in-size*.
However,

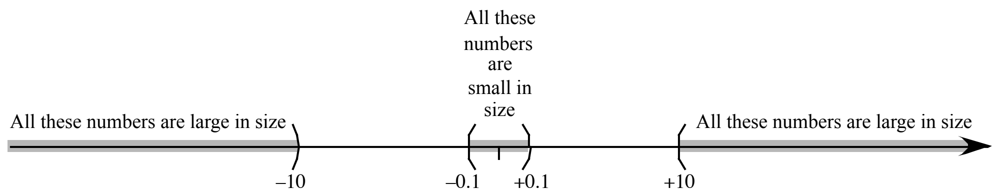
- $+0.49 = +0.5 + (...)$, the second power of -0.7 is *in-the-tenths* and therefore of the same order of magnitude as the original.

in-the-tens
in-the-thousands
in-the-ten-thousandths
small
large
near 0
near ∞

- $+0.04$, the second power of -0.2, is *in-the-hundredths* and therefore of a *smaller* order of magnitude than the original.

So, there is a *qualitative* difference between -0.7 and -0.2: While both are *small-in-size*, their second powers are *not* of the same order of magnitude. We shall say that while -0.7 and -0.2 are both *small-in-size*, they are so *by different orders of magnitudes.*

Thus, finally, the endpoints that we shall use in practice are:



Small In Size and Large In Size (Practically).

What this will do is to ensure that:

- Copies of a large-in-size original will multiply to results that are not only larger-in-size than the original but are so *by an order of magnitude.*
- Copies of a small-in-size original will multiply to results that are not only smaller-in-size than the original but are so *by an order of magnitude.*

Thus for instance, consider $+20$ and -70 which are both **in-the-tens**. When we multiply 2 copies, we get $(+20)(+20) = +400$ which is *in-the-hundreds* while $(-70)(-70) = +4900$ which is **in-the-thousands**. Of course, we still have a qualitative difference but, at least, both are *larger* than the originals *by an order of magnitude.*

Similarly, consider -0.02 and -0.07 which are both *in-the-hundredths.* When we multiply 2 copies, we get $(-0.02)(-0.02) = +0.0004$ which is **in-the-ten-thousandths** while $(-0.07)(-0.07) = +0.0049 = +0.005 + (...)$ which is *in-the-thousandths.* Again, we still have a qualitative difference but, at least, both are *smaller* than the originals *by an order of magnitude.*

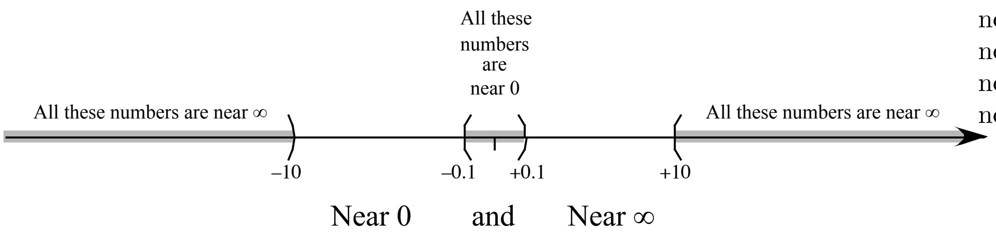
From now on, we shall use:

- **small** as short for *small-in-size*
- **large** as short for *large-in-size*

We shall also use the following language:

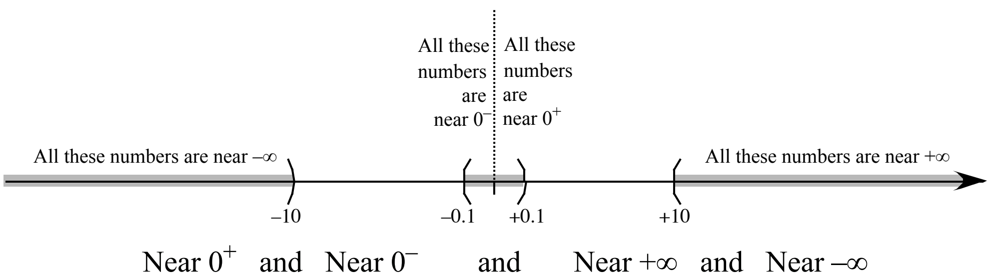
- x is **near** 0 will mean the same as x is *small* (in size).
- x is **near** ∞ will mean the same as x is *large* (in size).

near 0^+
near 0^-
near $+\infty$
near $= \infty$



and, when we want to include the sign,

- x is **near** 0^+ will mean the same as x is *small* (in size) and *positive*.
- x is **near** 0^- will mean the same as x is *small* (in size) and *negative*.
- x is **near** $+\infty$ will mean the same as x is *large* (in size) and *positive*.
- x is **near** $= \infty$ will mean the same as x is *large* (in size) and *negative*.



For most practical purposes, we can think of

- large numbers as positive powers of 10, possibly dilated by a single digit number such as, for instance, $3 \cdot 10^{+5} = 300000$ but also even $3 \cdot 10^{+1} = 30$,
- small numbers as negative powers of 10, possibly dilated by a single digit number such as, for instance, $3 \cdot 10^{-5} = 0.00003$ but also even $3 \cdot 10^{-1} = 0.3$,
- finite numbers as the zero power of 10 such as, for instance, $3 \cdot 10^{+0} = 3$.

