Chapter 1

Accounting For
Extended Collections
Of Money On A Counter

We now turn to the case of collections that are extended in that they have more objects than we have digits so that we cannot represent them on the board with just the digits we have.

1.1 Bundles and Exchanges

We begin by introducing two ideas relating to collections that we shall use systematically.

1. The first idea is that, given any collection of objects, we can bundle it into one object of a new kind.

a. For instance, banks wrap collections of fifty dimes into one roll-of-dimes.

♠ Similarly, we may thus have on the counter, say, a bundle-of-THREE-dimes as a new kind of object.

❖ We shall represent on the board this bundle, a new kind of object, by (3 Dimes), a new kind of denominator in which the parentheses represent the wrapping.

b. Then, of course, we can count collections of bundles.

♠ For instance, we may have on the counter bundle-of-TWO-dimes, bundle-of-TWO-dimes, bundle-of-TWO-dimes.

❖ We represent this on the board by writing (2 Dimes), (2 Dimes), (2 Dimes) and therefore 3(2 Dimes) in which 3 is the numerator and (2 Dimes) is
CHAPTER 1. EXTENDED COLLECTIONS OF MONEY

unpack multiplication tables shorthand exchange

the denominator.

\(c\). Now we look at what happens when we unpack the bundles.

\(\spadesuit\) For instance, unpacking \textit{bundle-of-TWO-dimes, bundle-of-TWO-dimes, bundle-of-TWO-dimes} gives \textit{dime, dime, dime, dime, dime, dime}.

\(\heartsuit\) On the board, we write, rather naturally

\[
3(2 \text{ Dimes}) = (2 \text{ Dimes}) \quad (2 \text{ Dimes}) \\
= \text{ Dimes, Dimes, Dimes, Dimes, Dimes, Dimes} \\
= 6 \text{ Dimes}
\]

where we obtained the numerator by \textit{counting} as above. But we can also write

\[
3(2 \text{ Dimes}) = 2 \text{ Dimes} + 2 \text{ Dimes} + 2 \text{ Dimes} \\
= 6 \text{ Dimes}
\]

where we used \textit{addition}. And, finally, we can just write

\[
3(2 \text{ Dimes}) = 6 \text{ Dimes}
\]

where we used the \textit{multiplication tables} that we memorized as children.

\(d\). There is of course nothing to prevent us to bundle collections that we cannot represent on the board.

\(\spadesuit\) For instance, we can bundle \textit{dime, dime, dime, dime, dime, dime, dime, dime, dime} as a \textit{bundle-of-TEN-dimes}.

\(\heartsuit\) And while we cannot represent \textit{dime, dime, dime, dime, dime, dime, dime, dime, dime} on the board, we can represent a \textit{bundle-of-TEN-dimes} as \textit{(Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime)}. Instead however, we shall write \textit{(ten Dimes)} as a \textit{shorthand} for the bundle \textit{(Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime)}.

\(2\). The second idea is that since a \textit{bundle} is an object, we can exchange it for another object, of a different kind. This will be particularly useful as dealing with bundles can be quite cumbersome.

\(\spadesuit\) While we cannot represent \textit{ten dimes} on the board, we can bundle it as \textit{bundle-of-TEN-dimes}, and then exchange the bundle for a new object, a \textit{dollar}, which is an object of another kind that we can also represent on the board.

\(\heartsuit\) On the board, we will write \textit{(ten Dimes)} = \textit{1 Dollar}. 

1.2 Representing Extended Collections With Combinations of (Counting) Number-Phrases

The general idea will be to reduce the number of objects we have to deal with until we need only count basic collections.

We will achieve this by bundling the collections and then counting the bundles of which there are necessarily fewer than there were objects in the original collections. If the collection of bundles is basic, then we can count it. If not, then we bundle the collection of bundles. Etc.

