Chapter 1

Accounting For Money Changing Hands *Over* The Counter

We now deal with collections that, for whatever reason, are *marked* either one of *two* ways.

The simplest example comes up when we want to undo aggregating ***steps*** on a path. When we wanted to undo aggregating ***apples***, what we did was to remove them. But, when we want to undo aggregating ***steps*** on a path, we cannot do that by *removing* these steps—which would require going back in time—but only by walking the same number of steps in the *opposite* direction, that is *backward*. So we have to be able to count both ***steps forward*** and ***steps backward***. But then, for instance, after we have walked forward three steps forward, we can aggregate five steps backward because, altogether, this amounts to walking two steps backward. However, in the case of steps on a path we will need to deal both with positions and walking steps.

1.1 States

We will call **state** a collection of objects that, *as a whole*, can be on *this-side* or *that-side* of some *benchmark*.

1. First, a few real-world examples.

• Being in such and such *solar year*. Thus, with Christ as benchmark, we can have THREE HUNDRED FORTY FIVE ***years after*** (345 AD) as well as THREE HUNDRED FORTY FIVE ***years before*** (345 BC).

signed-number-phrase
(side-) sign
signed-numerator
standard side
opposite side
transaction
direction

• Being at such and such *temperature*. Thus, we can have $+15°$C as well as $-15°$C with the temperature at which water starts *freezing* as benchmark.
• Being in such and such *financial state*. Thus, FIVE ***dollars*** "ahead of the game" and FIVE ***dollars*** "in the hole" are examples of states a gambler can be in while FIVE ***dollars*** "in the black" and FIVE ***dollars*** "in the red" are examples of states a business can be in.
• Being at such and such *point* on a path.

2. On the board, we will represent a *state* by a **signed-number-phrase** that consists of:

- a **(side-) sign** to represent the *side* of the benchmark the collection is,
- the *numerator* that represents the *number* of objects in the collection,
- the *denominator* that represents the *kind* of objects in the collection.

However, because this will make *procedures* on the *board* a lot simpler, we will lump the *side-sign* together with the *numerator* of the number-phrase that represents the number of objects in the state and speak of a **signed-numerator** which we will separate from the *denominator*.
First, we record on the board, once and for all, which *side* of the benchmark is to be the **standard side**. States on the *other* side of the benchmark will be said to be on the **opposite side**. Then we need only use, say, ↑ to represent the *standard* side and ↓ for the *opposite* side.
For instance, say that ***in-the-black*** is on the *standard* side so that ***in-the-red*** is on the *opposite* side. Then,

♠ On the *counter*, we *look* at:	❖ On the *board*, we *write*:
FIVE ***dollars*** ***in-the-black***	(5 ↑) **Washingtons**
THREE ***dollars*** ***in-the-red***	(3 ↓) **Washingtons**

where (5 ↑) and (3 ↓) are the *signed-numerators* and **Washingtons** is the *denominator*. Thus, *signed-number-phrases* will be to *states* what *number-phrases* are to *collections*.

1.2 Transactions

We will call **transaction** a collection of objects that, *as a whole*, can go *this-way* or *that-way* over the counter. Then, just as with *states*, together with the *number* and *kind* of objects in the collection, we will need to represent the **direction** of the transaction, that is the *way* the collection is going over the counter which we do very much in the same manner as with *states*. First we record on the board, once and for all, which way is to be in the **standard direction**. Transactions going the *other* way will be in the **opposite direction**. Then we need only use, say, $\rightarrow$ to represent the *standard* direction and $\leftarrow$ for the *opposite* direction.

standard direction
opposite direction

For instance, say that going from **Jack** to **Jill** is in the *standard* direction so that going from **Jill** to **Jack** is in the *opposite* direction.

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:
	Jack **Jill**
FIVE ***dollars*** ***from-Jack-to-Jill***	$(5 \rightarrow)$ **Washingtons**
THREE ***dollars*** ***from-Jill-to-Jack***	$(3 \leftarrow)$ **Washingtons**

where $(5 \rightarrow)$ and $(3 \leftarrow)$ are the *signed-numerators* and **Washingtons** is the *denominator*. Thus, *signed-number-phrases* will be to *transactions* the same as what they are to *states* and what *number-phrases* are to *collections*.

1.3 Standard Representations: Signed-Number-Phrases versus T-Accounts

Of course, in practice, we do not use *arrows* but the ways *mathematicians* and *accountants* represent *states* and *transactions* are quite different.

