extended
bundle
parentheses

Chapter 1

Accounting For *Extended* Collections Of Money *On* A Counter

We now turn to the case of collections that are **extended** in that they have more *objects* than we have *digits* so that we cannot represent them on the board with *just* the digits we have.

1.1 Bundles and Exchanges

We begin by introducing two ideas relating to collections that we shall use *systematically.*

1. The first idea is that, given any *collection* of objects, we can **bundle** it into ONE object of a *new* kind.

a. For instance, banks *wrap* collections of FIFTY ***dimes*** into ONE ***roll-of-dimes***.

♠ Similarly, we may thus have on the counter, say, a ***bundle-of-THREE-dimes*** as a new kind of object.

❖ We shall represent on the board this *bundle*, a new kind of *object*, by (3 Dimes), a new kind of *denominator* in which the **parentheses** represent the *wrapping.*

b. Then, of course, we can count *collections* of bundles.

♠ For instance, we may have on the counter ***bundle-of-TWO-dimes, bundle-of-TWO-dimes, bundle-of-TWO-dimes***.

❖ We represent this on the board by writing (2 Dimes), (2 Dimes), (2 Dimes) and therefore 3(2 Dimes) in which 3 is the *numerator* and (2 Dimes) is

unpack
multiplication tables
shorthand
exchange

the *denominator.*

c. Now we look at what happens when we **unpack** the bundles.

♠ For instance, unpacking ***bundle-of-TWO-dimes, bundle-of-TWO-dimes, bundle-of-TWO-dimes*** gives ***dime, dime, dime, dime, dime, dime***.

❖ On the board, we write, rather naturally

$$\begin{aligned}3(2\ \textsf{Dimes}) &= (2\ \textsf{Dimes}), (2\ \textsf{Dimes}), (2\ \textsf{Dimes})\\ &= \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}\\ &= 6\ \textsf{Dimes}\end{aligned}$$

where we obtained the numerator by *counting* as above. But we can also write

$$\begin{aligned}3(2\ \textsf{Dimes}) &= 2\ \textsf{Dimes} + 2\ \textsf{Dimes} + 2\ \textsf{Dimes}\\ &= 6\ \textsf{Dimes}\end{aligned}$$

where we used *addition.* And, finally, we can just write

$$3(2\ \textsf{Dimes}) = 6\ \textsf{Dimes}$$

where we used the **multiplication tables** that we memorized as children.

d. There is of course nothing to prevent us to bundle collections that we cannot represent on the board.

♠ For instance, we can bundle ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime*** as a ***bundle-of-TEN-dimes***.

❖ And while we *cannot* represent
dime, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***, ***dime***
on the board, we *can* represent a ***bundle-of-TEN-dimes*** as
(**Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**).
Instead however, we shall write (TEN **Dimes**) as a **shorthand** for the bundle (**Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**).

2. The second idea is that since a *bundle* is an object, we can **exchange** it for another object, of a different kind. This will be particularly useful as dealing with bundles can be quite cumbersome.

♠ While we *cannot* represent TEN ***dimes*** on the board, we can *bundle* it as ***bundle-of-TEN-dimes***, and then *exchange* the bundle for a new object, a ***dollar***, which is an object of another kind that we *can* also represent on the board.

❖ On the board, we will write (TEN **Dimes**) = 1 **Dollar**.

1.2 Representing Extended Collections With Combinations of (Counting) Number-Phrases

The general idea will be to reduce the number of objects we have to deal with until we need only count *basic* collections.

We will achieve this by *bundling* the collections and then counting the *bundles* of which there are necessarily fewer than there were objects in the original collections. If the collection of bundles is *basic*, then we can count it. If not, then we bundle the collection of bundles. Etc.

Actually, in the case of *money*, rather than to deal with the *bundles* themselves, we usually *exchange* the bundles for other *objects*. Then we have collections of these other objects we can bundle. Etc. This avoids having to deal with bundles of bundles of bundles ..., which, however, is what we shall have to do when, in Section **??**, we shall count extended collections of *discrete goods*.

In other words, when dealing with money, we will be using the two ideas that we discussed in Section 1.1.

1. We begin with an example.

♠ Suppose we have, say, the following collection on the counter:
dime, ***dime***.
As we have no digit to represent on the board TWENTY-THREE ***dimes*** on the counter,

- we bundle TEN ***dimes*** (out of the TWENTY-THREE ***dimes*** on the counter),
  we bundle another TEN ***dimes*** (out of the THIRTEEN ***dimes*** left on the counter),
- we then exchange each bundle of TEN ***dimes*** for ONE ***dollar***.