Actually, in the case of money, rather than to deal with the bundles themselves, we usually exchange the bundles for other objects. Then we have collections of these other objects we can bundle. Etc. This avoids having to deal with bundles of bundles of bundles . . . , which, however, is what we shall have to do when, in Section ??, we shall count extended collections of discrete goods.

In other words, when dealing with money, we will be using the two ideas that we discussed in Section 1.1.

1. We begin with an example.

♠ Suppose we have, say, the following collection on the counter:
\[ \text{dime, dime, dime, dime, dime, dime, dime, dime, dime, dime, dime, dime, dime, dime, dime, dime, dime.} \]

As we have no digit to represent on the board twenty-three dimes on the counter,
- we bundle ten dimes (out of the twenty-three dimes on the counter),
  we bundle another ten dimes (out of the thirteen dimes left on the counter),
- we then exchange each bundle of ten dimes for one dollar.

And so, instead of the original collection, we now have on the counter a bunch of two basic collections:
\[ \text{dollar, dollar} \]
\[ \text{dime, dime, dime.} \]
which we will be able to represent on the board as a combination.

❖ On the board, we write correspondingly:
\[ \text{Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime.} \]
\[ = (\text{ten Dimes}), (\text{ten Dimes}), \text{Dime, Dime, Dime.} \]
\[ = \text{Dollar, Dollar, Dime, Dime, Dime.} \]
= 2 Dollars & 3 Dimes.

where we wrote (TEN Dimes) as a shorthand for the bundle (Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime, Dime). (Section 1.1.)

2. Of course, the problem is that we will have to keep introducing ever-larger denominators. For the time being, we will stick to bills that change at a Ten to One exchange rate:

- Hamiltons:

- Franklins:

- Cleveland:

In other words, we shall disregard five-dollar-bills, twenty-dollar-bills and fifty-dollar-bills. (However, we will deal with these later on.)

To emphasize the distinction between the actual bills on the counter and the denominators on the board, Hamiltons, Franklins, Cleveland as denominators for ten-dollar-bills, hundred-dollar-bills and thousand-dollar-bills and also, since consistency is a virtue we will want to practice, Washingtons instead of Dollars. But, for lack of a better word, we will keep writing Dimes as denominator for dimes.

3. We are now in a position to deal with larger and larger collections, such as, for instance, occur when we aggregate bunches of collections.

- For instance, when we want to aggregate eight dollars and five dimes with six dollars and three dimes, we bundle ten of the resulting fourteen dollars on the counter and exchange the bundle for a ten-dollar-bill.

- The (board) representation of this is:

\[
[8 \text{ Washingtons} & 5 \text{ Dimes}] + [6 \text{ Washingtons} & 3 \text{ Dimes}]
= \text{FOURTEEN Washingtons} & 8 \text{ Dimes}
= 1 \text{ Hamilton} & 4 \text{ Washingtons} & 8 \text{ Dimes}
\]

4. As a result of all this, we can now represent collections of all the way up to NINETY NINE THOUSAND NINE HUNDRED NINETY NINE dimes which we write on the board as 9 Cleveland & 9 Franklins & 9 Hamiltons &
9 Washingtons & 9 Dimes. In order to go beyond that, we would have to continue to create ever “larger” denominators. Instead, we shall eventually develop another, even more systematic approach.

1.3 (Decimal) Headings

To simplify this representation, a somewhat natural idea would be to write the denominators only once and then just write the numerators, “as needed”. The problem, though, is how to indicate which numerator goes with which denominator.

What we do is to write the denominators into a heading such as

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklin</th>
<th>Hamilton</th>
<th>Washington</th>
<th>Dime</th>
</tr>
</thead>
</table>

and then write the numerators directly under the corresponding denominators.

For instance, we write the combination 3 Franklin & 1 Washington & 7 Dimes as follows:

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklin</th>
<th>Hamilton</th>
<th>Washington</th>
<th>Dime</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td></td>
<td>1</td>
<td>7</td>
</tr>
</tbody>
</table>

Thus, each column corresponds to a collection of a different kind of objects. While this may look like going from cumbersome to very cumbersome, we will now see how easy it is to work with.

