discrete
continuous

Chapter 1

Accounting For *Discrete* Goods

We now turn to the accounting for *goods*. While *essentially* the same as the accounting for *money*, there are a few technical issues and further developments. Here, though, we will deal with the accounting for

- **discrete** *goods* such as, say, ***apples*** or ***bananas***

as opposed to the accounting for

- **continuous** *goods* such as, say, ***lengths*** or ***liquids***

because the latter involve the *English* System of denominators which present problems of their own and which, therefore, we shall leave to a later chapter.

1.1 Counting And Adding Collections Of Discrete Goods

1. The main difference between counting *money* and *discrete goods* is that:

- When counting *money*, when we have more than TEN of a kind, we
  - *bundle* TEN of a kind, and then we
  - *exchange* the bundle of TEN of a kind for 1 of the next kind up—for which we already have a *denominator*.

  The question is how to read 23 **Apples**

discrete
worth
unit-worth
value
unit-value

- To count **discrete** *goods* such as, say, ***apples***, and while we will still *bundle* collections of TEN ***objects***, we will usually *not* be able to *exchange* these bundles the way we did with *money*, so that the *denominators* will usually have to represent *bundles* rather than *objects* as was the case with *money*.

Note. Bundling by TEN is rather recent. Historical remnants of a time when *discrete* goods were bundled by TWELVE include ***eggs*** and ***pencils***.

2. As far as *aggregating* collections of *different* goods goes, the situation is the same as in Section **??**, where we already saw that combinations can *always* be added.

♠ If, say Jack has THREE ***apples***, FIVE ***bananas*** and THIRTEEN ***cents*** and Jill has TWO ***apples***, EIGHT ***bananas*** and ELEVEN ***cents***, when they put everything together, they have FIVE ***apples***, THIRTEEN ***bananas*** and TWENTY-FOUR ***cents***.

❖ On the board we proceed as follows:

$$[3 \textsf{ Apples} \;\&\; 5 \textsf{ Bananas} \;\&\; 13 \textsf{ Cents}] \;\&\; [2 \textsf{ Apples} \;\&\; 8 \textsf{ Bananas} \;\&\; 11 \textsf{ Cents}]$$
$$= 5 \textsf{ Apples} \;\&\; 13 \textsf{ Bananas} \;\&\; 24 \textsf{ Cents}$$

1.2 Evaluating Collections of Discrete Goods: Co-multiplication

We seldom deal with *goods* alone and, often, we will want to know how much *money* a given collection of *goods* is **worth**. For that purpose, we shall need a **unit-worth** for the objects. On the board, we shall represent a *worth* by a **value** and a *unit-worth* by a **unit-value**.

Note. Just as we say "To *count* a collection" as a short for "To find the number-phrase that represents a collection", we shall say "To *evaluate* a collection" as a short for "To find the number-phrase that represents the *worth* of a collection".

1. For instance, given a collection of ***apples***, we may be interested in how many ***dimes*** it is *worth* and for that we need to know how many ***cents*** each ***apple*** is *worth*.

♠ Suppose Jack has THREE ***apples*** with a *unit-worth* of, say, SEVEN ***cents-per-apple***. Thus, at a unit-worth of SEVEN ***cents-per-apple***, ***Jack's worth*** would be TWENTY-ONE ***cents***.

co-multiply
multiplication tables
vector
co-vector
co-number-phrase
percentage
extend

❖ On the board, we write $7\ \frac{\textsf{Cents}}{\textsf{Apple}}$ to represent the unit-value SEVEN ***cents-per-apple*** and we **co-multiply**:

$$\begin{aligned}\textsf{Jack's Value} &= [3\ \cancel{\textsf{Apples}}] \times \left[7\ \frac{\textsf{Cents}}{\cancel{\textsf{Apple}}}\right]\\ &= [3 \times 7]\ \textsf{Cents}\\ &= 21\ \textsf{Cents}\end{aligned}$$

In other words, the (board) procedure for co-multiplication is as follows:
- Get the *denominator* by "canceling" the **Apples**, and
- Get the *numerator* from the **multiplication tables** we had to memorize as children.

2. *Co-multiplication* is in fact an extremely important concept.

a. The "cancelling" of denominators, as that of **Apples** above, is in fact something that *scientists* and *engineers* do all the time and it is in fact at the heart of a part of *mathematics* called DIMENSIONAL ANALYSIS.

b. Moreover, *co-multiplication* is also at the heart of LINEAR ALGEBRA, a part of mathematics already mentioned at the very beginning of this text, in which 3 **Apples** would be a **vector** while $7\ \frac{\textsf{Cents}}{\textsf{Apple}}$ would be a **co-vector**. Accordingly, since we call 3 **Apples** a *number-phrase*, we will call $7\ \frac{\textsf{Cents}}{\textsf{Apple}}$ a **co-number-phrase**.

c. More modestly, co-multiplication arises as at least one aspect of **percentage**:

$$[3\ \cancel{\textsf{Dollars}}] \times \left[7\ \frac{\textsf{Cents}}{\cancel{\textsf{Dollar}}}\right] = [3 \times 7]\ \textsf{Cents} = 21\ \textsf{Cents}$$

1.3 Effect of Transactions on States: Signed Co-Multiplication

We now **extend** the concept of *co-multiplication* to *signed*-number-phrases. The idea is of course to obtain the *gain* or *loss* caused by a transaction.