1. Instead of arrows, *mathematicians* "re-use" the + sign for the *standard* side and the *standard* direction and the − sign for the *opposite* side and the *opposite* direction and write the sign *ahead* of the numerator. Moreover, the parentheses are usually omitted. Thus,

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:
FIVE ***dollars*** ***in-the-black***	$+5$ **Washingtons**
THREE ***dollars*** ***in-the-red***	-3 **Washingtons**

and,

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:
	Jack **Jill**
FIVE ***dollars*** ***from-Jack-to-Jill***	$+5$ **Washingtons**
THREE ***dollars*** ***from-Jill-to-Jack***	-3 **Washingtons**

Note. Actually, the "usual way" is to let the + sign "go without saying" and to mark only, with the − sign, the states on the *opposite* side and the transactions in the *opposite* direction. The problem with this practice,

T-account
double-entry bookkeeping
balance

though, is that it tends to blur on the board the distinction between *states* on the *standard side* or transactions in the *standard direction* and *collections* just sitting on the counter. So, *we* shall always write +5.

2. *Accountants* use **T-accounts**. While the rules for operating with T-accounts can, at least initially, appear a bit intricate, **double-entry bookkeeping** is tremendously powerful and well worth the effort of understanding its basic principles. Here, we will just give a few indications. (For lack of space, we will use here $ instead of **Washington** as *denominator*.)

a. A *state*, called **balance** in ACCOUNTING, is represented by a line in the corresponding T-account.

In the following examples, money in the *black* is represented on the *left* side of the T-account and money in the *red* is represented on the *right* side of the T-account.

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:	
	Jill	
	Black	*Red*
Jill is THREE ***dollars*** in the ***red***		$3

and

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:	
	Jack	
	Black	*Red*
Jack is FIVE ***dollars*** in the ***black***	$5	

b. A *transaction* is represented by a line in the T-accounts of the *two* individuals involved in the *transaction*.

In the following example, money *coming into* the account is represented on the *black* side of the T-accounts while money *going out of* the account is represented on the *red* side of the T-accounts.

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:			
	Jack		**Jill**	
	Black (*In*)	*Red* (*Out*)	*Black* (*In*)	*Red* (*Out*)
FIVE ***dollars from-Jack-to-Jill***		$5	$5	
THREE ***dollars from-Jill-to-Jack***	$3			$3

Note. Instead of "*write the signed-number-phrase for* a transaction", accountants say "*enter* a transaction" just as, instead of saying "*write the number-phrase for* a collection", we say "*count* a collection".

However, with the advent of computerized accounting, T-accounts are increasingly giving way to signed-number-phrases.

merge
string
addition of signed-number-phrases
$\oplus$

1.4 Adding Signed-Number-Phrases.

Suppose that, just like we *aggregated* collections *on* the counter, we now

- **merge** *states*, each on this or that side of the same benchmark

or

- **string** *transactions*, each going one way or the other (but between the same two people).

followed by

Then, just like *addition of number-phrases* was the board procedure that gave us the number-phrase that represents the result of *aggregating collections*, **addition of signed-number-phrases** will be the procedure that will give us the signed-number-phrase that represents the *result* of *merging states* or *stringing transactions*.

We will need a new symbol to distinguish addition of *signed*-number-phrases from addition of *counting*-number-phrases. But, as usual, mathematicians dislike introducing new symbols! So, we will try to have it both ways by re-using, yet another time, the symbol $+$ but, at least for the time being, put it within a circle: $\oplus$. Later, we will learn to rely on the *context*.

1. In order to help us picture things while dealing with signed-numerators, we revert temporarily to the "arrow notation" that we used just above.
In what follows, we deal with *transactions* but everything applies to *states* (just use $\uparrow$ and $\downarrow$ instead of $\rightarrow$ and $\leftarrow$).
Then, for instance, we look at

$3 \rightarrow$ as standing for $\rightarrow \rightarrow \rightarrow$

$5 \leftarrow$ as standing for $\leftarrow \leftarrow \leftarrow \leftarrow \leftarrow$

In other words, we look at $\rightarrow$ and $\leftarrow$ as if they were *denominators* that, furthermore, "cancel each other":

$$\cancel{\rightarrow}\ \cancel{\leftarrow} \qquad \text{and} \qquad \cancel{\leftarrow}\ \cancel{\rightarrow}$$

so that

$$1\rightarrow \ \oplus\ 1\leftarrow \,=\, 0\leftarrow \,=\, 0\rightarrow \qquad \text{and} \qquad 1\leftarrow \ \oplus\ 1\rightarrow \,=\, 0\leftarrow \,=\, 0\rightarrow$$

2. When we string transactions, we must distinguish two cases.

a. The two transactions go in the *same* direction.