1.4 Adding Under A Heading

In Section ??, in the case of basic collections, we defined addition as the (board) procedure that represents aggregation of collections on the counter. Here, we focus on how addition works out, under a heading, for extended collections.

❖ Suppose, for instance, that we wanted to add the three combinations,

- 8 Hamiltons & 7 Dimes
- 8 Washingtons & 2 Dimes
- 5 Hamiltons & 3 Washingtons & 4 Dimes

First, we rewrite the three combinations under the heading:
## Chapter 1. Extended Collections of Money

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<table>
<thead>
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<td>Clevelands</td>
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</table>

Adding up the Dimes by “counting on our fingers”:

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<td>Clevelands</td>
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THIRTEEN

gives us THIRTEEN Dimes but we cannot write that. So, we bundle TEN of the THIRTEEN dimes and, if we can exchange the bundle for ONE dollar then we can write:

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<tbody>
<tr>
<td>Clevelands</td>
<td>Franklins</td>
<td>Hamiltons</td>
<td>Washingtons</td>
<td>Dimes</td>
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</table>

Adding up the Washingtons by “counting on our fingers”:

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</thead>
<tbody>
<tr>
<td>Clevelands</td>
<td>Franklins</td>
<td>Hamiltons</td>
<td>Washingtons</td>
<td>Dimes</td>
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</tbody>
</table>

TWELVE

gives us TWELVE Washingtons but we cannot write that. So, we bundle TEN of the TWELVE dollars and, if we can exchange the bundle for ONE ten-dollar-bill, then we can write:
Finally, adding the Hamiltons by “counting on our fingers”:

<table>
<thead>
<tr>
<th>Clevelands</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
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<td>8</td>
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</tbody>
</table>

gives us FOURTEEN Hamiltons but we cannot write that. So, we bundle ten of the FOURTEEN ten-dollar-bills and, if we can exchange the bundle for one hundred-dollar-bill, then we can write:

<table>
<thead>
<tr>
<th>Clevelands</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>7</td>
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</table>

Thus, if we have changing facilities, we can then carry out the addition and the result will be 1 Franklin & 4 Hamiltons & 2 Washingtons & 3 Dimes.

### 1.5 Subtracting Under A Heading

In Section ??, we defined subtraction, in the case of basic collections, as the (board) procedure that represents what is left on the counter of a collection after we have taken a number of objects. Here, we focus on how addition works out, under a heading, for extended collections.

1. Occasionally, we have enough of each kind of bills to take what we want.

   ♠ Suppose for instance that SEVENTY-EIGHT dollars, in the form of SEVEN ten-dollar-bills and EIGHT one-dollar-bills, are on the counter and
that we want twenty-five *dollars*, in the form of two *ten-dollar-bills* and five *one-dollar-bills*.

❖ On the board, what we want to do is the subtraction

\[
7 \text{ Hamiltons} \& 8 \text{ Washingtons} - 2 \text{ Hamiltons} \& 5 \text{ Washingtons}
\]

which we rewrite under a heading as follows:

<table>
<thead>
<tr>
<th>Clevelands</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
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</thead>
<tbody>
<tr>
<td></td>
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<td>7</td>
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<td>2</td>
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</tbody>
</table>

Then, counting 5 steps backward from 8, \(3, 4, 5, 6, 7, 8\), gives us 3 and counting 2 steps backward from 7, \(5, 7\), gives us 5.

<table>
<thead>
<tr>
<th>Clevelands</th>
<th>Franklins</th>
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<th>Washingtons</th>
<th>Dimes</th>
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<td>5</td>
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</table>

that is 5 Hamiltons & 3 Washingtons.