For illustration purposes, we will look at transactions occurring in an ***apple*** store. Suppose that, not only can ***apples***, for whatever reason to be left to the reader's imagination, *appear in* or *disappear from* the store, but the ***apples*** can *also* be ***good*** or ***bad***.

1. We begin with the issue of representing the *transactions* of ***apples***. This is no different from representing the *transactions* of ***dollars*** which we saw earlier on.

♠ We have ***apples*** that can *appear* or *disappear* from the store. (Just like, earlier on, we had ***dollars*** going from Jack to Jill and/or from Jill to Jack.)

unit-profit-worth
unit-loss-worth
co-signed-number-phrase
unit-profit
unit-loss-value

❖ We choose the *standard direction*, for which we use the + sign, to be that of ***apples*** that *appear in* the store and so the *opposite direction*, for which we use the − sign, is that of ***apples*** that *disappear out* of the store.

Thus, for instance,

♠ In the real world, we may have:	❖ We then write:
THREE ***apples*** *appear in* the store,	[+3 **Apples**]
or	
FIVE ***apples*** *disappear from* the store.	[−5 **Apples**]

2. Because the resulting *gain* or *loss* of such transactions will of course depend on what unit-worth the ***apples*** have, we now look at the kinds of unit-worth ***apples*** can have.

♠ The ***apples*** may be either ***good*** or ***bad*** and, if those that are ***good*** can be *sold* at a **unit-*profit*-worth**, those that are ***bad*** must be *disposed of* at a **unit-*loss*-worth**.

❖ To represent *unit-profit-worth* and *unit-loss-worth* on the board, we use **co-signed-number-phrase** as we did with *co-number-phrases* and we use a + sign for a **unit-*profit***-value and a − sign for a **unit-*loss*-value**.

Thus, for instance,

♠ In the store, we may have:	❖ We then write:
apples that are ***good*** and could be *sold* at a unit-*profit*-worth of, say, SEVEN ***cents-per-apple***	$+7\ \frac{\textsf{Cents}}{\textsf{Apple}}$
or	
apples that are ***bad*** and must be *disposed of* at a unit-*loss*-worth of, say, SEVEN ***cents-per-apple***	$-7\ \frac{\textsf{Cents}}{\textsf{Apple}}$

Note. The choice of symbols, + to represent ***good*** and − to represent ***bad***, is not really an arbitrary choice because of the way they will interact with the symbols for *appearing* and *disappearing*. We leave it as an exercise for the reader to investigate what happens when other choices are made.

3. We are now finally, in a position to look at the four kinds of *effect* that *transactions* of goods can have on money *states*.

make pictures of good in = good, good out = bad, bad in = bad, bad out = good

represent by $++=+$, $+-=-$, $-+=-$, $--=+$

♠ In the *real world*, we have	❖ On the *board*, we write
Case 1.	
THREE ***apples*** *appear* in the store.	$[+3 \text{ Apples}]$
They are ***good***, with a unit-*profit*-worth of SEVEN ***cents-per-apple***.	$\left[+7 \frac{\text{Cents}}{\text{Apple}}\right]$
The result is	$[+3 \cancel{\text{Apples}}] \times \left[+7 \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$
	$= [+3] \times [+7] \text{ Cents}$
a *profit*-worth of TWENTY-ONE ***cents***.	$= +21 \text{ Cents}$
Case 2.	
THREE ***apples*** *appear* in the store.	$[+3 \text{ Apples}]$
They are ***bad***, with a unit-*loss*-worth of SEVEN ***cents-per-apple***.	$\left[-7 \frac{\text{Cents}}{\text{Apple}}\right]$
The result is	$[+3 \cancel{\text{Apples}}] \times \left[-7 \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$
	$= [+3] \times [-7] \text{ Cents}$
a *loss*-worth of TWENTY-ONE ***cents***.	$= -21 \text{ Cents}$
Case 3.	
THREE ***apples*** *disappear* from the store.	$[-3 \text{ Apples}]$
They are ***good***, with a unit-*profit*-worth of SEVEN ***cents-per-apple***.	$\left[+7 \frac{\text{Cents}}{\text{Apple}}\right]$
The result is	$[-3 \cancel{\text{Apples}}] \times \left[+7 \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$
	$= [-3] \times [+7] \text{ Cents}$
a *loss*-worth of TWENTY-ONE ***cents***.	$= -21 \text{ Cents}$
Case 4.	
THREE ***apples*** *disappear* from the store.	$[-3 \text{ Apples}]$
They are ***bad***, with a unit-*loss*-worth of SEVEN ***cents-per-apple***.	$\left[-7 \frac{\text{Cents}}{\text{Apple}}\right]$
The result is	$[-3 \cancel{\text{Apples}}] \times \left[-7 \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$
	$= [-3] \times [-7] \text{ Cents}$
a *profit*-worth of TWENTY-ONE ***cents***.	$= +21 \text{ Cents}$

1.4 Common Denominator

Usually, in order to deal with collections of *goods* of *different* kinds, these must still have "something in common".

