form
data
specifying-phrase
specify

Chapter 1

Algebra

1.1 Unspecified Numbers

1. A **form** is a phrase in which not all the information has been filled in but with box(es) in which to enter the missing information. A **data** is given information to be entered in the relevant box(es) of the form. So, a *form* together with *data* will produce a **specifying-phrase**.
For instance, "The President of the United States in $\boxed{}$" is a *form* that, with the *data* "1998", produces the *specifying-phrase*:

The President of the United States in $\boxed{1998}$

Once we find out that this was

Bill Clinton

we can write the *sentence*:

The President of the United States in $\boxed{1998}$
is
Bill Clinton

Thus, a specifying-phrase is a phrase that **specify** somebody/something but without necessarily knowing who/what that body/thing is. Specifying-phrase are much used in mathematics but in a somewhat different *format* that we shall now introduce.

a. In mathematics, we could use *forms* just as we did above as, for instance, in the following example:

$$\frac{\boxed{}^2 - 13}{\boxed{} - 2}$$

brackets
parentheses
:=

where the boxes are meant to be filled in with *copies* of a number to be *given* by the *data*. Say the *data* is "5". So we *write* copies of 5 within the boxes:

$$\frac{\boxed{5}^2 - 13}{\boxed{5} - 2}$$

and, after we have *computed*

$$\frac{\boxed{5}^2 - 13}{\boxed{5} - 2} = \frac{25 - 13}{5 - 2} = \frac{12}{3} = 4$$

we can write the *sentence*

Together, the *form* $\frac{\boxed{}^2-13}{\boxed{}-2}$ and the *data* 5 specify the number 4.

b. Boxes, though, are quite difficult both to type and to type *in* and so we omit the top and bottom of the boxes and type **brackets** to fake the sides of the boxes or **parentheses** to fake circular boxes. Parentheses are actually much more common.

$$\frac{[\quad]^2 - 13}{[\quad] - 2} \qquad\qquad \frac{(\quad)^2 - 13}{(\quad) - 2}$$

Say the *data* is 5. Then we *write* copies of 5 within the parentheses:

$$\frac{(5)^2 - 13}{(5) - 2}$$

c. This would still be awkward to use in *algebra*. What we will do *there* is to use, instead of brackets or parentheses, a letter, usually x. So we write the above *form* as

$$\frac{x^2 - 13}{x - 2}$$

and the *instruction* to enter the *data* 5 as

$$\Big|_{\text{where } x:=5}$$

where the symbol **:=**, borrowed from a computer language called Pascal, is read as "is to be replaced by".

Altogether, the *specifying-phrase* is: substitute

$$\left.\frac{x^2-13}{x-2}\right|_{\text{where x:=5}}$$

Now, whereas we *entered* copies of the data *within the parentheses*, we shall now **substitute** copies of the data *for the letter* x wherever x appears. We thus write

$$\left.\frac{x^2-13}{x-2}\right|_{\text{where x:=5}} = \frac{5^2-13}{5-2} = \frac{25-13}{5-2} = \frac{12}{3} = 4$$

d. When substituting a *signed*-number, it is usually safer to surround it with parentheses!
For instance, if the data is -5, we write

$$\left.\frac{x^2-13}{x-2}\right|_{\text{where x:=-5}} = \frac{(-5)^2-13}{(-5)-2} = \frac{+25-13}{-5-2} = \frac{+12}{-7} = -\frac{12}{7}$$

because *not* surrounding -5 with parentheses would give us:

$$\frac{-5^2-13}{-5-2} = \frac{-25-13}{-5-2} = \frac{-38}{-7} = +\frac{38}{7}$$

e. The letter x alone can be considered as a *form* and, indeed,

$$x|_{\text{when } x:=5} = 5$$

This is often "abbreviated" as $x = 5$ but, as usual when cutting corners, and as we shall see when we deal with *equations*, this is a bit of an abuse of language and creates an *ambiguity.*