1.3 The Arithmetic of Signs and Sizes

It will be crucial to be able to operate with numbers in relation to multiplication and division on the sole basis of their sign or size.

1. Signs behave according to the so-called "rule of signs":

$$
\begin{aligned}
+ \text{ times } + &= + \\
+ \text{ times } - &= - \\
+ \text{ times } + &= - \\
- \text{ times } - &= +
\end{aligned}
$$

and

$$
\frac{+}{+} = + \qquad \frac{+}{-} = - \qquad \frac{-}{+} = - \qquad \frac{-}{-} = +
$$

2. Sizes behave logically and we must be careful about that logic.

a. Sizes behave obviously in the following cases:

$$\text{finite} \times \text{large} = \text{large}$$
$$\frac{\text{finite}}{\text{large}} = \text{small}$$
$$\text{finite} \times \text{small} = \text{small}$$
$$\frac{\text{finite}}{\text{small}} = \text{large}$$
$$\text{large} \times \text{large} = \text{large}$$
$$\frac{\text{large}}{\text{small}} = \text{large}$$
$$\text{small} \times \text{small} = \text{small}$$
$$\frac{\text{small}}{\text{large}} = \text{small}$$

b. However, sizes do not behave that simply in the following cases:

$$\text{large} \times \text{small} = ???$$
$$\frac{\text{large}}{\text{large}} = ???$$
$$\frac{\text{small}}{\text{small}} = ???$$

This is because here the matter of how small is small and how large is large comes in. For instance, the following three are all instances of large × small.

picture
ruler

We have

$$2000000. \times 0.0003 = 600.$$

but

$$20000. \times 0.00000 = 0.06$$

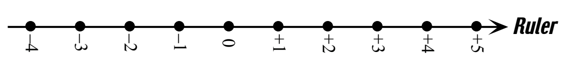
and

$$200000 \times 0.00003 = 6.$$

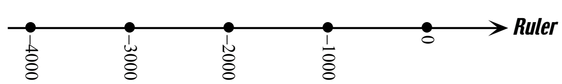
That is, this is a matter of order of magnitude and we will deal with this before we start investigating RATIONAL FUNCTIONS.

1.4 Rulers

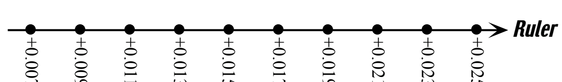
Up until now, we have *represented* numbers by writing *signed-number-phrases* but, quite often, it will be very convenient also to **picture** numbers. To this purpose, we shall use **rulers** by which we mean essentially what goes by that name in the real world, like, for instance,



or

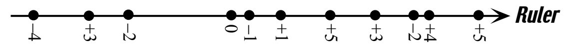


or



as opposed, for instance, to :

number lines
extremities
extent



which, for a variety of reasons, is *not* a ruler.

Note. In school environments, rulers are usually called **number lines**.

1.4.1 How Well Rulers Represent Numbers

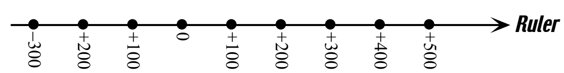
Given a bunch of numbers, it is too much to expect to be able to picture these numbers on any *given* ruler. What is not as immediately obvious is that, given a bunch of numbers, it is not always possible to come up with a particular ruler on which we can picture them.

In order to discuss this, we need to introduce some terminology.

1. By the **extremities** of a ruler, we shall mean the *smallest* number and the *largest* number tick-marked on the ruler. Thus, to picture a given bunch of numbers, the first requirement is that all the numbers in the given bunch fall

the extremities of the ruler encompass the *smallest* and the *largest* of the bunch of numbers.

For instance, if we want to *picture*, say, a bunch of numbers whose largest number is +470 and whose smallest number is −284, we need a ruler whose *extremities* are −300 and +500:



On the other hand, we cannot picture numbers such as $-1000, +700, -1400$ on the above ruler: for numbers to be pictured on a given ruler, they have to fall in-between the *extremities* of the ruler.

2. The second problem is that even if a number is between the extremities of a ruler, it may not be able to picture it if doesn't happen to be one of the tick-marks.

By the **extent** of a ruler, we shall mean the *distance* between the *extremities.* The problem is that, if we use a ruler with a large extent, that is if the distance between the smallest and largest numbers tick-marked on the ruler is large, the tick-marks cannot be consecutive numbers so that the second problem is that there may not be any tick-mark for the number to be pictured.

For instance, it is impossible to represent on a ruler the numbers 200, +400, +500 and +501.

scale
subdivide
tick-mark

The concept that is involved here is the **scale** of the ruler, that is, the amount of blackboard space used to represent one unit. In other words, this is the ratio of the *length* of the ruler to that of its *extent*.

Find the scale of the following rulers, both 5 inches in length.

Since the length of the extent is 500, the scale is 5500 = 1100 . This is a relatively small scale.

Since the length of the extent is 0.5, the scale is 50.5 = 101 . This is a relatively large scale.