For instance, say ***Jack*** has SIX ***apples*** and FIVE ***bananas*** while ***Jill*** has TWO ***apples*** and EIGHT ***bananas***. The question, say, of who has more and who has less is meaningless as we cannot compare ***apples*** and ***bananas***.

Collections can have "something in common" in two ways.

1. When both kinds of goods can be *exchanged* for a *same* kind of goods.
♠ For instance, say Jack has SIX ***apples*** and FIVE ***bananas*** while Jill has TWO ***apples*** and EIGHT ***bananas*** but that ONE ***apple*** can be exchanged for FOUR ***nuts*** and ONE ***banana*** can be exchanged for SEVEN ***nuts***.
Jack can then exchange his SIX ***apples*** for TWENTY-FOUR ***nuts*** and his FIVE ***bananas*** for THIRTY-FIVE ***nuts*** for a total of FIFTY-NINE ***nuts*** while ***Jill*** can exchange her TWO ***apples*** for EIGHT ***nuts*** and her EIGHT ***bananas*** for FIFTY-SIX ***nuts*** for a total of SIXTY-FOUR ***nuts***.
❖ On the board we use the *multiplication tables* to write:

$$\begin{aligned}\textsf{Jack's Holding} &= 6\ \textsf{Apples}\ \&\ 5\ \textsf{Bananas}\\ &= 6(4\ \textsf{Nuts}) + 5(7\ \textsf{Nuts})\\ &= 24\ \textsf{Nuts} + 35\ \textsf{Nuts}\\ &= 59\ \textsf{Nuts}\end{aligned}$$

and

$$\begin{aligned}\textsf{Jill's Holding} &= 2\ \textsf{Apples}\ \&\ 8\ \textsf{Bananas}\\ &= 2(4\ \textsf{Nuts}) + 8(7\ \textsf{Nuts})\\ &= 8\ \textsf{Nuts} + 56\ \textsf{Nuts}\\ &= 64\ \textsf{Nuts}\end{aligned}$$

2. Usually, though, we don't deal with just collections of *goods* but also with *money.* So, even being able to exchange the given kinds of *goods* for a "common" kind of *goods* would still leave us with both these goods *and* money.
♠ Suppose, for instance, that ***Jack*** not only has SIX ***apples*** and FIVE ***bananas*** as above but also THIRTEEN ***cents*** and that ***Jill*** not only has TWO ***apples*** and EIGHT ***bananas*** as above but also THIRTY-ONE ***cents***. Even if both ***apples*** and ***bananas*** could be exchanged for ***nuts*** as above, this would still leave ***Jack*** and ***Jill*** holding both ***nuts*** and ***cents***:

- ***Jack*** would be holding FIFTY-NINE ***nuts*** and THIRTEEN ***cents***
- ***Jill*** would be holding SIXTY-FOUR ***nuts*** and THIRTY-ONE ***cents***

In this kind of situations, it will be convenient to be able to *evaluate* both kinds of *goods* in terms of *money* so that, eventually, we will be left only with money. This will require that, corresponding to each *good*, we have a *unit-worth* for that *good.*

For instance, if, say, ***apples*** have a ***unit-worth*** of SEVEN ***cents-per-apple*** and ***bananas*** have a ***unit-worth*** of FOUR ***cents-per-banana***, *then*:

common denominator

- ***Jack***'s SIX ***apples*** will be worth FORTY-TWO ***cents*** and his FIVE ***bananas*** will be worth TWENTY ***cents*** so that, with the THIRTEEN ***cents*** he already had, ***Jack's worth*** will be SEVENTY-FIVE ***cents***.
- ***Jill***'s TWO ***apples*** will be worth FOURTEEN ***cents*** and her EIGHT ***bananas*** will be worth THIRTY-TWO ***cents*** so that, with the THIRTY-ONE ***cents*** she already had, ***Jill's worth*** will be SEVENTY-SEVEN ***cents***.