f. On the other hand, in order to conform with usual mathematical practice, and even though technically this is not correct, we *shall* use "when $x =$" instead of "where $x :=$". (The word "when" in fact goes usually without saying but *we shall* say it.) For instance, we shall write

$$\left.\frac{x^2-13}{x-2}\right|_{\text{when x=5}} = 4$$

g. Quite often, the instruction to enter the *data*, say $|_{\text{when } x=+5}$, will remain unstated for a while during which time we will think of, and operate with, the *form* as if it were a *number-phrase.* This can save us a lot or work. Suppose for instance that we had a form , say $\frac{x^2-9}{x-3}$, and a whole lot of data. With, say the data $|_{\text{when } x=+5}$ we would go the following route:

commutative diagram

$$\frac{x^2-9}{x-3} \longrightarrow \left.\frac{x^2-9}{x-3}\right|_{\text{when x=5}} \longrightarrow \frac{25-9}{5-3} \longrightarrow 8$$

with the last step being arithmetically complicated.
On the other hand, we could have gone the following route:

$$\frac{x^2-9}{x-3} \longrightarrow x+3 \longrightarrow x+3|_{\text{when x=5}} \longrightarrow 8$$

with the first step being algebraically complicated but the last step arithmetically simple. However, in case we have a whole lot of datas, the first step will have been done once and for all and the second route will thus result in an economy with, the larger the number of datas, the larger the economy.
It is customary to put the two routes in the shape of a **commutative diagram**:

$$\begin{array}{ccccc} \frac{x^2-9}{x-3} & \xrightarrow{\textit{enter data}} & \left.\frac{x^2-9}{x-3}\right|_{\text{when x=5}} & \longrightarrow & \frac{5^2-9}{5-3} \\ \textit{algebra}\Big\downarrow & & & & \Big\downarrow\textit{arithmetic} \\ x+3 & \xrightarrow{\textit{enter data}} & x+3|_{\text{when x=5}} & \longrightarrow & 8 \end{array}$$

Altogether then, we may think of a *form* as a "partially-specified" number-phrase with which it can be useful to compute *algebraically*.

2. Not only can we have unspecified *numerators* but we can also have unspecified *denominators*.
For instance, just like we can have

x **Apples** $+ 5$ **Apples** $= (x+5)$ **Apples**

we can also have

3 **Things** + 5 **Things** = 8 **Things**

We can even have combinations

3 **Gizmos** & 5 **Things**

and in fact we can even think of

$3x + 5y$

as having exactly the same meaning as 3 **Gizmos** & 5 **Things** even though it would not be a very good idea to use the same letters for unspecified *numerators* and for unspecified *denominators*.

3. Powers of x are read just as powers of TEN are read in arithmetic:

a. For ease of comparison, in arithmetic, we had:

- +3(5)+2 is a positive power of 5 to be read as +3 multiplied by 2 copies of 5 which tells us to write +3 (5) (5) which we can then compute to +75.
- +3(5)2 is a negative power of 5 to be read as +3 divided by 2 copies of 5 which tells us to write which we can then compute to = 0.12.
- In particular, we read +3(5)+1 as +3 multiplied by 1 copy of 5 which tells us to write +3 (5). It is usual to say that the exponent +1 can go without saying: +3(5) is the same as +3(5)+1 which we write as +3 (5).
- On the other hand, we read +3(5)1 as +3 divided by 1 copy of 5 which tells us to write so that the exponent 1 cannot go without saying as +3(5)1 is not the same as +3(5).
- In (4)+3 the coefficient +1 goes without saying and so we write (4)+3 = (4) (4) (4) where the coefficient +1 still goes without saying. We shall often just speak of 3 copies of 4 with the multiplication going without saying.
- Note that (4)+3 and 4+3 are not the same since the coefficient in 4+3 is 1 and the 3 copies are of copies of 4. In other words, 4+3 stands for (1) 4+3.
- In (4)3 the coefficient +1 goes without saying and so we write (4)3 = but here of course the coefficient +1 does not go without saying.
- Finally, we read +3(5)0 as +3 multiplied/divided by 0 copy of 5 which tells us to write simply +3.