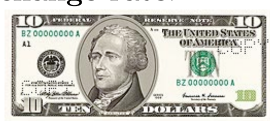
And so, instead of the original collection, we now have on the counter a *bunch* of two *basic* collections:
dollar, ***dollar***
dime, ***dime***, ***dime***.
which we will be able to represent on the board as a *combination*.

❖ On the board, we write correspondingly:
Dime, **Dime**.
= (TEN **Dimes**), (TEN **Dimes**), **Dime**, **Dime**, **Dime**.
= **Dollar**, **Dollar**, **Dime**, **Dime**, **Dime**.

= 2 **Dollars** & 3 **Dimes**.

where we wrote (TEN **Dimes**) as a *shorthand* for the bundle (**Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**). (Section 1.1.)

2. Of course, the problem is that we will have to keep introducing *ever-larger* denominators. For the time being, we will stick to bills that change at a TEN to ONE **exchange rate**:

- ***Hamiltons***: 
- ***Franklins***: 
- ***Cleveland***: 

In other words, we shall disregard ***five-dollar-bills***, ***twenty-dollar-bills*** and ***fifty-dollar-bills***. (However, we will deal with these later on.)

To emphasize the distinction between the actual *bills* on the counter and the denominators on the board, **Hamiltons**, **Franklins**, **Clevelands** as denominators for ***ten-dollar-bills***, ***hundred-dollar-bills*** and ***thousand-dollar-bills*** and also, since consistency is a virtue we will want to practice, **Washingtons** instead of **Dollars**. But, for lack of a better word, we will keep writing **Dimes** as denominator for ***dimes***.

3. We are now in a position to deal with larger and larger collections, such as, for instance, occur when we aggregate *bunches* of collections.

♠ For instance, when we want to aggregate EIGHT ***dollars*** and FIVE ***dimes*** with SIX ***dollars*** and THREE ***dimes***, we bundle TEN of the resulting FOURTEEN ***dollars*** on the counter and exchange the bundle for a ***ten-dollar-bill***.

❖ The (board) representation of this is:

[8 **Washingtons** & 5 **Dimes**] + [6 **Washingtons** & 3 **Dimes**]
= FOURTEEN **Washingtons** & 8 **Dimes**
= 1 **Hamilton** & 4 **Washingtons** & 8 **Dimes**

4. As a result of all this, we can now represent collections of all the way up to NINETY NINE THOUSAND NINE HUNDRED NINETY NINE ***dimes*** which we write on the board as 9 **Clevelands** & 9 **Franklins** & 9 **Hamiltons** &

9 **Washingtons** & 9 **Dimes**. In order to go beyond that, we would have to continue to create ever “larger” denominators. Instead, we shall eventually develop another, even more systematic approach. heading

1.3 (Decimal) Headings

To simplify this representation, a somewhat natural idea would be to write the denominators only *once* and then just write the numerators, “as needed”. The problem, though, is how to indicate which numerator goes with which denominator.

What we do is to write the *denominators* into a **heading** such as

Clevelands	Franklins	Hamiltons	Washingtons	Dimes

and then write the *numerators* directly under the corresponding denominators.

For instance, we write the combination 3 **Franklins** & 1 **Washington** & 7 **Dimes** as follows:

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
	3		1	7

Thus, each *column* corresponds to a collection of a *different kind* of objects. While this may look like going from cumbersome to *very* cumbersome, we will now see how easy it is to *work* with.

1.4 Adding Under A Heading

In Section **??**, in the case of *basic* collections, we defined *addition* as the (board) procedure that represents *aggregation* of collections on the counter. Here, we focus on how addition works out, under a heading, for *extended* collections.

❖ Suppose, for instance, that we wanted to add the three combinations,

8 **Hamiltons** & 7 **Dimes**
8 **Washingtons** & 2 **Dimes**
5 **Hamiltons** & 3 **Washingtons** & 4 **Dimes**

First, we rewrite the three combinations under the heading:

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
		8		7
			8	2
		5	3	4

Adding up the **Dimes** by "counting on our fingers":

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
		8		7
			8	2
		5	3	4
				THIRTEEN

gives us THIRTEEN **Dimes** but we cannot *write* that. So, we *bundle* TEN of the THIRTEEN ***dimes*** and, *if* we can *exchange* the bundle for ONE ***dollar*** then we can write:

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
			1	
		8		7
			8	2
		5	3	4
				3

Adding up the **Washingtons** by "counting on our fingers":