❖ On the board we proceed as follows

$$\begin{aligned}\textsf{Jack's Value} &= [3\ \cancel{\textsf{Apples}}] \times \left[7\ \frac{\textsf{Cents}}{\cancel{\textsf{Apple}}}\right] + [5\ \cancel{\textsf{Bananas}}] \times \left[4\ \frac{\textsf{Cents}}{\cancel{\textsf{Banana}}}\right] + 13\ \textsf{Cents}^{1} \\ &= [3 \times 7]\ \textsf{Cents} + [5 \times 4]\ \textsf{Cents} + 13\ \textsf{Cents} \\ &= 21\ \textsf{Cents} + 20\ \textsf{Cents} + 13\ \textsf{Cents} \\ &= 75\ \textsf{Cents}\end{aligned}$$

and

$$\begin{aligned}\textsf{Jill's Value} &= [2\ \cancel{\textsf{Apples}}] \times \left[7\ \frac{\textsf{Cents}}{\cancel{\textsf{Apple}}}\right] + [8\ \cancel{\textsf{Bananas}}] \times \left[4\ \frac{\textsf{Cents}}{\cancel{\textsf{Banana}}}\right] + 11\ \textsf{Cents} \\ &= [2 \times 7]\ \textsf{Cents} + [8 \times 4]\ \textsf{Cents} + 11\ \textsf{Cents} \\ &= 14\ \textsf{Cents} + 32\ \textsf{Cents} + 31\ \textsf{Cents} \\ &= 77\ \textsf{Cents}\end{aligned}$$

We shall refer to **Cents** as a **common denominator**.

1.5 Comparing Collections of Different Kinds of Goods

To compare collections of goods, they *must* consist of the *same* kind of objects. On the board, this means that we must have a *common* denominator. But, once we do, the comparison proceeds quite smoothly.

1. Indeed, if the collections consist of the *same* kinds of objects, we need of course concern ourselves only with the *numbers* of objects in the collections as we did in Section **??** .

• Say Jack has THREE ***apples*** and Jill has FIVE ***apples***. Who's ***worth*** is *smaller*?

[1] In other words, $\textsf{Jack's Value} = \begin{bmatrix} 3\ \textsf{Apples} & 5\ \textsf{Bananas}\end{bmatrix} \circ \begin{bmatrix} 7\ \frac{\textsf{Cents}}{\textsf{Apple}} \\ 4\ \frac{\textsf{Cents}}{\textsf{Banana}}\end{bmatrix} + 13\ \textsf{Cents}$ as Educologists will be sure to note.

one-to-one
one-to-one

♠ After matching their ***apples*** **one-to-one**, we find that ***Jill*** is left with un-matched ***apples*** which means that ***Jack*** has fewer ***apples*** than ***Jill***. So, ***Jack's worth*** is *smaller* than ***Jill's worth*** regardless of the ***unit-worth*** of ***apples***.
❖ On the board, since we go from 3 **Apples** to 5 **Apples** by counting *forward*, $\xrightarrow{3,\,4,\,5}$, we write that

$$3 \text{ Apples} < 5 \text{ Apples}$$

and therefore

$$\text{Jack's Value} < \text{Jill's Value}$$

• Say Jack has EIGHT ***apples*** and Jill has TWO ***apples***. Who's worth is *larger*?
♠ After matching their ***apples*** **one-to-one**, ***Jack*** has ***apples*** left over and so *more* ***apples*** than Jill. Thus, ***Jack's worth*** is *larger* than ***Jill's worth*** regardless of the worth of ***apples***.
❖ On the board, since we go from 8 **Apples** to 2 **Apples** by counting *backward*, $\xleftarrow{2,\,3,\,4,\,5,\,6,\,7,\,8}$, we write that

$$8 \text{ Apples} > 2 \text{ Apples}$$

so that

$$\text{Jack's Value} > \text{Jill's Value}$$

2. Usually, though, we need to compare collections of *different* goods and we first need to get a *common denominator*.

♠ For instance, in the last example of Section **??** we found that ***Jack's worth*** would be SEVENTY-FIVE ***cents*** and that ***Jill's worth*** would be SEVENTY-SEVEN ***cents***. So, after matching ***cents*** *one-to-one*, we find that ***Jack's worth*** is smaller than ***Jill's worth***

❖ On the board, after we get to the point where we have:

$$\text{Jack's Value} = 75 \text{ Cents}$$
$$\text{Jill's Value} = 77 \text{ Cents}$$

we proceed as in Section **??** and since we go from 75 **Cents** to 77 **Cents** by counting forward, $\xrightarrow{76,\,77}$, we write that

$$\text{Jack's Value} < \text{Jill's Value}$$

1.6 Computing Unit-Values: Co-division

Division
co-division
quotient
algebra
solve
equation

Division comes up in at least two different types of situation but, as we shall see, from the *procedural* viewpoint, these situations will turn out to be really the same. Here we shall deal with the type of situations in which we want to arrive at *unit-values*. We shall refer to is as **co-division**.