Thus, a small scale is used to represent numbers that are spread far apart while a large scale is used to represent numbers that are clustered close together.

Represent the following numbers on a ruler: 1.1, 2.7, 5.1, 6.1, 7.8

We can use a 5 inch ruler extending from 1.0 to 8.0 that is tickmarked every 0.1

Represent the following numbers on a ruler: 297, 302, 302, 306, 312.

We can use a 5 inch ruler extending from 295 to 315 that is tickmarked every 1.

3.

scale of a ruler extent of a ruler

For instance, given a ruler determined by 20 and 30

figure

we can picture the number +60 by translating

For instance, to picture the numbers 180, +20, +160, +490, on a ruler, we would need, say a five inch ruler tick-marked every 10 from 200 to +500.

figure

The other way is to **subdivide** as follows.

Draw another ruler through the first tick-mark,

figure

translate as many time as you want to subdivide

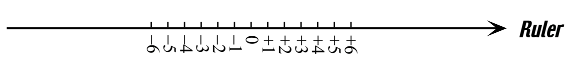
figure

draw a line through the second tick-marks and parallel lines through the intermediate tick-marks

Here again, we are obviously limited in how many times we can subdivide.

which is a straight line drawn on the blackboard with **tick-marks**

marked on it to *picture* numbers. For instance, to picture the number +4 on the ruler we start from 0, we count up to +4 on the side of +1.



1. One problem is that we may not be able to represent all the numbers we have on the same ruler

2.

3. Finally, another problem has to do with our ability to separate and distinguish inputs on a ru-ler and that is measured by the resolution of a ruler, namely the reciprocal of the length bet-ween two con-secutive tick-marks.

The following rulers all have the same length, approximatively 2 1/2 inches.

TABLE

Observe that there seems to be a sort of conservation of information: for rulers of a given length, the larger we make the scale, the higher the resolution but the smaller the extent. Thus, given that we have only so much space on a page as this one, the larger the extent we need, the smaller the scale will have to be and the lower the resolution will be.

Note. Changing scales often raises problems. For instance, railroad modelists are well aware that, if they scale the speed of a model train the same way as they scale the train itself, the resulting speed feels very wrong. Problems of scale are the object of Similitude Theory but, in this text, we shall mostly avoid such problems.

One difficulty though that we shall not be able to avoid is that it is not always possible to adjust the scale for the extent to include all the numbers that we are interested in because, then, the resolution might become too low for us to see what we are doing. In some cases though, it will be possible to work around the problem.

EXAMPLE. Represent 1, 2, 3, 4, and 197, 198, 199, 200, on a ruler. In order to distinguish consecutive numbers, we need to separate them by, say, 1/16 of an inch. But then, we need one inch to represent the numbers from 1 to 16 and twelve inches to represent the numbers from 1 to 192 and so, at that scale, it is impossible to represent the given numbers on a page like this one. On the other hand, if we are not interested in any of the numbers between 4 and 197, we can, in a way, repre-sent the given numbers and even use a larger scale:

The scale here is about 1/2 inch between two consecutive numbers but the dotted part of the ruler is at a much smaller scale.

In other cases, we shall have to use different scales in different parts of the ruler. EXAMPLE. Represent 0.01, 0.02, 0.03, 0.04, 0.05, 1 000, 2 000, 3 000, 4 000, 5 000 on a ruler.

In order to represent 0.01, 0.02, 0.03, 0.04, 0.05, we need to separate them by, say, 1/16 of an inch but then, at this scale, we will not be able to represent the large numbers. And if we separate 1 000, 2 000, 3 000, 4 000, 5 000 by even two inches, we will not be able to represent the small numbers. On the other hand, if we are not interested in any other numbers, we can use two different scales:

The scale of the heavy line between the parentheses is much larger than the scale of the rest of the ruler. It is as if we were looking at this part of the ruler through a microscope.

When two numbers are on a ruler, we can see that they are different but when numbers get so big that they do not appear on the ruler, we cannot see that they are different and, for all practical purposes, their difference is irrelevant.

Any number that can be marked on the ruler will be said to be finite and any number too large to be marked on the ruler will be said to be infinite that is, too large to be taken into consideration.

For instance, given the following ruler,

numbers such as 180, +20, +160, +490 will be considered infinite,

1.4.2 How to Construct a Ruler

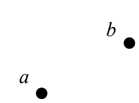
The question of how a ruler is *constructed* is an interesting one but not a crucial one in this context because purely a GEOMETRY problem. So, this section can be safely omitted.

straightedge
compass
solid angle
solid angle

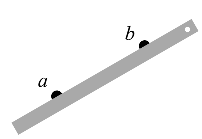
Rulers are usually constructed with a **straightedge** and a **compass**as *tools* but, as we shall discuss presently, this is a something of an overkill and here we shall use a *straightedge* and, instead of a *compass*, a **solid angle**.