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
			1	
		8		7
			8	2
		5	3	4
			TWELVE	3

gives us TWELVE **Washingtons** but we cannot *write* that. So, we *bundle* TEN of the TWELVE ***dollars*** and, *if* we can *exchange* the bundle for ONE ***ten-dollar-bill***, then we can write:

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
		1	*1*	
		8		7
			8	2
		5	3	4
			2	3

Finally, adding the **Hamiltons** by "counting on our fingers":

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
		1	*1*	
		8		7
			8	2
		5	3	4
		FOURTEEN	2	3

gives us FOURTEEN **Hamiltons** but we cannot *write* that. So, we *bundle* TEN of the FOURTEEN ***ten-dollar-bills*** and, *if* we can *exchange* the bundle for ONE ***hundred-dollar-bill***, then we can write:

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
	1	*1*	*1*	
		8		7
			8	2
		5	3	4
	1	4	2	3

Thus, *if* we have *changing facilities*, we can then carry out the addition and the result will be 1 **Franklin** & 4 **Hamiltons** & 2 **Washingtons** & 3 **Dimes**.

1.5 Subtracting Under A Heading

In Section **??**, we defined *subtraction*, in the case of *basic* collections, as the (board) procedure that represents what is left on the counter of a collection after we have taken a number of objects. Here, we focus on how addition works out, under a heading, for *extended* collections.

1. Occasionally, we have enough of each kind of bills to take what we want.

♠ Suppose for instance that SEVENTY-EIGHT ***dollars***, in the form of SEVEN ***ten-dollar-bills*** and EIGHT ***one-dollar-bills***, are on the counter and that we want TWENTY-FIVE ***dollars***, in the form of TWO ***ten-dollar-bills*** and FIVE ***one-dollar-bills***.

❖ On the board, what we want to do is the subtraction

7 **Hamiltons** & 8 **Washingtons** − 2 **Hamiltons** & 5 **Washingtons**

which we rewrite under a heading as follows:

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		7	8	
		2	5	

Then, counting 5 steps backward from 8, $\xleftarrow{3,\,4,\,5,\,6,\,7,\,8}$, gives us 3 and counting 2 steps backward from 7, $\xleftarrow{5,\,6,\,7}$, gives us 5.

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		7	8	
		2	5	
		5	3	

that is 5 **Hamiltons** & 3 **Washingtons**.

2. Most of the time, though, we need to *exchange* and *unpack*.

♠ Suppose for instance that, instead of SEVENTY-EIGHT ***dollars***, there is only SEVENTY-THREE ***dollars*** on the counter, in the form of SEVEN ***ten-dollar-bills*** and THREE ***one-dollar-bills***. and that we want TWENTY-FIVE ***dollars***, in the form of TWO ***ten-dollar-bills*** and FIVE ***one-dollar-bills***. The problem we immediately run into is that THREE ***one-dollar-bills*** is not enough from which to take FIVE ***one-dollar-bills***. However, if we can *exchange* ONE ***ten-dollar-bill*** for a bundle of TEN ***one-dollar-bills***, then we can unpack the bundle so that we now have THIRTEEN ***one-dollar-bills*** from which we can take the FIVE ***one-dollar-bills***. But now, of course, we have only SIX ***ten-dollar bills*** from which to take the TWO ***ten-dollar-bills***. Altogether, this leaves us with FOUR ***ten-dollar-bills*** and EIGHT ***one-dollar-bills***.

❖ On the board, we write:

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		7	3	
		2	5	

and then

specify
reconstruct

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		~~7~~ 6	THIRTEEN	
		2	5	

from which we get

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
		~~7~~ 6	THIRTEEN	
		2	5	
		4	8	

and so the result of the subtraction is 4 **Hamiltons** & 8 **Washingtons**.

1.6 Decimal Number-Phrases

1. While headings are convenient when we want to *work* with *several* combinations, they are much too cumbersome if all we want is just to *write* *one* combination. So we will now develop yet another way in which it will be convenient *both* to *write* a single combination and to *add* several combinations.

a. Suppose we wanted to **specify** by mail a whole lot of combinations. A natural idea would be to mail the *heading*

Clevelands	Franklins	Hamiltons	Washingtons	Dimes

once and for all so that, in later mails, the heading could now go "without saying" and we would just have to mail the *numerators*.
For instance, rather than mailing the combination