We begin by looking at *basic* collections, that is collections involving *less* than TEN ***objects*** (See Section **??**) because our purpose here is only to show how *division* is connected with *co-multiplication*. We leave the more technical aspects of the procedure to a later chapter.

1. We begin with the somewhat rare case in which "the division comes out even".

a. Suppose we wanted to buy TWO ***apples*** at an auction and that we had EIGHT ***dimes*** available. We want to know what is the ***highest possible bid*** that we could make.

♠ Essentially, we proceed by *trial and error*.

• Bidding ONE ***dime-per-apple*** would require TWO ***dimes*** to buy TWO ***apples*** with SIX ***dimes*** left over.

• Bidding TWO ***dime-per-apple*** would require FOUR ***dimes*** to buy TWO ***apples*** with FOUR ***dimes*** left over.

• Bidding THREE ***dimes-per-apple*** would require SIX ***dimes*** to buy TWO ***apples*** with TWO ***dimes*** left over.

• Bidding FOUR ***dimes-per-apple*** would require EIGHT ***dimes*** to buy TWO ***apples*** with NO ***dime*** left over.

So, our ***highest possible bid*** would be FOUR ***dime-per-apple***.

❖ On the board, we use the "2 times" *multiplication table* to accelerate the trial and error process. We start with 1

• $2 \times 1 = 2$ which is *less* than 8 so we try the next one, 2,
• $2 \times 2 = 4$ which is *less* than 8 so we try the next one, 3,
• $2 \times 3 = 6$ which is *less* than 8 so we try the next one, 4,
• $2 \times 4 = 8$ which is *equal* to 8 so we stop with 4.

The **quotient**, namely what we write on the *board* to represent our ***highest possible bid***, is thus $4\ \frac{\mathsf{Dimes}}{\mathsf{Apples}}$.

b. From the **algebra** viewpoint, we are trying to **solve** the **equation**

$$2\ \mathsf{Apples} \times ?\ \frac{???}{¿¿¿} = 8\ \mathsf{Dimes}.$$

that is, to figure out what *co-number-phrase* would fit.

solution
÷
divided by

- ¿¿¿ must be **Apple** to allow for the cancellation and ??? must be **Dimes** because of the right hand side:

$$2\ \cancel{\textbf{Apples}} \times ?\ \frac{\textbf{Dimes}}{\cancel{\textbf{Apple}}} = 8\ \textbf{Dimes}.$$

$$(2 \times ?)\ \textbf{Dimes} = 8\ \textbf{Dimes}$$

- Since the *denominators* are now the same, we saw in Section **??** that we must solve:

$$2 \times ?\ = 8$$

  Then, using the "2 times" multiplication table, we find that 4 fits in the place of ?.

The *procedure* to solve the equation $2\ \textbf{Apples} \times ?\ \frac{???}{¿¿¿} = 8\ \textbf{Dimes}$ is thus to "divide both sides by 2 **Apples**":

$$\frac{2\ \textbf{Apples}}{2\ \textbf{Apples}} \times ?\ \frac{???}{¿¿¿} = \frac{8\ \textbf{Dimes}}{2\ \textbf{Apples}}$$

and, since, $\frac{2\ \textbf{Apples}}{2\ \textbf{Apples}} = 1$, we get

$$?\ \frac{???}{¿¿¿} = \frac{8\ \textbf{Dimes}}{2\ \textbf{Apples}}$$
$$= 4\ \frac{\textbf{Dimes}}{\textbf{Apples}}$$

So, $4\ \frac{\textbf{Dimes}}{\textbf{Apple}}$ is the **solution** of the equation $2\ \textbf{Apples} \times ?\ \frac{???}{¿¿¿} = 8\ \textbf{Dimes}$ and we will say that it is "given" by

$$8\ \textbf{Dimes} \div 2\ \textbf{Apples} = \ 4\ \frac{\textbf{Dimes}}{\textbf{Apple}}$$

which we read as "8 **Dimes divided by** 2 **Apples** is equal to $4\ \frac{\textbf{Dimes}}{\textbf{Apple}}$" and where the procedure for ÷ is the one above.
In other words,

$$2\ \textbf{Apples}\ \times 4\ \frac{\textbf{Dimes}}{\textbf{Apple}} = 8\ \textbf{Dimes}.$$

and

$$8\ \textbf{Dimes} \div 2\ \textbf{Apples} = \ 4\ \frac{\textbf{Dimes}}{\textbf{Apple}}$$

are *dual* sentences.