2. Most of the time, though, we need to exchange and unpack.

♠ Suppose for instance that, instead of seventy-eight *dollars* on the counter, in the form of seven *ten-dollar-bills* and three *one-dollar-bills* and that we want twenty-five *dollars*, in the form of two *ten-dollar-bills* and five *one-dollar-bills*. The problem we immediately run into is that three *one-dollar-bills* is not enough from which to take five *one-dollar-bills*. However, if we can exchange one *ten-dollar-bill* for a bundle of ten *one-dollar-bills*, then we can unpack the bundle so that we now have thirteen *one-dollar-bills* from which we can take the five *one-dollar-bills*. But now, of course, we have only six *ten-dollar bills* from which to take the two *ten-dollar-bills*. Altogether, this leaves us with four *ten-dollar-bills* and eight *one-dollar-bills*.

❖ On the board, we write:

<table>
<thead>
<tr>
<th>Clevelands</th>
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<th>Washingtons</th>
<th>Dimes</th>
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<td>2</td>
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</table>

and then
1.6 Decimal Number-Phrases

1. While headings are convenient when we want to work with several combinations, they are much too cumbersome if all we want is just to write one combination. So we will now develop yet another way in which it will be convenient both to write a single combination and to add several combinations.

   a. Suppose we wanted to specify by mail a whole lot of combinations. A natural idea would be to mail the heading

   \[
   \begin{array}{ccccc}
   \text{Clevelands} & \text{Franklins} & \text{Hamiltons} & \text{Washingtons} & \text{Dimes} \\
   \hline
   7/6 & \text{THIRTEEN} & 2 & 5 & \\
   \hline
   \end{array}
   \]

   from which we get

   \[
   \begin{array}{ccccc}
   \text{Clevelands} & \text{Franklins} & \text{Hamiltons} & \text{Washingtons} & \text{Dimes} \\
   \hline
   7/6 & \text{THIRTEEN} & 2 & 5 & \\
   \hline
   4 & 8 & \\
   \hline
   \end{array}
   \]

   and so the result of the subtraction is 4 Hamiltons & 8 Washingtons.

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   a. Suppose we wanted to specify by mail a whole lot of combinations. A natural idea would be to mail the heading

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   \hline
   7/6 & \text{THIRTEEN} & 2 & 5 & \\
   \hline
   \end{array}
   \]

   once and for all so that, in later mails, the heading could now go “without saying” and we would just have to mail the numerators.

   For instance, rather than mailing the combination

   7 Franklins & 2 Hamiltons & 4 Washingtons

   we would just mail the numerators 7, 2, 4.

   The problem with that, though, is that the recipient would not be able to reconstruct the specified combination as s/he wouldn’t know under which denominators in the heading to write these numerators and thus wouldn’t be able to recover the combination.

   For instance, s/he wouldn’t know if the numerators 7, 2, 4 should be placed under the heading this way

   \[
   \begin{array}{ccccc}
   \text{Clevelands} & \text{Franklins} & \text{Hamiltons} & \text{Washingtons} & \text{Dimes} \\
   \hline
   7 & 2 & 4 & \\
   \hline
   \end{array}
   \]
or that way

<p>| | | | | |</p>
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<td>Cleveland</td>
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</tbody>
</table>

or any other way.

So, we cannot just mail the numerators and, along with the numerators, we must also mail some information as to how the numerators are to be placed under the heading.

b. To that purpose, we introduce a new type of number-phrases, (decimal) number-phrase, which consists of:

- the numerators that we want to mail,
- a (select) denominator that we can “pick” from among the denominators in the heading, that is whose choice is entirely ours.
- a (decimal) pointer to indicate which of the numerators is to be placed under the (select) denominator, the agreement being that the other numerators are to be placed “accordingly”.