♠ Say we have two transactions:	❖ On the *board*, we *write*:
FIVE ***dollars from-Jack-to-Jill***	$5 \rightarrow$ **Washingtons**
THREE ***dollars from-Jack-to-Jill***	$3 \rightarrow$ **Washingtons**
Stringing the transactions	Adding the signed-numerators
	$5 \rightarrow \oplus\ 3 \rightarrow$
gives	$\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow$
	$8 \rightarrow$
EIGHT ***dollars from-Jack-to-Jill***	$+8$ **Washingtons**

Accountants would represent this as follows:

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*: **Jack** *Black* (*In*)	**Jack** *Red* (*Out*)	**Jill** *Black* (*In*)	**Jill** *Red* (*Out*)
FIVE ***dollars from-Jack-to-Jill***		\$5	\$5	
THREE ***dollars from-Jack-to-Jill***		\$3	\$3	
EIGHT ***dollars from-Jack-to-Jill***		\$8	\$8	

In other words, when we add signed-numerators that have the *same* sign, we *add* the numerators and the sign of the resulting signed-numerator is of course the sign common to the signed-numerators being added.

b. The two transactions go in *opposite* directions.

♠ Say we have the two transactions:	❖ On the *board*, we *write*:
THREE ***dollars from-Jack-to-Jill***	$3 \rightarrow$ **Washingtons**
FIVE ***dollars from-Jill-to-Jack***	$5 \leftarrow$ **Washingtons**
Stringing the transactions	Adding the signed-numerators
	$3 \rightarrow \oplus\ 5 \leftarrow$
	$\rightarrow \rightarrow \rightarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow$
gives	$\rightarrow \rightarrow \cancel{\rightarrow\leftarrow} \leftarrow \leftarrow \leftarrow \leftarrow$
	$\rightarrow \cancel{\rightarrow\leftarrow}\ \cancel{\rightarrow\leftarrow} \leftarrow \leftarrow \leftarrow$
	$\cancel{\rightarrow\leftarrow}\ \cancel{\rightarrow\leftarrow} \leftarrow \leftarrow$
	$2 \leftarrow$
TWO ***dollars Jill-to-Jack***	-2 **Washingtons**

Accountants would represent the above as follows:

subtraction of signed-number-phrases
incorrect
subtract
$\ominus$

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:			
	Jack		**Jill**	
	Black (*In*)	*Red* (*Out*)	*Black* (*In*)	*Red* (*Out*)
THREE ***dollars from-Jack-to-Jill***		\$3	\$3	
FIVE ***dollars from-Jill-to-Jack***	\$5			\$5
TWO ***dollars from-Jill-to-Jack***	\$2			\$2

In other words, when we add signed-number-phrases that have *opposite* signs, we *subtract* one numerator from the other and since this can only be done *one* way, this gives us the sign of the resulting signed-numerator.

1.5 Subtracting Signed-Number-Phrases.

The next issue is the **subtraction of signed-number-phrases**. However, (i) what a subtraction *represents*, and, (ii) what the *procedure* should be, are not immediately obvious. So, first, here is an example of how subtraction could come up. Suppose we had just added a long string of signed-number-phrases, say

$$-2 \textbf{ Dollars} \oplus -7 \textbf{ Dollars} \oplus +5 \textbf{ Dollars} \oplus \ldots \oplus +3 \textbf{ Dollars}$$

and say, for the sake of the argument, that we had found that the total was, say, -132 **Dollars**.

Now suppose we then realized that, somewhere along the line, one of the signed-number-phrases, say the second one, -7 **Dollars**, was **incorrect** in that it should not have appeared in the addition, so that the total too is incorrect. A priori, to obtain the new, *corrected total*, we have the following three choices.

1. We could *strike out* the incorrect signed-number-phrase and *redo* the entire addition:

$$-2 \textbf{ Dollars} \oplus \cancel{-7 \textbf{ Dollars}} \oplus +5 \textbf{ Dollars} \oplus \ldots \oplus +3 \textbf{ Dollars}$$

Of course, if the addition is really long, this is going to involve a lot of unnecessary work, redoing a lot that had been done correctly.