c. If, instead of writing 8 **Dimes** ÷ 2 **Apples** we move 8 **Dimes** in place of the upper dot and 2 **Apples** in place of the lower dot, we get

$$\frac{8\ \textbf{Dimes}}{2\ \textbf{Apples}}$$

divide into
quotient
remainder

as another way to write 8 **Dimes** ÷ 2 **Apple** and thus

$$\frac{8\ \textsf{Dimes}}{2\ \textsf{Apples}} = 4\ \frac{\textsf{Dimes}}{\textsf{Apple}}$$

d. In anticipation of the more complicated cases to be investigated later on, we set up the division as follows:

$$\begin{array}{r|l} & 4\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}} \\ \hline 2\ \cancel{\textsf{Apples}}\ \big) & 8\ \textsf{Dimes} \\ & 8\ \textsf{Dimes} \\ \cline{2-2} & 0\ \textsf{Dime} \end{array}$$

which we read as **divide** 2 **Apples into** 8 **Dimes**.
Note. Observe that the order of the two number-phrases in "*divide* 2 **Apples** *into* 8 **Dimes**" is the opposite of their order in "8 **Dimes** *divided by* 2 **Apples**" but that both result in the same co-number-phrase, namely 4 $\frac{\textsf{Dimes}}{\textsf{Apple}}$.
This is another instance of *linguistic duality*.

2. We now look at a case where "the division does *not* come out even"

a. Suppose we wanted to buy THREE ***apples*** but that we had only SEVEN ***dimes***. We want to know what would be our ***highest possible bid***.
♠ Again, we proceed by *trial and error*.
• Bidding ONE ***dime-per-apple*** would require THREE ***dimes*** to buy THREE ***apples*** with FOUR ***dimes*** leftover.
• Bidding TWO ***dime-per-apple*** would require SIX ***dimes*** to buy THREE ***apples*** with ONE ***dime*** leftover.
• Bidding THREE ***dimes-per-apple*** would require NINE ***dimes*** to buy THREE ***apples*** which is more than the SEVEN ***dimes*** we have.
So, our ***highest possible bid*** is TWO ***dime-per-apple*** at which unit-value we would be able to buy TWO ***apples*** with ONE ***dime*** left over.
❖ On the board, we use the "3 times" *multiplication table* to accelerate the trial and error process. We start with 1.

- 3 × 1 = 3 which is *less* than 7 so we try the next one, 2,
- 3 × 2 = 6 which is *less* than 7 so we try the next one, 3,
- 3 × 3 = 9 which is *more* than 7 so we stop with 2.

Since 2 $\frac{\textsf{Dimes}}{\textsf{Apple}}$ represents the ***highest possible bid***, we will again call it the **quotient** and, since 1 **Dimes** represents the ***lowest possible leftover***, we will call it the **remainder** of the division.

b. From the *algebra* viewpoint, we are trying to solve the equation

$$3\ \textsf{Apples} \times ?\ \frac{???}{¿¿¿} = 7\ \textsf{Dimes}$$

≈
approximately equal
(...)

However, even though the quotient is $2\ \frac{\textsf{Dimes}}{\textsf{Apple}}$, we cannot just replace $?\ \frac{???}{¿¿¿}$ by $2\ \frac{\textsf{Dimes}}{\textsf{Apple}}$ because the way *division* is related to *co-multplication* is now complicated by the *remainder* and we can only write

$$7\ \textsf{Dimes} \approx 3\ \cancel{\textsf{Apples}} \times 2\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}}$$

where ≈ means that $7\ \textsf{Dimes}$ is **approximately equal** to $3\ \cancel{\textsf{Apples}} \times 2\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}}$ by which we mean that the difference is "too small to matter". However, we shall not use the symbol ≈ because we shall prefer to keep track of this difference, even though it is "too small to matter" and we shall write

$$7\ \textsf{Dimes} = 3\ \cancel{\textsf{Apples}} \times 2\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}} + (...)\ \textsf{Dimes}$$

where **(...) Dimes** represents a *number* of ***dimes*** "too small to matter". In fact, there will be situations where we will have to be more precise about what is being ignored and, in particular, we may even replace (...) by the *remainder*:

$$7\ \textsf{Dimes} = 3\ \cancel{\textsf{Apples}} \times 2\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}} + 1\ \textsf{Dimes}$$

Since the *denominators* are the same, we can focus on the *numerators* and write:

$$7 = 3 \times 2 + (...)$$

as well as

$$7 = 3 \times 2 + 1$$

Note. More generally, keeping track of things even though they are too small to be taken into account, one way or the other, will turn out to be an extremely powerful approach called ASYMPTOTIC ANALYSIS, one that we shall make extensive and systematic use of in Part 2 and 3 of this proto-text.

c. It is imperative to realize how this extends to the use of ÷.
First recall that, in the previous case, we wrote

$$8\ \textsf{Dimes}\ \div\ 2\ \textsf{Apples} =\ (4\ \frac{\textsf{Dimes}}{\textsf{Apple}})$$

as just the *dual* of

$$8\ \textsf{Dimes}\ = 2\ \textsf{Apples}\ \times\ (4\ \frac{\textsf{Dimes}}{\textsf{Apple}})$$