Before we can construct a *ruler*, though, we shall describe three basic constructions done with just a *straightedge* and a **solid angle** that we shall use over and over again:

A. Draw a straight line through two given points.

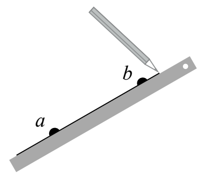


1. We place the *straightedge* through the two points

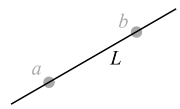


ruler
tick-marks

2. We draw a straight line along the *straightedge*:



3. The straight line resulting from construction **A**:

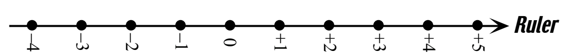


B. Mark a point at the intersection of two straight lines.

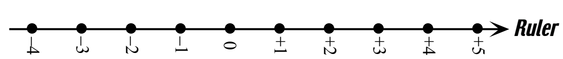
C. Draw a straight line through a given point that is parallel to a given straight line.

on a **ruler**, which, given any two numbers, is a straight line drawn on the blackboard with two **tick-marks**, one for each one of the two numbers.

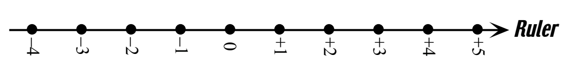
For example, the two numbers -5 and $+3$ give us the following *ruler*



Of course, we usually need more than two tick-marks but once we do have two tick-marks, we can always tick-mark any other number and, for instance, the above ruler could look like



and should we need to tick-mark $+2.5$, the ruler would then look like

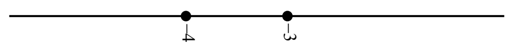


1. We begin with the case where the two given numbers are *consecutive whole numbers*, say -4 and -3.

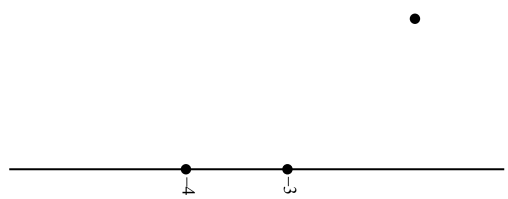
i.

auxiliary point
parallel

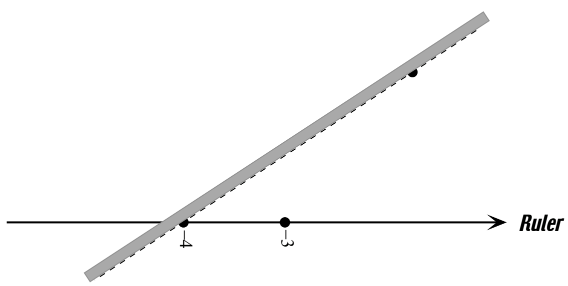
ii. We *pick* two points on the straight line and label them -4 and -3.



iii. We pick an **auxiliary point** away from the straight line

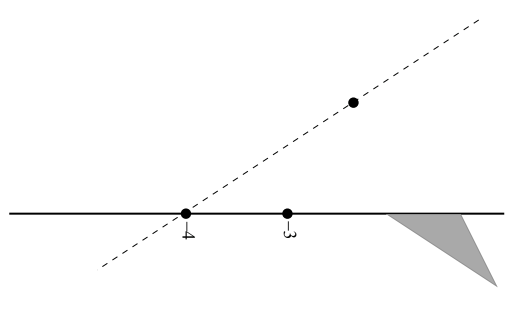


iv. We use the straightedge to draw a straight line through the point -4 and the auxiliary point

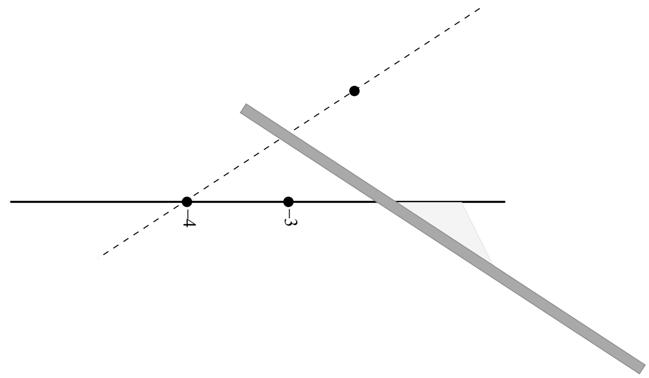


2. We draw a line **parallel** to the *straight line* through the *auxiliary point*:

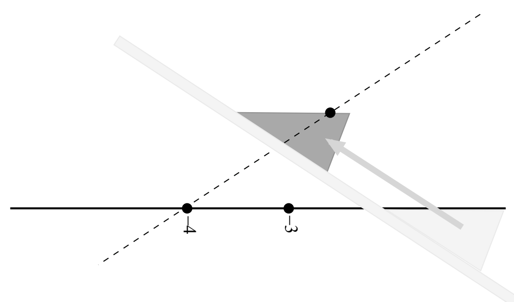
a. we place the solid triangle along the ruler



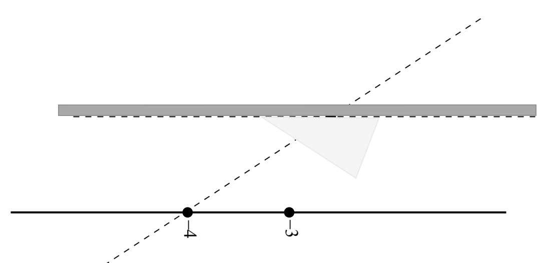
b. we place the straightedge along the other side of the solid triangle