7 **Franklins** & 2 **Hamiltons** & 4 **Washingtons**

we would just mail the numerators 7, 2, 4.
The problem with that, though, is that the recipient would not be able to **reconstruct** the specified combination as s/he wouldn't know under which denominators in the heading to write these numerators and thus wouldn't be able to recover the combination.
For instance, s/he wouldn't know if the numerators 7, 2, 4 should be placed under the heading this way

Clevelands	Franklins	Hamiltons	Washingtons	Dimes
7	2	4		

(decimal) number-phrase
(select) denominator
pick
(decimal) pointer
decimal point
(decimal) numerator

or that way

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
		7	2	4

or *any* other way.
So, we cannot just mail the *numerators* and, along with the numerators, we must also mail some information as to *how* the numerators are to be placed under the heading.

b. To that purpose, we introduce a new type of number-phrases, **(decimal) number-phrase**, which consists of:

- the *numerators* that we want to mail,
- a **(select) denominator** that *we* can " **pick**" from among the denominators in the heading, that is whose choice is entirely *ours*.
- a **(decimal) pointer** to indicate which of the numerators is to be placed under the (select) denominator, the agreement being that the other numerators are to be placed "accordingly".

Note. For a little while, rather than a *dot*, we shall use $_{\leftarrow}$ for a *decimal pointer*. This is to remind us that, by tradition, a decimal pointer points to its *left* even though we read from left to right. However, after this section, we shall conform and use a **decimal point**. (The use of a dot as decimal pointer is not universal and, for instance, many languages use a *comma* instead of a *dot*.) The regrettable thing about decimal *points*, as opposed to decimal *pointers*, is that we will have to *remember* that the digit being pointed at is to the *left* of the decimal point.

All the individual numerators, together with the decimal pointer, will be collectively referred to as **(decimal) numerator**. Thus, just like a (counting) number-phrase consists of a *numerator* and a *denominator*, a (decimal) number-phrase consists of a (*decimal*) *numerator* and a (*select*) *denominator*.

For instance, after we have mailed the heading, we can specify the combination 7 **Franklins** & 2 **Hamiltons** & 4 **Washingtons** by mailing the (decimal) *number-phrase*

$$72_{\leftarrow}4 \text{ Hamiltons}$$

where we "picked" **Hamiltons** as our (select) denominator and where $72_{\leftarrow}4$ is the corresponding (decimal) *numerator*. The recipient would then *reconstruct* the specified combination as follows:

- since 2 is being pointed at, s/he would place 2 under **Hamiltons** since it is the (select) denominator,
- since 7 is left of 2, s/he would place 7 under **Franklins**, the denominator

left of **Hamiltons**, and, 0

- since 4 is right of 2, s/he would place 4 under **Washingtons**, the denominator right of **Hamiltons**.

that is, s/he would write

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
	7	2	4	

which is indeed 7 **Franklins** & 2 **Hamiltons** & 4 **Washingtons**.

c. There is however a problem with combinations such as 5 **Franklins** & 3 **Hamiltons** & 8 **Dimes** which, under a heading, looks like

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
	5	3		8

Say we "pick" **Franklins** as our (select) denominator. The problem is that what is immediately to the right of **Hamiltons** is **Washingtons** and not **Dimes**. So, we cannot write $5_{\leftarrow}38$ **Franklins** because the recipient would reconstruct *that* (decimal) number-phrase as

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
	5	3	8	

and thus as specifying 5 **Franklins** & 3 **Hamiltons** & 8 **Washingtons**.
In order to specify 5 **Franklins** & 3 **Hamiltons** & 8 **Dimes** we must indicate that there are no **Washingtons**. To that purpose, we use the digit **0** which we introduced in Section **??** and we write $5_{\leftarrow}308$ **Hamiltons** and the recipient will reconstruct *that* as

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
	5	3	0	8

which is 5 **Franklins** & 3 **Hamiltons** & 0 **Washingtons** & 8 **Dimes** and indeed the same as 5 **Franklins** & 3 **Hamiltons** & 8 **Dimes**.
Note. This is in fact what 0 was invented for.

2. There are two default rules. (We now use the decimal *point* . instead of the decimal *pointer* $_{\leftarrow}$.)

- When the decimal point is to the right of the (decimal) numerator, as in 7204. **Dimes**, it is customary not to write the decimal point at all and just to write 7204 **Dimes**. The corresponding default rule is:

  When there is no decimal point, it goes without saying that the decimal *point is to the right of the (decimal) numerator.*

- When there is *no doubt* as to what the (select) denominator is, say that it is **Hamiltons**, it is customary not to write it and, for instance, just to write 72.04 instead of 72.04 **Hamiltons**. The corresponding default rule is:

  When there is no (select) denominator, it goes without saying that it is the (select) denominator that was picked earlier on.

