atoms

Chapter 5

Division of Money *On* the Counter

Here we deal with three issues involving *division*:

- Division proper. For instance, If we have THIRTEEN ***dollars*** and want to split the money among THREE ***persons***, how many ***dollars*-per-*person*** can we give? Answer: FOUR ***dollars*-per-*person***.
- Distribution. For instance, If we have THIRTEEN ***dollars*** and we want to give THREE ***dollars*-per-*person***, to how many ***persons*** can we give money? Answer: FOUR ***persons***.
- How far down can we divide? That is, are there **atoms**, that is objects that cannot be divided?

5.1 Dividing Money Among People

We recall that number-phrases are the (board) representation of collections on a counter.

What we are after is division, that is, a (board) procedure that gives the result of splitting fairly the objects in a collection among objects in another collection.

As always when the situation is a bit complicated, we will go here through a succession of progressively more complicated examples:

1. $6 \div 3$ in order to introduce the *quotient* of a division.

2. $8 \div 3$ in order to introduce the *remainder* of a division.

3. $639 \div 3$ in order to divide a *combination* with *no* remainder.

4. $800 \div 20$

5. $715 \div 3$ in the absence of exchange facilities.

6. $715 \div 3$ in the presence of exchange facilities.

7. $715 \div 23$

1.

2. We now look at a slightly more general example in which the division is *not* exact. We use "$8 \div 3$" as our example,

♠ Suppose we have EIGHT ***one-dollar-bills*** on the counter and that we want to split them among THREE ***people***. What we want to know is what the share of each ***people*** will be. We proceed exactly as before, that is we make ***rounds*** during each of which we hand out ONE ***one-dollar-bill*** to each and every ***people***.

- First round: We hand out ONE ***one-dollar-bill*** to each and every ***people*** which uses up THREE ***one-dollar-bills*** and leaves us with FIVE ***one-dollar-bills***.
- Second round: We hand out another ONE ***one-dollar-bill*** to each and every ***people*** which uses up another THREE ***one-dollar-bills*** and leaves us now with TWO ***one-dollar-bills***.
- TWO ***one-dollar-bill*** are not enough for a third round.

Altogether, since each ***people*** gets ONE ***one-dollar-bill*** for each ***round***, there can be only TWO ***rounds*** since the ***share*** multiplied by the number of ***rounds*** cannot exceed the available ***one-dollar-bills***.

Because there were TWO rounds, each ***people*** gets TWO ***one-dollar-bills*** with TWO ***one-dollar-bill*** that we cannot split among the THREE ***people***.

❖ The (board) representation of this is the division 8 **Washingtons** $\div$ 3 **People**.

a. Again, what we do on the board does not parallel exactly what we did on the counter: Instead of proceeding round by round and see after each round if we have enough for another round, we use the *multiplication tables* to figure what increasing shares would use up:

- First try: Since "1 times 3 is 3", we figure that handing out $1 \, \frac{\textsf{Washingtons}}{\textsf{People}}$ to each of 3 **People** will use up

  $$3 \, \cancel{\textsf{People}} \times \left(1 \, \frac{\textsf{Washingtons}}{\cancel{\textsf{People}}}\right) = 3 \, \textsf{Washingtons}.$$

  which we then subtract from 8 **Washingtons**:

  $$8 \, \textsf{Washingtons} - 3 \, \textsf{Washingtons} = 5 \, \textsf{Washingtons}$$

  which tells us to proceed with another try.

quotient
remainder
+ (...)

- <u>Second try</u>: Since "2 times 3 is 6", we figure that handing out 2 $\frac{\mathsf{Washingtons}}{\mathsf{People}}$ to each of 3 **People** will use up
$$3\ \cancel{\mathsf{People}} \times (2\ \frac{\mathsf{Washingtons}}{\cancel{\mathsf{People}}}) = 6\ \mathsf{Washingtons}.$$
which we then subtract from 8 **Washingtons**:
$$8\ \mathsf{Washingtons} - 6\ \mathsf{Washingtons} = 2\ \mathsf{Washingtons}$$
- <u>Third try</u>: Since "3 times 3 is 9", we figure that handing out 3 $\frac{\mathsf{Washingtons}}{\mathsf{People}}$ to each of 3 **People** will use up
$$3\ \cancel{\mathsf{People}} \times (3\ \frac{\mathsf{Washingtons}}{\cancel{\mathsf{People}}}) = 9\ \mathsf{Washingtons}.$$
which, since we cannot subtract it from 8 **Washingtons**, tells us that we should have stopped with the *previous* try.

b. In other words, when we try to figure out what will fit in place of the question mark in
$$3\ \cancel{\mathsf{People}} \times (?\ \frac{\mathsf{Washingtons}}{\cancel{\mathsf{People}}}) = 8\ \mathsf{Washingtons}.$$
$$(3 \times ?)\ \mathsf{Washingtons} = 8\ \mathsf{Washingtons}$$
we find that neither 2 nor 3 fits exactly: 2 is too small and 3, the next number up, is too large. Since 2 $\frac{\mathsf{Washingtons}}{\mathsf{People}}$ represents what we can *actually* hand out, we *still* call it the **quotient** but we then say that 2 **Washingtons** is the **remainder**.

