Chapter 1

Accounting For Money Changing Hands Over The Counter

We now deal with collections that, for whatever reason, are *marked* either one of two ways.

The simplest example comes up when we want to undo aggregating *steps* on a path. When we wanted to undo aggregating *apples*, what we did was to remove them. But, when we want to undo aggregating *steps* on a path, we cannot do that by *removing* these steps—which would require going back in time—but only by walking the same number of steps in the *opposite* direction, that is *backward*. So we have to be able to count both *steps forward* and *steps backward*. But then, for instance, after we have walked forward three steps forward, we can aggregate five steps backward because, altogether, this amounts to walking two steps backward. However, in the case of steps on a path we will need to deal both with positions and walking steps.

1.1 States

We will call *state* a collection of objects that, *as a whole*, can be on *this-side* or *that-side* of some *benchmark*.

1. First, a few real-world examples.

- Being in such and such *solar year*. Thus, with Christ as benchmark, we can have THREE HUNDRED FORTY FIVE *years after* (345 AD) as well as THREE HUNDRED FORTY FIVE *years before* (345 BC).
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• Being at such and such temperature. Thus, we can have $15\degree C$ as well as $-15\degree C$ with the temperature at which water starts freezing as benchmark.
• Being in such and such financial state. Thus, five dollars “ahead of the game” and five dollars “in the hole” are examples of states a gambler can be in while five dollars “in the black” and five dollars “in the red” are examples of states a business can be in.
• Being at such and such point on a path.

2. On the board, we will represent a state by a signed-number-phrase that consists of:

- a (side-) sign to represent the side of the benchmark the collection is,
- the numerator that represents the number of objects in the collection,
- the denominator that represents the kind of objects in the collection.

However, because this will make procedures on the board a lot simpler, we will lump the side-sign together with the numerator of the number-phrase that represents the number of objects in the state and speak of a signed-numerator which we will separate from the denominator.

First, we record on the board, once and for all, which side of the benchmark is to be the standard side. States on the other side of the benchmark will be said to be on the opposite side. Then we need only use, say, ↑ to represent the standard side and ↓ for the opposite side.

For instance, say that in-the-black is on the standard side so that in-the-red is on the opposite side. Then,

<table>
<thead>
<tr>
<th>♠ On the counter, we look at:</th>
<th>♦ On the board, we write:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FIVE</strong> dollars <em>in-the-black</em></td>
<td><em>(5 ↑)</em> Washingtons</td>
</tr>
<tr>
<td><strong>THREE</strong> dollars <em>in-the-red</em></td>
<td><em>(3 ↓)</em> Washingtons</td>
</tr>
</tbody>
</table>

where (5 ↑) and (3 ↓) are the signed-numerators and Washingtons is the denominator. Thus, signed-number-phrases will be to states what number-phrases are to collections.

1.2 Transactions

We will call transaction a collection of objects that, as a whole, can go this-way or that-way over the counter. Then, just as with states, together with the number and kind of objects in the collection, we will need to represent the direction of the transaction, that is the way the collection is going over the counter which we do very much in the same manner as with states. First we record on the board, once and for all, which way is to be
in the standard direction. Transactions going the other way will be in the opposite direction. Then we need only use, say, → to represent the standard direction and ← for the opposite direction.

For instance, say that going from Jack to Jill is in the standard direction so that going from Jill to Jack is in the opposite direction.

<table>
<thead>
<tr>
<th>♠ Over the counter, we look at:</th>
<th>♦ On the board, we write:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FIVE dollars from-Jack-to-Jill</strong></td>
<td>Jack</td>
</tr>
<tr>
<td><strong>THREE dollars from-Jill-to-Jack</strong></td>
<td>(5 →) Washingtons</td>
</tr>
</tbody>
</table>

where (5 →) and (3 ←) are the signed-numerators and Washingtons is the denominator. Thus, signed-number-phrases will be to transactions the same as what they are to states and what number-phrases are to collections.

### 1.3 Standard Representations:

#### Signed-Number-Phrases versus T-Accounts

Of course, in practice, we do not use arrows but the ways mathematicians and accountants represent states and transactions are quite different.