2. We could **subtract** the incorrect signed-number-phrase from the incorrect *total*:

$$-132 \textbf{ Dollars} \qquad \ominus \qquad -7 \textbf{ Dollars}$$

The trouble, though, is that we have no idea what *procedure* to use for $\ominus$!

cancel
adjustment

3. We can **cancel** the *effect* of the incorrect signed-number-phrase on the incorrect total by *adding the opposite* of the incorrect signed-number-phrase to the incorrect total. Accountants call this entering an **adjustment**. That this *must* give us the same correct result as would choice **1.** is easy to see by comparing:

- The addition in which -7 **Dollars**, the incorrect signed-number-phrase, was *struck out*:

-2 **Dollars** $\oplus$ ~~-7 **Dollars**~~ $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars**

- The addition in which -7 **Dollars**, the incorrect signed-number-phrase has been *left in* but has been *cancelled* by the adjustment $+7$ **Dollars** that was *added* at the end:

-2 **Dollars** $\oplus$ ~~-7 **Dollars**~~ $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars** $\oplus$ ~~$+7$ **Dollars**~~

Either way, the signed-number-phrases that are *actually* involved are:

-2 **Dollars** $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars**

which makes the case.

Accountants would represent the above as follows:

♠ Over the *counter*, we *look* at:	❖ On the *board*, we *write*:			
	Striking out		Cancelling	
	Black (*In*)	*Red* (*Out*)	*Black* (*In*)	*Red* (*Out*)
TWO ***dollars*** out		\$2		\$2
SEVEN ***dollars*** out		~~\$7~~		\$7
FIVE ***dollars*** in	\$5		\$5	
...	...		...	
THREE ***dollars*** in	\$3		\$3	
SEVEN ***dollars*** in (Adjustment)			\$7	

In other words,

- *Subtracting* the incorrect signed-number-phrase (choice **2.**):

-132 **Dollars** $\ominus$ -7 **Dollars**

amounts to exactly the same as

- *Adding the opposite* of the incorrect signed-number-phrase (choice **3.**):

-132 **Dollars** $\oplus$ $+7$ **Dollars**

Since, as already pointed out, we have no ready-made *procedure* for *subtraction*, we will say that "adding the opposite" *is* the procedure and that, *in the case of signed-number-phrases*, to **subtract** something *is just short for* "to add its opposite".

subtract
effect
initial state
final state
gain
loss
change of state

1.6 Effect of Transactions on States

We now look at the **effect** of a *transaction* on *state*s. Given an **initial state** and a transaction involving that state, we will call **final state** the state *after* the transaction. (This will fit exactly what we did in Section **??**.)

For instance,

♠ Looking at ***Jill***, suppose that:

- In the *initial* state, ***Jill*** is THREE ***dollars in-the-red***.
- Then, a *transaction* occurs, say FIVE ***dollars from-Jack-to-Jill***.
- Now, in the *final* state, ***Jill*** is TWO ***dollars in-the-black***.

Thus, the *effect* of a FIVE ***dollars from-Jack-to-Jill*** *transaction* is a FIVE ***dollars*** **gain** on ***Jill***'s *state*—as well as a FIVE ***dollars*** **loss** on ***Jack***'s *state*. A transaction in the *opposite* direction would have the *opposite* effects.

❖ On the board, to find the **change of state**, we *subtract* the *initial* state from the *final* state to *remove* from the final state the effect of all *previous* transactions.

$$\begin{aligned}
\textsf{Change of State} &= \textsf{Final State} \ominus \textsf{Initial State} \\
&= +2 \textsf{ Washingtons} \ominus -3 \textsf{ Washingtons} \\
&= +2 \textsf{ Washingtons} \oplus +3 \textsf{ Washingtons} \\
&= +5 \textsf{ Washingtons} \\
&= \xrightarrow{\to 5 \textsf{ Washingtons}}
\end{aligned}$$

We thus have that

$$\textit{Effect of a Transaction} = \textit{Change of State}$$

This seemingly trivial statement will have in fact far-reaching generalizations.

1.7 Comparing States

Essentially, we compare *states* just the way we compared *collections* in Section **??**: Where we used *forward* and *backward* counts, here we use transactions in the *standard* and *opposite* directions:

algebra-smaller
algebra-larger
algebra-equal

- When it takes a transaction in the *standard* direction to go from an *initial* state to a *final* state, we say that the *initial* state is **algebra-smaller** than the *final* state.
- When it takes a transaction in the *opposite* direction to go from an *initial* state to a *final* state, we say that the *initial* state is **algebra-larger** than the *final* state.
- When it takes *no* transaction to go from an *initial* state to a *final* state, we say that the initial state is **algebra-equal** to the final state.

1.8 Equations and Inequations

We can now return to the problems we encountered in Section **??**.