Altogether, while in the previous case, we wrote
$$6\ \mathsf{Washingtons} \div 3\ \mathsf{People} = 2\ \frac{\mathsf{Washingtons}}{\mathsf{People}}.$$
here, because the division is *not* exact in that there is the *remainder* 2 **Washingtons**, we must write
$$8\ \mathsf{Washingtons} \div 3\ \mathsf{People} = 2\ \frac{\mathsf{Washingtons}}{\mathsf{People}} + (\dots)$$
where the purpose of writing **+ (...)** is to alert us to the fact that the division is *not* exact and that while the quotient is
$$2\ \frac{\mathsf{Washingtons}}{\mathsf{People}}$$
there is a *remainder*.

Note. It is imperative to realize that $(\dots)$ absolutely does *not* stand for the remainder. To see why not, recall that while, in the previous case, we could write
$$6\ \mathsf{Washingtons} \div 3\ \mathsf{People} = 2\ \frac{\mathsf{Washingtons}}{\mathsf{People}}$$
because we had
$$6\ \mathsf{Washingtons} = 3\ \cancel{\mathsf{People}} \times (2\ \frac{\mathsf{Washingtons}}{\cancel{\mathsf{People}}})$$
But now we have
$$8\ \mathsf{Washingtons} = 3\ \cancel{\mathsf{People}} \times 2\ \frac{\mathsf{Washingtons}}{\cancel{\mathsf{People}}} + 2\ \mathsf{Washingtons}$$
so that
$$8\ \mathsf{Washingtons} \neq 3\ \cancel{\mathsf{People}} \times 2\ \frac{\mathsf{Washingtons}}{\cancel{\mathsf{People}}}$$
and therefore

(...)
into
by

$$8\ \textsf{\textbf{Washingtons}} \div 3\ \textsf{\textbf{People}} \neq 2\ \frac{\textsf{\textbf{Washingtons}}}{\textsf{\textbf{People}}}\ .$$

However, we *can* write either

$$8\ \textsf{\textbf{Washingtons}} \div 3\ \textsf{\textbf{People}} = 2\ \frac{\textsf{\textbf{Washingtons}}}{\textsf{\textbf{People}}} + \frac{2\ \textsf{\textbf{Washingtons}}}{3\ \textsf{\textbf{People}}}$$

or

$$8\ \textsf{\textbf{Washingtons}} \div 3\ \textsf{\textbf{People}} = 2\ \frac{\textsf{\textbf{Washingtons}}}{\textsf{\textbf{People}}} + 2\ \frac{\textsf{\textbf{Washingtons}}}{3\ \textsf{\textbf{People}}}$$

which, compared with

$$8\ \textsf{\textbf{Washingtons}} \div 3\ \textsf{\textbf{People}} = 2\ \frac{\textsf{\textbf{Washingtons}}}{\textsf{\textbf{People}}} + (...)$$

shows that

$$(...) = \frac{2\ \textsf{\textbf{Washingtons}}}{3\ \textsf{\textbf{People}}} = 2\ \frac{\textsf{\textbf{Washingtons}}}{3\ \textsf{\textbf{People}}}$$

Thus, while the purpose of + (...) *as a whole* is *only* to alert us to the fact that the division is *not* exact, *by itself* **(...)** does in fact stand for something, if something *too small to be divided.*

Actually, keeping track of things even though they are too small to be taken into account, one way or the other, will turn out to be an extremely powerful approach called ASYMPTOTIC ANALYSIS, one that we shall make extensive and systematic use of in Part 3 of this proto-text.

c. Finally, here is, without comment, division as it is usually set up.

$$\begin{array}{r|l}
 & 2\ \frac{\textsf{\textbf{Washingtons}}}{\textsf{\textbf{People}}} \\ \cline{2-2}
3\ \textsf{\textbf{People}} & 8\ \textsf{\textbf{Washingtons}} \\
 & 6\ \textsf{\textbf{Washingtons}} \\ \cline{2-2}
 & 2\ \textsf{\textbf{Washingtons}}
\end{array}$$

In an attempt to save time and energy, though, it is usual *not* to write the denominators and to write only

$$\begin{array}{r|r}
 & 2 \\ \cline{2-2}
3 & 8 \\
 & 6 \\ \cline{2-2}
 & 2
\end{array}$$

Note. In this case, by the way, one usually talks of "dividing 3 **into** 8" rather than "dividing 8 **by** 3".

3. Now that we have seen the basic *ideas* governing division, we want to split *more* than NINE ***dollars*** but still among *fewer* than TEN ***people***. We will thus have to split a *collection of collections* of ***bills*** among a collection of ***people***. We use $639 \div 3$ as our example.

♠ Suppose we have SIX HUNDRED THIRTY NINE ***dollars*** on the counter in the form of SIX ***hundred-dollar-bills***, THREE ***ten-dollar-bills***, and NINE ***one-dollar-bills*** that we want to split among THREE ***people***.