1. Instead of arrows, mathematicians “re-use” the + sign for the standard side and the standard direction and the − sign for the opposite side and the opposite direction and write the sign ahead of the numerator. Moreover, the parentheses are usually omitted. Thus,

<table>
<thead>
<tr>
<th>♠ Over the counter, we look at:</th>
<th>♦ On the board, we write:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FIVE dollars in-the-black</strong></td>
<td>+5 Washingtons</td>
</tr>
<tr>
<td><strong>THREE dollars in-the-red</strong></td>
<td>−3 Washingtons</td>
</tr>
</tbody>
</table>

and,

<table>
<thead>
<tr>
<th>♠ Over the counter, we look at:</th>
<th>♦ On the board, we write:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FIVE dollars from-Jack-to-Jill</strong></td>
<td>Jack</td>
</tr>
<tr>
<td><strong>THREE dollars from-Jill-to-Jack</strong></td>
<td>+5 Washingtons</td>
</tr>
</tbody>
</table>

Note. Actually, the “usual way” is to let the + sign “go without saying” and to mark only, with the − sign, the states on the opposite side and the transactions in the opposite direction. The problem with this practice,
though, is that it tends to blur on the board the distinction between states on the standard side or transactions in the standard direction and collections just sitting on the counter. So, we shall always write +5.

2. Accountants use T-accounts. While the rules for operating with T-accounts can, at least initially, appear a bit intricate, double-entry bookkeeping is tremendously powerful and well worth the effort of understanding its basic principles. Here, we will just give a few indications. (For lack of space, we will use here $ instead of Washington as denominator.)

a. A state, called balance in accounting, is represented by a line in the corresponding T-account.

In the following examples, money in the black is represented on the left side of the T-account and money in the red is represented on the right side of the T-account.

<table>
<thead>
<tr>
<th>♠ Over the counter, we look at:</th>
<th>❖ On the board, we write:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Jill</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Black</td>
</tr>
<tr>
<td><strong>Jill</strong> is three dollars in the red</td>
<td>$3</td>
</tr>
</tbody>
</table>


and

<table>
<thead>
<tr>
<th>♠ Over the counter, we look at:</th>
<th>❖ On the board, we write:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Jack</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Black</td>
</tr>
<tr>
<td><strong>Jack</strong> is five dollars in the black</td>
<td>$5</td>
</tr>
</tbody>
</table>

b. A transaction is represented by a line in the T-accounts of the two individuals involved in the transaction.

In the following example, money coming into the account is represented on the black side of the T-accounts while money going out of the account is represented on the red side of the T-accounts.

<table>
<thead>
<tr>
<th>♠ Over the counter, we look at:</th>
<th>❖ On the board, we write:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Jack</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Black</td>
</tr>
<tr>
<td>FIVE dollars from-Jack-to-Jill</td>
<td>($5</td>
</tr>
<tr>
<td>THREE dollars from-Jill-to-Jack</td>
<td>$3</td>
</tr>
</tbody>
</table>

Note. Instead of “write the signed-number-phrase for a transaction”, ac-
countants say “enter a transaction” just as, instead of saying “write the number-phrase for a collection”, we say “count a collection”.

However, with the advent of computerized accounting, T-accounts are increasingly giving way to signed-number-phrases.

### 1.4 Adding Signed-Number-Phrases.

Suppose that, just like we aggregated collections on the counter, we now

- **merge states**, each on this or that side of the same benchmark

or

- **string transactions**, each going one way or the other (but between the same two people).

followed by

Then, just like addition of number-phrases was the board procedure that gave us the number-phrase that represents the result of aggregating collections, addition of signed-number-phrases will be the procedure that will give us the signed-number-phrase that represents the result of merging states or stringing transactions.

We will need a new symbol to distinguish addition of signed-number-phrases from addition of counting-number-phrases. But, as usual, mathematicians dislike introducing new symbols! So, we will try to have it both ways by re-using, yet another time, the symbol + but, at least for the time being, put it within a circle: $\oplus$. Later, we will learn to rely on the context.