But, *here*, we have

$$7\ \textsf{Dimes}\ =\ 3\ \cancel{\textsf{Apples}}\ \times\ (2\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}}) + (...)\ \textsf{Dimes}$$

mixed-numbers notation

so that

$$7\ \textsf{Dimes}\ \neq\ 3\ \cancel{\textsf{Apple}}\ \times\ (2\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}})$$

and therefore

$$7\ \textsf{Dimes} \div 3\ \textsf{Apple}\ \neq\ 2\ \frac{\textsf{Dimes}}{\textsf{Apple}}$$

However, we *can* write

$$7\ \textsf{Dimes} \div 3\ \textsf{Apples} = 2\ \frac{\textsf{Dimes}}{\textsf{Apple}} + (\ldots)\frac{\textsf{Dime}}{\textsf{Apple}}$$

and, if we want to be more specific,

$$7\ \textsf{Dimes} \div 3\ \textsf{Apples} = 2\ \frac{\textsf{Dimes}}{\textsf{Apple}} + \frac{1\ \textsf{Dime}}{3\ \textsf{Apple}}$$

where the 3 **Apples** under the remainder 1 **Dime** are to remind us that the remainder is yet to be divided among 3 **Apples**.

d. If we rewrite 7 **Dimes** ÷ 3 **Apples** as $\frac{7\ \textsf{Dimes}}{3\ \textsf{Apples}}$ then we can rewrite the above as

$$\frac{7\ \textsf{Dimes}}{3\ \textsf{Apple}} = 2\ \frac{\textsf{Dimes}}{\textsf{Apple}} + \frac{1\ \textsf{Dime}}{3\ \textsf{Apple}}$$

and, with a slight abuse of language,

$$\frac{7}{3}\ \frac{\textsf{Dimes}}{\textsf{Apple}} = 2\ \frac{\textsf{Dimes}}{\textsf{Apple}} + \frac{1}{3}\ \frac{\textsf{Dimes}}{\textsf{Apple}}$$

and therefore:

$$\frac{7}{3}\ \frac{\textsf{Dimes}}{\textsf{Apple}} = \left(2\ + \frac{1}{3}\right)\frac{\textsf{Dimes}}{\textsf{Apple}}$$

whence the so-called **mixed-numbers notation**:

$$\frac{7}{3} = 2\ \tfrac{1}{3}$$

which we shall encounter in a later chapter, in a somewhat different context.

e. In anticipation of the more complicated cases to be investigated later on, we set up the division as follows:

$$\begin{array}{r|l} & 2\ \frac{\cancel{\textsf{Dimes}}}{\textsf{Apple}} \\ \hline 3\ \cancel{\textsf{Apples}}\ ) & 7\ \textsf{Dimes} \\ & 6\ \textsf{Dimes} \\ \hline & 1\ \textsf{Dime} \end{array}$$

f. In an attempt to save time and energy, though, it is usual *not* to write the denominators and to write only the numerators

$$\begin{array}{r|l} & 2 \\ \hline 3\ ) & 7 \\ & 6 \\ \hline & 1 \end{array}$$

1.7 Signed Co-division

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TO DO

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1.8 Equations and Inequations For Collections of Goods

We saw in Section **??** that, given *collections* of *money*, a type of problem one often encounters in the real-world is not only how they *compare* but how they *differ*. Here we shall investigate how collections of *goods* differ in *worth*.

1. We begin with the case where ***Jack*** has a collection of ***apples*** and ***Jill*** has ***money***.

♠ Say that ***Jack*** has TWO ***apples*** and that ***Jill*** has EIGHT ***dimes***. We want to know for which ***unit-worth*** will ***Jack's worth*** be the same as ***Jill's worth***.
This is exactly the problem we dealt with in Section **??**: If ONE ***apple*** is worth FOUR ***cents***, then ***Jack's worth*** will be the same as ***Jill's worth***.

❖ On the board, we write, again as in Section **??**,

$$\begin{array}{cc} \textsf{Jack's Value} & \textsf{Jill's Value} \\ 2\ \textsf{Apples} \times ?\ \frac{???}{¿¿¿} = & 8\ \textsf{Cents} \end{array}$$

Dividing both sides by 2 **Apples** gives

$$\frac{2\ \cancel{\textsf{Apples}}}{2\ \cancel{\textsf{Apples}}} \times ?\ \frac{???}{¿¿¿} = \frac{8\ \textsf{Cents}}{2\ \textsf{Apples}}$$

$$1 \times ?\ \frac{???}{¿¿¿} = \frac{8\ \textsf{Cents}}{2\ \textsf{Apple}}$$

and therefore

$$?\ \frac{???}{¿¿¿} = 4\ \frac{\textsf{Cents}}{\textsf{Apple}}$$

2. We now look at the case where both ***Jack*** and ***Jill*** hold both a collection of ***apples*** and ***money***. What we will do is to change ***Jack's holding*** and ***Jill's holding*** so as to obtain a situation such as the previous one, that is where ***Jack*** has *only* a collection of ***apples*** and ***Jill*** has *only* ***money***. Of course, we will have to proceed in an **equitable manner**, that is, whatever we do onto either one, we must do onto the other. equitable manner separate

Since this is in fact a very general approach called *Separation of Variables*, we will say that we **separate** the ***apples*** from the ***money***.