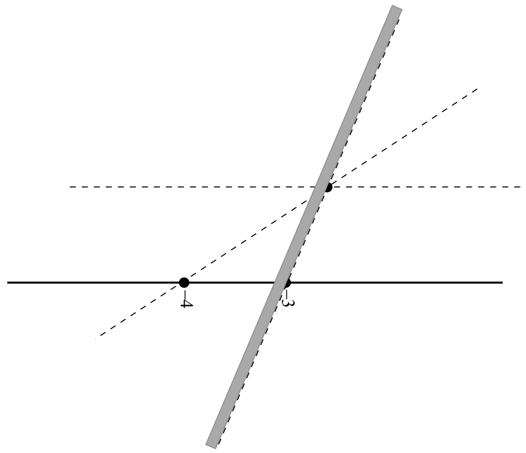
c. we slide the solid triangle until its other side goes through the auxiliary point



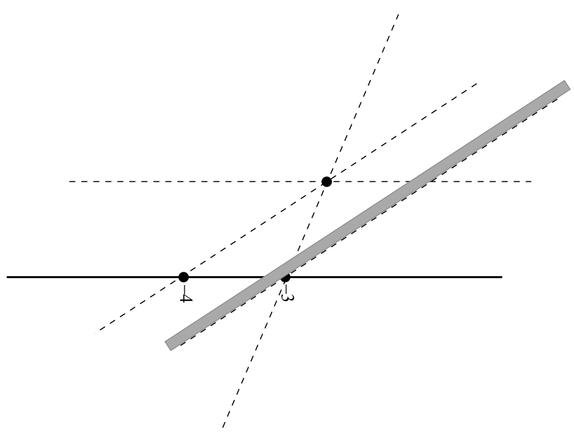
d. we use the straightedge to draw a line parallel to the ruler through the auxiliary point



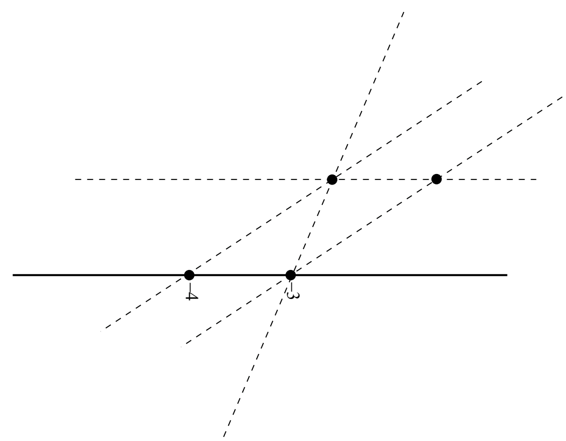
3. we use the straightedge to draw a straight line through the point -3 and the auxiliary point parallel



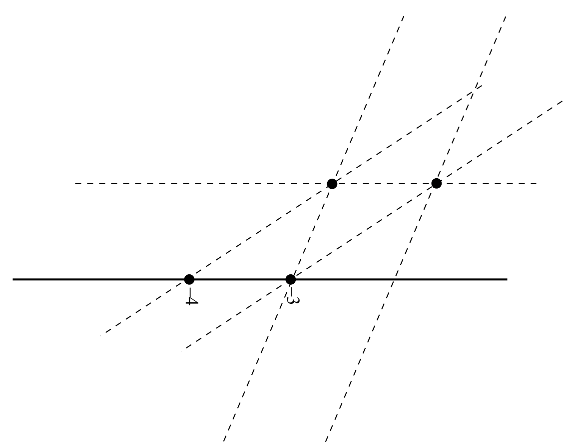
4. we draw a line **parallel** to the *straight line* through the *auxiliary point*:



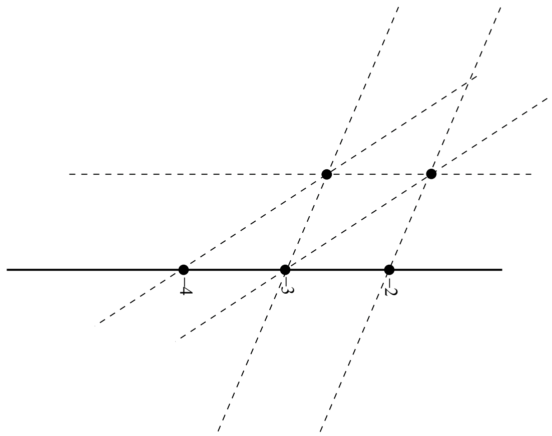
5. we mark the point at the intersection:



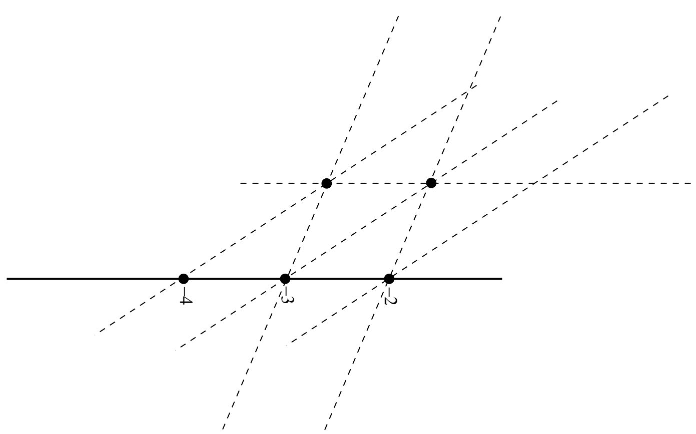
6. we mark the point at the intersection:



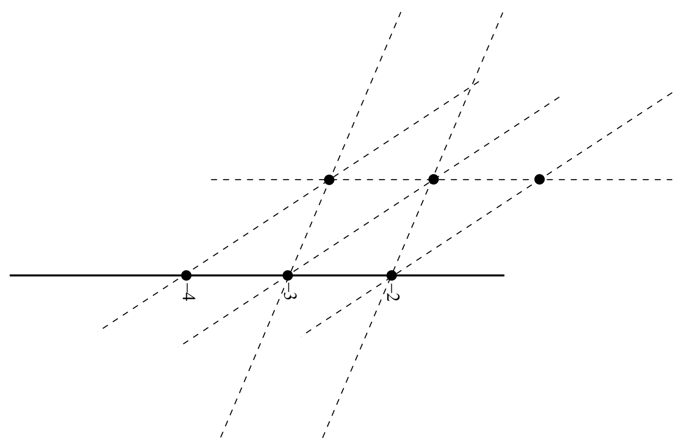
7. we mark the point at the intersection:



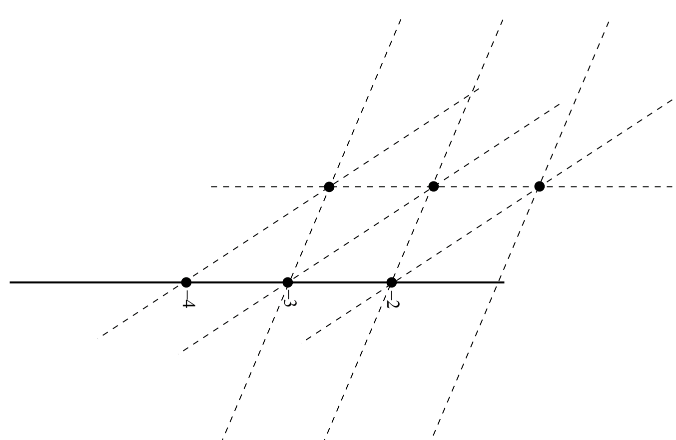
8. we mark the point at the intersection:



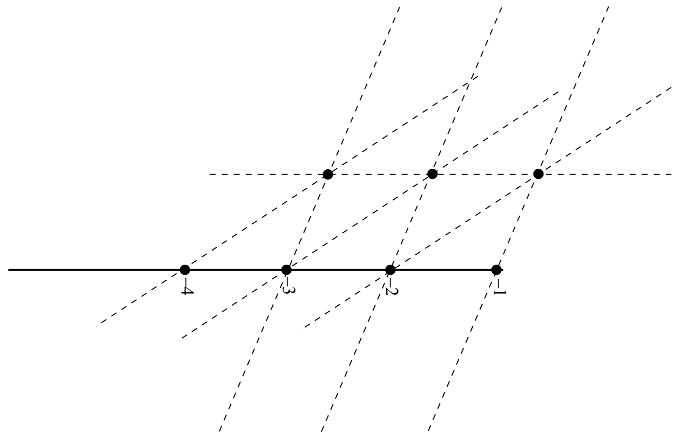
9. we mark the point at the intersection:



10. we mark the point at the intersection:



11. we mark the point at the intersection:



translate
window
screen
first ruler
second ruler
grey-space

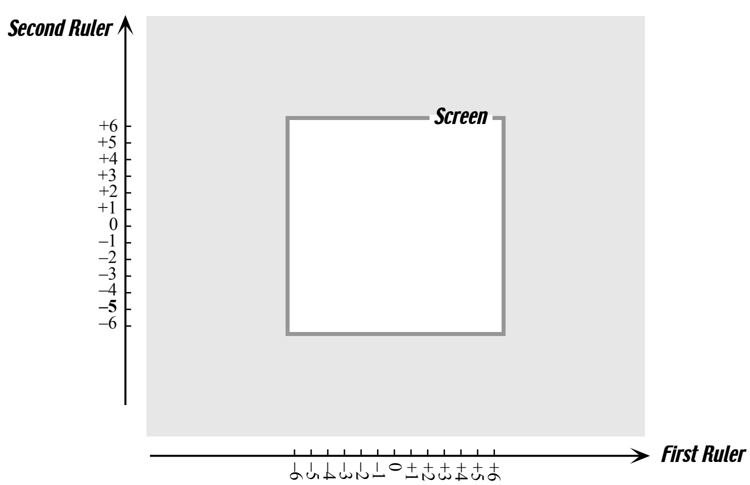
One way is to **translate** the distance between the two tick-marks.