Note. For a little while, rather than a dot, we shall use ← for a decimal pointer. This is to remind us that, by tradition, a decimal pointer points to its left even though we read from left to right. However, after this section, we shall conform and use a decimal point. (The use of a dot as decimal pointer is not universal and, for instance, many languages use a comma instead of a dot.) The regrettable thing about decimal points, as opposed to decimal pointers, is that we will have to remember that the digit being pointed at is to the left of the decimal point.

All the individual numerators, together with the decimal pointer, will be collectively referred to as (decimal) numerator. Thus, just like a (counting) number-phrase consists of a numerator and a denominator, a (decimal) number-phrase consists of a (decimal) numerator and a (select) denominator.

For instance, after we have mailed the heading, we can specify the combination 7 Franklins & 2 Hamiltons & 4 Washingtons by mailing the (decimal) number-phrase

72←4 Hamiltons

where we “picked” Hamiltons as our (select) denominator and where 72←4 is the corresponding (decimal) numerator. The recipient would then reconstruct the specified combination as follows:

- since 2 is being pointed at, s/he would place 2 under Hamiltons since it is the (select) denominator,
- since 7 is left of 2, s/he would place 7 under Franklins, the denominator
left of Hamiltons, and,

- since 4 is right of 2, s/he would place 4 under Washingtons, the denominator right of Hamiltons.

that is, s/he would write

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>7</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

which is indeed 7 Franklins & 2 Hamiltons & 4 Washingtons.

c. There is however a problem with combinations such as 5 Franklins & 3 Hamiltons & 8 Dimes which, under a heading, looks like

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5</td>
<td>3</td>
<td>8</td>
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</tbody>
</table>

Say we “pick” Franklins as our (select) denominator. The problem is that what is immediately to the right of Hamiltons is Washingtons and not Dimes. So, we cannot write 5...38 Franklins because the recipient would reconstruct that (decimal) number-phrase as

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>8</td>
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</tbody>
</table>

and thus as specifying 5 Franklins & 3 Hamiltons & 0 Washingtons & 8 Dimes. In order to specify 5 Franklins & 3 Hamiltons & 8 Dimes we must indicate that there are no Washingtons. To that purpose, we use the digit 0 which we introduced in Section ?? and we write 5...308 Hamiltons and the recipient will reconstruct that as

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>8</td>
</tr>
</tbody>
</table>

which is 5 Franklins & 3 Hamiltons & 0 Washingtons & 8 Dimes and indeed the same as 5 Franklins & 3 Hamiltons & 8 Dimes.

Note. This is in fact what 0 was invented for.

2. There are two default rules. (We now use the decimal point . instead of the decimal pointer ⎯.)

- When the decimal point is to the right of the (decimal) numerator, as in 7204. Dimes, it is customary not to write the decimal point at all and just to write 7204 Dimes. The corresponding default rule is:

  When there is no decimal point, it goes without saying that the decimal
point is to the right of the (decimal) numerator.

- When there is no doubt as to what the (select) denominator is, say that it is Hamiltons, it is customary not to write it and, for instance, just to write 72.04 instead of 72.04 Hamiltons. The corresponding default rule is:

  When there is no (select) denominator, it goes without saying that it is the (select) denominator that was picked earlier on.

This, though, is extremely dangerous because it depends on us remembering what the (select) denominator is that was picked earlier on. So, while it is being done all the time, we shall not.

Note. In the U.S., it is usual to write, for instance, .56 Hamiltons rather than 0.56 Hamiltons. We shall not do so in this text, if only because we don’t like the idea of a pointer pointing, at least apparently, at nothing. In any case, the “energy saving” would not be worth it.

3. To add and subtract (decimal) number-phrases we can just place them back under a heading. When the (decimal) number-phrases have the same (select) denominators, we need not even write the heading but just make sure that the pointed digits are lined up and that, in fact, is exactly what we were taught to do in school.

When the (decimal) number-phrases have different (select) denominators, we may begin by changing to a common (select) denominator but, keeping the heading in mind and with a little bit of practice, it is almost as easy to line up the numerators “as if” they were under the heading.