b. In algebra,

- (3) x+2 is a positive power of x to be read (3) multiplied by 2 copies of x. (regardless of the sign of the coefficient)
- (5) x2 is a negative power of x to be read (5) divided by 2 copies of x. (regardless of the sign of the coefficient).
- In particular, we read +3x+1 as +3 multiplied by 1 copy of x which tells us to write +3x. It is usual to say that the exponent +1 can go without saying: +3x+1 is the same as +3x.
- On the other hand, we read +3x1 as +3 divided by 1 copy of x which tells us to write so that the exponent 1 cannot go without saying: +3x. is not the same as +3x.
- In x+3 the coefficient is +1 goes without saying and so we write x+3 = x x x where the co-efficient +1 still goes without saying. We shall often just speak of 3 copies of x with the multipli-cation too going without saying.

(Laurent) polynomials

- Note that in x+3 the coefficient is 1 and the 3 copies are copies of x. In other words, 4+3 stands for (1) 4+3. In (x)+3 on the other hand, the coefficient is (+1) and the 3 copies are copies of x, that is the opposite of x.
- In x 3 the coefficient +1 goes without saying and so we write +x 3 = but here of course the coefficient +1 does not go without saying.
- Finally, we read $+3x^0$ as +3 multiplied/divided by 0 copy of x which tells us to write simply +3.

$3x^{+2}$ is read as 3 *multiplied* by 2 copies of x and $3x^{-2}$ as 3 *divided* by 2 copies of x where 3 is the coefficient and the number of copies is the power. On the other hand, looking at $3x^{\pm 2}$ as 3 stacks of 2 copies of a book called x, that is looking at 3 as a *numerator* and at $x^{\pm 2}$ as a *denominator*, we get
$3x^{\pm 2} + 5x^{\pm 2} = 8x^{\pm 2}$
while
$3x^{\pm 2} + 5x^{\mp 2}$ is a combination.
Linear combinations of powers of x are called **(Laurent) polynomials**.

4. The four operations on (*Laurent*) *polynomials.*

a. *Addition* and *subtraction* are simpler than in arithmetic in that there is no exchange among powers and therefore neither carryover nor borrowing.

b. *Multiplication* and *division* present no particular difficulty but, in anticipation of later considerations of size, we will look at dividing in *ascending* powers as well as in *descending* powers and, if only by analogy with arithmetic, we will push divisions to pre-specified powers, including negative ones.

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1.2 Comparing

In Chapter 3, we compared collections when unit-values were *given.* We did so in each of the following three cases:

- Jack has ***apples*** and Jill has ***cents***,
- Jack has ***apples*** and Jill has ***bananas***,

- Jack has ***apples*** and ***bananas*** and Jill has ***cents***.

requirement
break-even

Here, when unit-values are *not* given, we will ask in the same three cases the question of *which* unit-values will result in any given one of the following three **requirement** being met by the collections:

- ***Jack's value*** is *more* than ***Jill's value***,
- ***Jack's value*** is the *same* as ***Jill's value***,
- ***Jack's value*** is *less* than ***Jill's value***,

As we shall see, it will make sense to start with the middle requirement because, even with the other two requirements, we shall always start by looking for the **break-even** point, that is the ***cents-value-of-an-apple*** for which ***Jack's value*** is the *same* as ***Jill's value***.