Obviously we are limited as to how many times we can *translate* the two tick-marks by the blackboard length of the ruler and thus

1.5 Windows

In order to *picture* a *number-pair*, we will need a **window** which consists of:

- a **screen**
- a **first ruler** placed under the *screen*,
- a **second ruler** placed left of the *screen*,
- **grey-space** between the *screen* and the *rulers*. We shall see in what it is for.)



first-number point
first-number level line
second-number point
second number level line
plot point
good picture

Note. The above arrangement is quite *arbitrary*. In particular, the rulers need not go from left to right and from bottom to top. More generally, for instance, the rulers need not be at a 90 degree angle.

Then, for instance, to represent the *number-pair* $(-2, +5)$, (see Figure below),

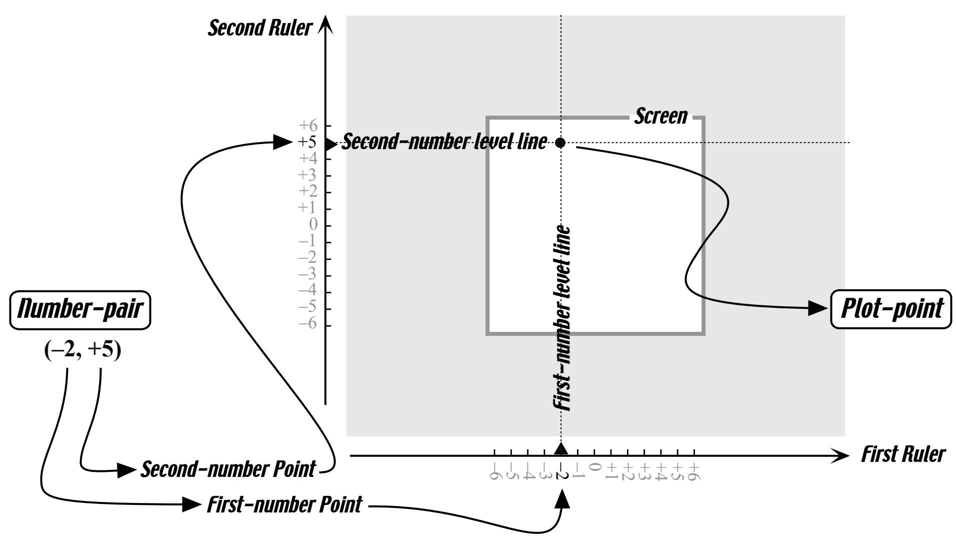
i. we represent the *first number* in the pair, -2, by a **first-number point** on the *first ruler*,

ii. we draw the **first-number level line**—a *vertical* line, through the *first-number point*,

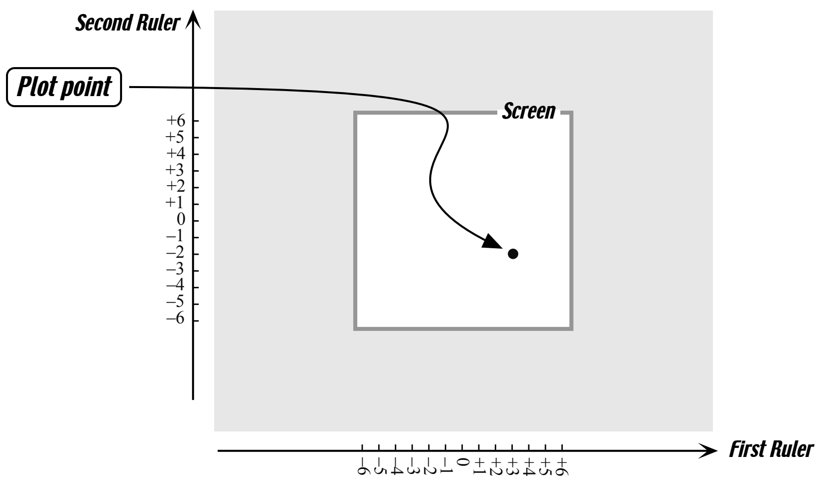
iii. we represent the *second number* in the pair, $+5$ by a **second-number point** on the *second* ruler,

iv. we draw the **second number level line**—a *horizontal* level line, through the *second-number point*,

v. Then, the **plot point**, that is the point that represents the *pair* $(-2, +5)$ on the *screen* is where the first number level line and the second-number level level line intersect.



Observe that a *plot-point* is a **good picture** of a *number-pair* because, once we have drawn the plot-point, we can erase the two number points as well as the two level lines *without loss of information*. In other words, given, for instance, the following *plot-point*



we can recover the *number-pair* of which this plot-point is the picture. All we have to do is to go backwards through the above steps:

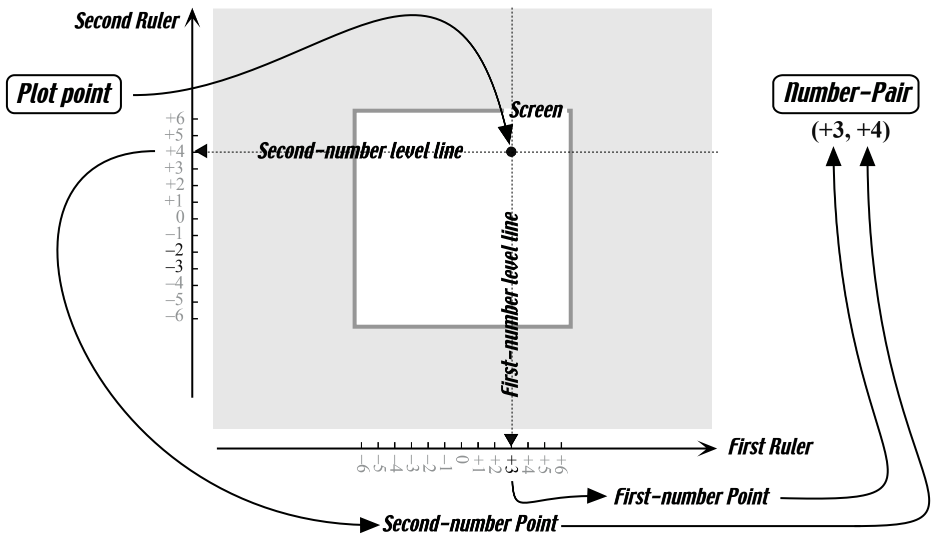
i. we draw the *first-number level line* (vertical) through the given plot point,

ii. the point that represent the *first number* in the pair is where the first-number level line intersects the first-number ruler,

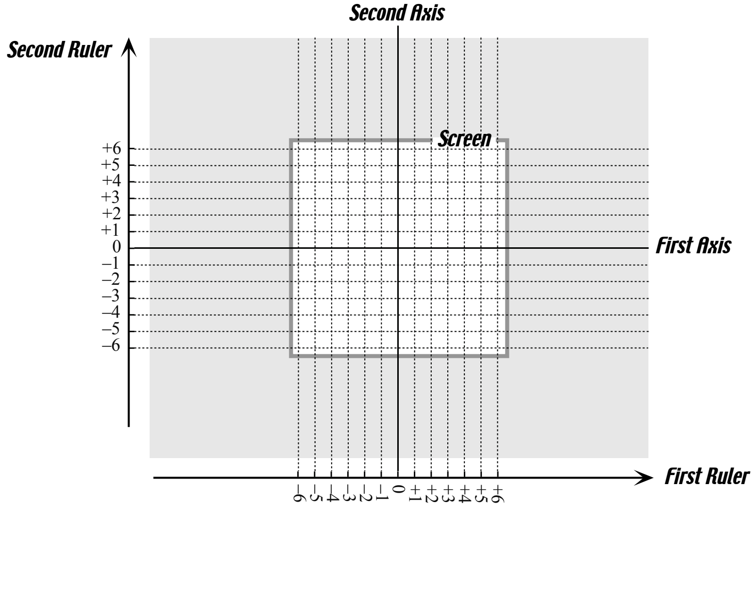
iii. we draw the *second-number level line* (horizontal) through the given plot point,

iv. the point that represent the *second number* in the pair is where the second-number level line intersects the second-number ruler.

graph-paper



Sometimes, to facilitate plotting, the window comes as **graph-paper**, that is already equipped with level lines that form a grid:



first-axis
second-axis
infinite
finite
infinitesimal

Note. Very often, the level lines for 0 are singled out as for instance on the graph paper and are called **first-axis** and **second-axis**. Sometimes, only the *axes* are drawn without the rest of the graph paper and/or the rulers. We shall always draw the rulers but, when doing *qualitative* investigations, we will only draw those levels lines that are relevant to the investigation.

We will distinguish:

- **infinite** is what cannot be seen because it is outside the window regardless of the *extent* of the ruler. This is most of the ruler.
- **finite** is what can be seen. So, first it has to be in the window and then it has to appear as a tick-mark on the corresponding ruler. Thus, whether something is finite depends on the *extent* of the ruler and then on the *resolution* of the ruler. See Section 1.4
- **infinitesimal** is what cannot be seen in the window regardless of the *resolution* of the ruler

First we note that:

- When *positive* numbers get smaller and smaller *algebraically* they also get smaller and smaller *in size*:

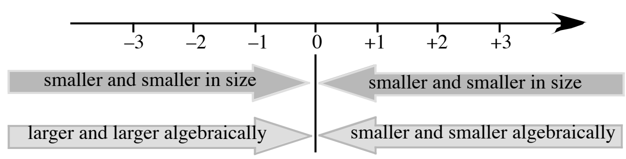


Figure 1.1: Smaller and Smaller In Size.

- But when *negative* numbers get smaller and smaller *algebraically*, they get *larger and larger in size.*

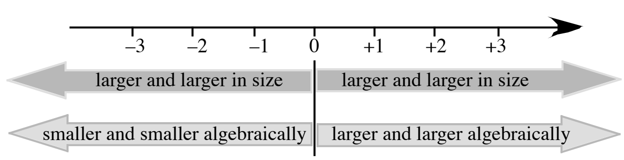


Figure 1.2: Larger and Larger In Size.

- When positive numbers get larger and larger algebraically, they get larger and larger in size.
- When negative numbers get larger and larger algebraically, they get smaller and smaller in size.