♠ Say that ***Jack*** has SEVEN ***apples*** and THREE ***cents*** and that ***Jill*** has FIVE ***apples*** and ELEVEN ***cents***. We want to know for which ***unit-worth*** will ***Jack's worth*** be the same as ***Jill's worth***.
  - Since we do not want ***Jack*** to have any ***money*** and since he has THREE ***cents***, we take them away from him and, to be equitable, we also take THREE ***cents*** away from ***Jill*** which leaves her with only EIGTH ***cents***.
  - Since we do not want ***Jill*** to have any ***apple*** and since she has FIVE ***apples***, we take them away from her and, to be equitable, we also take FIVE ***apples*** away from ***Jack*** which leaves him with only TWO ***apples***.

❖ On the board, we can proceed in either one of two ways:
  - We can start by separating the **Combinations**

Jack's Combination	Jill's Combination
7 **Apples** & 3 **Cents**	5 **Apples** & 11 **Cents**

Subtracting 3 **Cents** from each **Combination** leaves

7 **Apples**	5 **Apples** & 8 **Cents**

Subtracting 5 **Apples** from each **Combination** leaves

2 **Apples**	8 **Cents**

Moving to **Values** now gives the *equation*

$$2 \text{ Apples} \times ?\,\frac{???}{¿¿¿} = 8 \text{ Cents}$$

  - Alternatively, we can start with **Values** up front:

Jack's Value **Jill's Value**

$$7 \text{ Apples} \times ?\,\frac{???}{¿¿¿} + 3 \text{ Cents} = 5 \text{ Apples} \times ?\,\frac{???}{¿¿¿} + 11 \text{ Cents}$$

Subtracting 3 **Cents** from each side of the *equation* gives

$$7 \text{ Apples} \times ? \frac{???}{¿¿¿} = 5 \text{ Apples} \times ? \frac{???}{¿¿¿} + 8 \text{ Cents}$$

Subtracting 5 **Apples** $\times$? $\frac{???}{¿¿¿}$ from each side of the *equation* gives

$$2 \text{ Apples} \times ? \frac{???}{¿¿¿} = 8 \text{ Cents}$$

So, either way, by *separating* the ***apples*** from the ***money***, we have reduced the problem to the preceding one.

3. Inequation

Test

Test at infinity

1.9 Equations and Inequations For States

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TO DO

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The exact same approach works for *states* as long as one keeps in mind that:

- "*to subtract*" means "*to add the opposite*" Section **??**
- "law of signs for signed co-multiplication" Section **??**
- "law of signs for signed division" Section **??**

1. -2 **Apples** $\times$? $\frac{???}{¿¿¿} = -14$ **Cents**

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2. xxxxxxxxxxxxxx

♠ Say that ***Jack*** is SEVEN ***apples in-the-red*** and THREE ***cents in-the-black*** and that ***Jill*** is FIVE ***apples in-the-red*** and ELEVEN ***cents in-the-red***. We want to know for which ***profit/loss-unit-worth*** will ***Jack's worth*** be the same as ***Jill's worth***.

- Since we do not want ***Jack*** to have any ***money***, and since he is THREE ***cents in-the-black***, we take them away from him and, to be equitable, we also take THREE ***cents*** away from ***Jill*** which, since she already was ELEVEN ***cents in-the-red***, puts her now FOURTEEN ***cents in-the-red***.

- Since we do not want ***Jill*** to have any ***apple***, and since she is FIVE ***apples in-the-red***, we give her FIVE ***apples*** and, to be equitable, we also give FIVE ***apples*** to ***Jack*** which, since he was SEVEN ***apples in-the-red***, puts him now only TWO ***apples in-the-red***.

❖ Again, on the board, we can proceed in either one of two ways:
- We can separate the **Combinations** and then move to **Values**

Jack's Combination	Jill's Combination
-7 **Apples** & $+3$ **Cents**	-5 **Apples** & -11 **Cents**

To *subtract* $+3$ **Cents** from each side, we *add the opposite*, -3 **Cents**, to each side which results in

-7 **Apples**	-5 **Apples** & -14 **Cents**

To subtract -5 **Apples** from each side, we *add the opposite*, $+5$ **Apples**, to each side which results in

-2 **Apples**	-14 **Cents**

Moving to **Values** now gives the *equation*

$$-2 \textbf{ Apples} \times ? \frac{???}{¿¿¿} = -14 \textbf{ Cents}$$

- Alternatively, we can move to **Values** up front:

3. Inequation

Test

Test at infinity