1. Consider the case when Jack has, say THREE ***apples*** and Jill has FIFTEEN ***cents***. What we want to know is for *what* ***cents-values*** will Jack have, say, *less* than Jill?
♠ We could of course proceed by trial and error. For instance, we could give Jack ONE ***cent*** for each one of his ***apples*** and let him compare with Jill. Then we would give him TWO ***cents*** for each one of his ***apples*** and let him compare with Jill. Etc. As usual, it will turn out a lot easier to find out on the *board* what the solution is.
❖ Since we don't have a unit-value for ***apples***, we make up a temporary name for it, $\left[x\ \frac{\text{Cents}}{\text{Apple}}\right]$. It is customary, but by no means necessary, to use x and we write

$$\begin{aligned}\text{Jack's Value} &= [3\ \cancel{\text{Apples}}] \times \left[x\ \frac{\text{Cents}}{\cancel{\text{Apple}}}\right] \\ &= [3 \times x]\ \text{Cents}\end{aligned}$$

$$\text{Jill's Value} = 15\ \text{Cents}$$

What we want is

$$\text{Jack's Value} < \text{Jill's Value}$$

that is

$$[3 \times x] \textsf{ Cents} < 15 \textsf{ Cents}$$

Since the *denominators* are now the same we can focus on the *numerators*.

$$3 \times x < 15$$

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which tells us that when apples are worth 5 $\frac{\textsf{Cents}}{\textsf{Apple}}$ then Jack and Jill will have the same value.

Now we "pick" a unit-value that is *less* than the break-even point 5 $\frac{\textsf{Cents}}{\textsf{Apple}}$ and we ask what the values of Jack and Jill are. Say we pick 2 $\frac{\textsf{Cents}}{\textsf{Apple}}$

Then Jack's value is 3 **Apples** $\times$ 2 $\frac{\textsf{Cents}}{\textsf{Apple}}$ = 6 **Cents**

Comparing with Jill's value which is 15 **Cents**, we conclude that as long as the unit-value is less than 5 $\frac{\textsf{Cents}}{\textsf{Apple}}$ then Jack will have less value than Jill.

Now we "pick" a unit-value that is *more* than the break-even point 5 $\frac{\textsf{Cents}}{\textsf{Apple}}$ and we ask what the values of Jack and Jill are. Say we pick 8 $\frac{\textsf{Cents}}{\textsf{Apple}}$

Then Jack's value is 3 **Apples** $\times$ 11 $\frac{\textsf{Cents}}{\textsf{Apple}}$ = 33 **Cents**

Comparing with Jill's value which is 15 **Cents**, we conclude that as long as the unit-value is more than 5 $\frac{\textsf{Cents}}{\textsf{Apple}}$ then Jack will have more value than Jill.

2. Consider the case where Jack has THREE ***apples*** and Jill has FOUR ***bananas***. Who has *more*?

If we don't have a unit-value for ***apples*** and ***bananas***, then we can't tell.

If we know, say, that ***apples*** have a *unit-value* of, say, SEVEN ***cents-per-apple*** and ***bananas*** have a unit-value of, say, FIVE ***banana*** then we can tell:

♠ Jack exchanges his THREE ***apples*** for TWENTY-ONE ***cents*** and Jill exchanges her FOUR ***bananas*** for TWENTY ***cents*** so that now Jack has "more than" Jill.

❖ On the board we write

$$\textsf{Jack's Value} = [3\ \cancel{\textsf{Apples}}] \times \left[7\ \frac{\textsf{Cents}}{\cancel{\textsf{Apple}}}\right] = [3 \times 7]\ \textsf{Cents} = 21\ \textsf{Cents}$$

$$\textsf{Jill's Value} = [4\ \cancel{\textsf{Bananas}}] \times \left[5\ \frac{\textsf{Cents}}{\cancel{\textsf{Banana}}}\right] = [4 \times 5]\ \textsf{Cents} = 20\ \textsf{Cents}$$

$$\textsf{Jack's Value} > \textsf{Jill's Value}$$

3. COMPARING TWO KINDS OF GOODS WITH MONEY. From Jack has Jill has 2 apples and 8 bananas and 7 dollars 5 apples and 3 bananas and 13 dollars to Jack has Jill has 3 apples and 5 bananas 6 dollars to Jack has Jill has 3 apples and 5 bananas and 6 dollars 0

Fact 1 If we pick a value for the apples and then pick a value for the bananas, the chance that Jack = Jill is about nil. Fact 2 If we pick a value for the apples and then compute a matching value for the bananas, we can have Jack = Jill.

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