1. In order to help us picture things while dealing with signed-numerators, we revert temporarily to the “arrow notation” that we used just above. In what follows, we deal with transactions but everything applies to states (just use ↑ and ↓ instead of → and ←).

Then, for instance, we look at

- $3 \rightarrow$ as standing for $\rightarrow \rightarrow \rightarrow$
- $5 \leftarrow$ as standing for $\leftarrow \leftarrow \leftarrow \leftarrow \leftarrow$

In other words, we look at $\rightarrow$ and $\leftarrow$ as if they were denominators that, furthermore, “cancel each other”:

```
###  and  ###
```

so that

```
1 \rightarrow \oplus 1 \leftarrow = 0 \leftarrow = 0 \rightarrow \quad \text{and} \quad 1 \leftarrow \oplus 1 \rightarrow = 0 \leftarrow = 0 \rightarrow
```
2. When we string transactions, we must distinguish two cases.

   a. The two transactions go in the same direction.

   Say we have two transactions:

   ▶ On the board, we write:

   | FIVE dollars from-Jack-to-Jill | 5 ← Washingtons |
   | THREE dollars from-Jack-to-Jill | 3 ← Washingtons |

   Stringing the transactions Adding the signed-numerators

   5 → ⊕ 3 →

   gives

   → → → → → → → → → →

   8 →

   EIGHT dollars from-Jack-to-Jill

   Accountants would represent this as follows:

   | Over the counter, we look at: |
   | Jack | Jill |
   | Black (In) | Red (Out) | Black (In) | Red (Out) |
   | FIVE dollars from-Jack-to-Jill | $5 | $5 |
   | THREE dollars from-Jack-to-Jill | $3 | $3 |
   | EIGHT dollars from-Jack-to-Jill | $8 | $8 |

   In other words, when we add signed-numerators that have the same sign, we add the numerators and the sign of the resulting signed-numerator is of course the sign common to the signed-numerators being added.

   b. The two transactions go in opposite directions.

   Say we have the two transactions:

   ▶ On the board, we write:

   | THREE dollars from-Jack-to-Jill | 3 ← Washingtons |
   | FIVE dollars from-Jill-to-Jack | 5 ← Washingtons |

   Stringing the transactions Adding the signed-numerators

   3 → ⊕ 5 ←

   gives

   → → ### ### ← ← ← ←

   → ###  ### ← ← ← ←

   → ###  ### ← ← ← ←

   → → → → → → → → → →

   2 ←

   TWO dollars Jill-to-Jack

   −2 Washingtons

   Accountants would represent the above as follows:
1.5 Subtracting Signed-Number-Phrases.

The next issue is the subtraction of signed-number-phrases. However, (i) what a subtraction represents, and, (ii) what the procedure should be, are not immediately obvious. So, first, here is an example of how subtraction could come up. Suppose we had just added a long string of signed-number-phrases, say

$-2 \text{ Dollars} \oplus -7 \text{ Dollars} \oplus +5 \text{ Dollars} \oplus \ldots \oplus +3 \text{ Dollars}$

and say, for the sake of the argument, that we had found that the total was, say, $-132 \text{ Dollars}$.

Now suppose we then realized that, somewhere along the line, one of the signed-number-phrases, say the second one, $-7 \text{ Dollars}$, was incorrect in that it should not have appeared in the addition, so that the total too is incorrect. A priori, to obtain the new, corrected total, we have the following three choices.

1. We could strike out the incorrect signed-number-phrase and redo the entire addition:

$-2 \text{ Dollars} \oplus \cancel{7 \text{ Dollars}} \oplus +5 \text{ Dollars} \oplus \ldots \oplus +3 \text{ Dollars}$

Of course, if the addition is really long, this is going to involve a lot of unnecessary work, redoing a lot that had been done correctly.