4. The use of (decimal) number-phrases has several advantages:

   a. As we already saw, we can pick as our (select) denominator any denominator we want. For instance, a rich person might pick Cleveland as her/his (select) denominator while the rest of us would probably pick Washington as our (select) denominator.

   b. Not only does using (decimal) number-phrases allow us to pick as our (select) denominator any denominator we want but, by placing the (decimal) number-phrase back under the heading, we can easily change the denominator we want to use as our (select) denominator. Of course we then need to adjust the decimal point accordingly.

For instance, placing 85.7 Hamiltons under a heading,

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklins</th>
<th>Hamiltons</th>
<th>Washingtons</th>
<th>Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>5</td>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

we see that the (decimal) number-phrase 85.7 Hamiltons could just as well
1.6. DECIMAL NUMBER-PHRASES

be written as any of the following (decimal) number-phrases:

\[
\begin{align*}
0.857 & \text{ Cleveland} \ldots \\
8.57 & \text{ Franklin} \\
857. & \text{ Washington} \\
8570. & \text{ Dime}
\end{align*}
\]

Any of the above means the same as 8 Franklins & 5 Hamiltons & 7 Dimes.

c. Another advantage of (decimal) number-phrases is that if, for whatever reason, we needed to have a “smaller” or a “larger” (decimal) numerator, it would be easy to do. We would “move the decimal point” by placing the number-phrase back under a heading, change the digit being pointed and change the (select) denominator accordingly.

d. We can now count beyond 9: \( \frac{1}{9} \), \( \frac{2}{9} \), \( \frac{3}{9} \), \( \ldots \), \( \frac{9}{9} \), \( \frac{10}{9} \), \( \frac{11}{9} \), \( \ldots \), \( \frac{99999}{9} \).

e. Finally, there is another, intriguing, advantage to the use of (decimal) number-phrases. While we can now count all the way up to 9 Cleveland & 9 Franklin & 9 Hamilton & 9 Washington & 9 Dime, that is all the way up to 99999 Dime, what if we wanted to add 1 Dime? Let us do it under a heading:

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklin</th>
<th>Hamilton</th>
<th>Washington</th>
<th>Dime</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

We have no trouble reaching the following stage (in the presence of exchange facilities):

<table>
<thead>
<tr>
<th>Cleveland</th>
<th>Franklin</th>
<th>Hamilton</th>
<th>Washington</th>
<th>Dime</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>T E N</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

At this point, although we can bundle the ten thousand-dollar-bills, we cannot exchange the bundle for any bill. Nevertheless, as long as we “pick” any of the existing denominators for our (select) denominator, then we can write a number-phrase. For instance, if we “pick” Franklin, we can write the result as 100 Franklin, or, if we “pick” Dime, as 100000. Dime.

Note. To some extent this would be cheating because we still do no have
CHAPTER 1. EXTENDED COLLECTIONS OF MONEY

in the hundreds
in the ones
in the tenths
in the hundredths
in the thousands
principal unit

a denominator corresponding to the leftmost 1 but, other than that, everything looks fine and we could even say that we wrote the (decimal) number phrases pending the creation of that new denominator! But, in case you should worry, we shall in fact develop several ways in Section ?? to get automatically as many denominators as we need.

1.7 Comparing Orders of Magnitude

While both 3 Franklin and 7 Washington are larger than 8 Dimes, they are not so in the same manner. We shall say that
• 3 Franklin is in the hundreds,
• 7 Washington is in the ones,
• 8 Dimes is in the tenths.

More generally, we shall say that
0.00875 Hamiltons is in the hundredths as we can see from
0.00875 Hamiltons = 8.75 Cents
and that 42187.53 Dimes is in the thousands as we can see from
42187.53 Dimes = 4.218753 Cleveland

All of the above is in terms of Washington being the principal unit.