  This, though, is *extremely dangerous* because it depends on us *remembering* what the (select) denominator is that was picked earlier on. So, while it is being done all the time, *we* shall not.

Note. In the U.S., it is usual to write, for instance, .56 **Hamiltons** rather than 0.56 **Hamiltons**. *We* shall *not* do so in this text, if only because we don't like the idea of a pointer pointing, at least apparently, at nothing. In any case, the "energy saving" would not be worth it.

3. To add and subtract (decimal) number-phrases we can just place them back under a heading. When the (decimal) number-phrases have the same (select) denominators, we need not even write the heading but just make sure that the pointed digits are lined up and that, in fact, is exactly what we were taught to do in school.
When the (decimal) number-phrases have different (select) denominators, we may begin by changing to a *common* (select) denominator but, keeping the heading in mind and with a little bit of practice, it is almost as easy to line up the numerators "as if" they were under the heading.

4. The use of (decimal) number-phrases has several *advantages*:

a. As we already saw, *we* can pick as our (select) denominator any denominator *we* want. For instance, a rich person might pick **Clevelands** as her/his (select) denominator while the rest of us would probably pick **Washingtons** as our (select) denominator.

b. Not only does using (decimal) number-phrases allow us to *pick* as our (select) denominator any denominator we want but, by placing the (decimal) number-phrase back under the heading, we can easily *change* the denominator we want to use as our (select) denominator. Of course we then need to adjust the decimal point accordingly.
For instance, placing 85.7 **Hamiltons** under a heading,

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
	8	5	7	

we see that the (decimal) number-phrase 85.7 **Hamiltons** could just as well be written as any of the following (decimal) number-phrases:

0.857 **Clevelands**
8.57 **Franklins**
857. **Washingtons**
8570. **Dimes**

Any of the above means the same as 8 **Franklins** & 5 **Hamiltons** & 7 **Dimes**.

c. Another advantage of (decimal) number-phrases is that if, for whatever reason, we needed to have a "smaller" or a "larger" (decimal) numerator, it would be easy to do. We would "move the decimal point" by placing the number-phrase back under a heading, change the *digit* being pointed and change the (select) *denominator* accordingly.

d. We can now *count* beyond 9: $\xrightarrow{1,\,2,\,3,\,\ldots,\,9,\,10,\,11,\,\ldots,\,99999.}$

e. Finally, there is another, intriguing, advantage to the use of (decimal) number-phrases. While we can now count all the way up to 9 **Clevelands** & 9 **Franklins** & 9 **Hamiltons** & 9 **Washingtons** & 9 **Dimes**, that is all the way up to 99999 **Dimes**, what if we wanted to add 1 **Dime**? Let us do it under a heading:

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
9	9	9	9	9 1

We have no trouble reaching the following stage (in the presence of exchange facilities):

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**
1	*1*	*1*	*1*	
9	9	9	9	9 1
TEN	0	0	0	0

At this point, although we can *bundle* the TEN ***thousand-dollar-bills***, we cannot *exchange* the bundle for any bill. Nevertheless, as long as we "pick" any of the *existing* denominators for our (select) denominator, then we *can* write a number-phrase. For instance, if we "pick" **Franklins**, we can write the result as 100. **Franklins** or, if we "pick" **Dimes**, as 100000. **Dimes**.

Note. To some extent this would be cheating because we *still* do no have

in the hundreds
in the ones
in the tenths
in the hundredths
in the thousands
principal unit

a denominator corresponding to the leftmost 1 but, other than that, everything looks fine and we could even say that we wrote the (decimal) number phrases *pending the creation* of that new denominator! But, in case you should worry, we shall in fact develop *several* ways in Section **??** to get *automatically* as many denominators as we need.

1.7 Comparing Orders of Magnitude

While both 3 **Franklins** and 7 **Washingtons** are larger than 8 **Dimes**, they are not so in the same manner. We shall say that

- 3 **Franklins** is **in the hundreds**,
- 7 **Washingtons** is **in the ones**,
- 8 **Dimes** is **in the tenths**.

More generally, we shall say that
0.00875 **Hamiltons** is **in the hundredths** as we can see from
0.00875 **Hamiltons** = 8.75 **Cents**
and that 42187.53 **Dimes** is **in the thousands** as we can see from
42187.53 **Dimes** = 4.218753 **Clevelands**
All of the above is in terms of **Washingtons** being the **principal unit**.