2. We could subtract the incorrect signed-number-phrase from the incorrect total:

$-132 \text{ Dollars} \ominus \cdot \cdot \cdot \ominus -7 \text{ Dollars}$

The trouble, though, is that we have no idea what procedure to use for $\ominus$!
3. We can cancel the effect of the incorrect signed-number-phrase on the incorrect total by adding the opposite of the incorrect signed-number-phrase to the incorrect total. Accountants call this entering an adjustment. That this must give us the same correct result as would choice 1, is easy to see by comparing:

- The addition in which \(-7\) Dollars, the incorrect signed-number-phrase, was struck out:

  \[-2\text{ Dollars } \oplus \overset{\text{Strike-out}}{7\text{ Dollars}} \oplus +5\text{ Dollars } \oplus \ldots \oplus +3\text{ Dollars}\]

- The addition in which \(-7\) Dollars, the incorrect signed-number-phrase has been left in but has been cancelled by the adjustment +7 Dollars that was added at the end:

  \[-2\text{ Dollars } \oplus \underset{\text{Cancelled}}{-7\text{ Dollars}} \oplus +5\text{ Dollars } \oplus \ldots \oplus +3\text{ Dollars} \oplus \overset{\text{Added}}{7\text{ Dollars}}\]

Either way, the signed-number-phrases that are actually involved are:

\[-2\text{ Dollars } \oplus +5\text{ Dollars } \oplus \ldots \oplus +3\text{ Dollars}\]

which makes the case.

*Accountants* would represent the above as follows:

<table>
<thead>
<tr>
<th>Striking out</th>
<th>Canceling</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{Black In})</td>
<td>(\text{Red Out})</td>
</tr>
<tr>
<td>TWO dollars out</td>
<td>$2</td>
</tr>
<tr>
<td>SEVEN dollars out</td>
<td>$7</td>
</tr>
<tr>
<td>FIVE dollars in</td>
<td>$5</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>THREE dollars in</td>
<td>$3</td>
</tr>
<tr>
<td>SEVEN dollars in</td>
<td>$7</td>
</tr>
</tbody>
</table>

In other words,

- *Subtracting* the incorrect signed-number-phrase (choice 2.):

  \[-132\text{ Dollars } \ominus -7\text{ Dollars}\]

amounts to exactly the same as

- *Adding the opposite* of the incorrect signed-number-phrase (choice 3.):

  \[-132\text{ Dollars } \oplus +7\text{ Dollars}\]
Since, as already pointed out, we have no ready-made procedure for subtraction, we will say that “adding the opposite” is the procedure and that, in the case of signed-number-phrases, to subtract something is just short for “to add its opposite”.

1.6 Effect of Transactions on States

We now look at the effect of a transaction on states. Given an initial state and a transaction involving that state, we will call final state the state after the transaction. (This will fit exactly what we did in Section ??.)

For instance,

♠ Looking at Jill, suppose that:
- In the initial state, Jill is THREE dollars in-the-red.
- Then, a transaction occurs, say FIVE dollars from-Jack-to-Jill.
- Now, in the final state, Jill is TWO dollars in-the-black.

Thus, the effect of a FIVE dollars from-Jack-to-Jill transaction is a FIVE dollars gain on Jill’s state—as well as a FIVE dollars loss on Jack’s state. A transaction in the opposite direction would have the opposite effects.

❖ On the board, to find the change of state, we subtract the initial state from the final state to remove from the final state the effect of all previous transactions.

\[
\text{Change of State} = \text{Final State} \ominus \text{Initial State} \\
= +2 \text{ Washingtons} \ominus -3 \text{ Washingtons} \\
= +2 \text{ Washingtons} \oplus +3 \text{ Washingtons} \\
= +5 \text{ Washingtons} \\
= -5 \text{ Washingtons}
\]

We thus have that

\[
\text{Effect of a Transaction} = \text{Change of State}
\]

This seemingly trivial statement will have in fact far-reaching generalizations.

1.7 Comparing States

Essentially, we compare states just the way we compared collections in Section ??: Where we used forward and backward counts, here we use transactions in the standard and opposite directions:
• When it takes a transaction in the standard direction to go from an initial state to a final state, we say that the initial state is algebra-smaller than the final state.

• When it takes a transaction in the opposite direction to go from an initial state to a final state, we say that the initial state is algebra-larger than the final state.

• When it takes no transaction to go from an initial state to a final state, we say that the initial state is algebra-equal to the final state.

1.8 Equations and Inequations

We can now return to the problems we encountered in Section ??.