Giving TWO ***hundred-dollar-bills***, ONE ***ten-dollar-bills***, and THREE ***one-dollar-bills*** to each ***people*** does it.

❖ On the board, the division would be:

$$639\ \textsf{\textbf{Washingtons}} \div 3\ \textsf{\textbf{People}}$$

which we will eventually set up as

$$3\ \textsf{\textbf{People}}\,\overline{)\ 639\ \textsf{\textbf{Washingtons}}}$$

To see in detail what we do, though, we will proceed in three stages: **a.** we will set up the division as if under a heading and go through the steps of the division, **b.** we will make a couple of considerations about how to write the result and **c.** we will rewrite the division as it is usually done, with and without denominators.

a. We set up the division

$$6\ \textsf{\textbf{Franklins}}\ \&\ 3\ \textsf{\textbf{Hamiltons}}\ \&\ 9\ \textsf{\textbf{Washingtons}} \div 3\ \textsf{\textbf{People}}$$

as if we were using headings:

$$3\ \textsf{\textbf{People}}\,\overline{)\ 6\ \textsf{\textbf{Franklins}}\ \&\ 3\ \textsf{\textbf{Hamiltons}}\ \&\ 9\ \textsf{\textbf{Washingtons}}}$$

We now go through the division step by step:

- Dividing 6 **Franklins** among 3 **People** gives us 2 $\frac{\textsf{\textbf{Franklins}}}{\textsf{\textbf{People}}}$ as quotient and multiplying 3 $\cancel{\textsf{\textbf{People}}}$ × 2 $\frac{\textsf{\textbf{Franklins}}}{\cancel{\textsf{\textbf{People}}}}$ gives us that this uses up 6 **Franklins**. We write:

$$\begin{array}{r|l} & 2\ \frac{\textsf{\textbf{Franklins}}}{\cancel{\textsf{\textbf{People}}}} \\ \hline 3\ \cancel{\textsf{\textbf{People}}} & 6\ \textsf{\textbf{Franklins}}\ \&\ 3\ \textsf{\textbf{Hamiltons}}\ \&\ 9\ \textsf{\textbf{Washingtons}} \\ & 6\ \textsf{\textbf{Franklins}} \end{array}$$

- Subtracting 6 **Franklins** from 6 **Franklins** & 3 **Hamilton** & 9 **Washington** leaves us with 0 **Franklin** & 3 **Hamilton** & 9 **Washington** remaining to be divided. We write

$$\begin{array}{r|ll} & 2\ \frac{\textsf{\textbf{Franklins}}}{\textsf{\textbf{People}}} & \\ \hline 3\ \textsf{\textbf{People}} & 6\ \textsf{\textbf{Franklins}} & \&\ 3\ \textsf{\textbf{Hamiltons}}\ \&\ 9\ \textsf{\textbf{Washingtons}} \\ & 6\ \textsf{\textbf{Franklins}} & \\ \cline{2-3} & 0\ \textsf{\textbf{Franklin}} & \&\ 3\ \textsf{\textbf{Hamiltons}}\ \&\ 9\ \textsf{\textbf{Washingtons}} \end{array}$$

  Note. In fact, as is usual, we will *not* write 0 **Franklin** but we will leave the corresponding space empty so as to respect the alignment.
  Also, in elementary schools, presumably out of a desire of maximal economy, it is usual as well to write *only* 3 **Hamiltons**, the beginning of the next step Of course, *we* will *not* do so.

- Dividing 3 **Hamiltons** among 3 **People** gives us 1 $\frac{\textsf{\textbf{Hamilton}}}{\textsf{\textbf{People}}}$ as quotient and multiplying 3 $\cancel{\textsf{\textbf{Hamiltons}}}$ × 1 $\frac{\textsf{\textbf{Hamilton}}}{\cancel{\textsf{\textbf{People}}}}$ gives us that this uses up 3 **Hamiltons**. We write:

$$
\begin{array}{rl}
 & 2\,\frac{\textsf{Franklins}}{\textsf{People}} \;\&\; 1\,\frac{\textsf{Hamilton}}{\cancel{\textsf{People}}} \\
\cline{2-2}
3\,\cancel{\textsf{People}} & \big)\; 6\,\textsf{Franklins} \;\&\; 3\,\textsf{Hamiltons} \;\&\; 9\,\textsf{Washingtons} \\
 & \; 6\,\textsf{Franklins} \\
\cline{2-2}
 & \qquad\qquad 3\,\textsf{Hamiltons} \;\&\; 9\,\textsf{Washington} \\
 & \qquad\qquad 3\,\textsf{Hamiltons}
\end{array}
$$

- Subtracting 3 **Hamiltons** from 3 **Hamiltons** & 9 **Washington** leaves us with 0 **Hamilton** & 9 **Washington** remaining to be divided. We write

$$
\begin{array}{rl}
 & 2\,\frac{\textsf{Franklins}}{\textsf{People}} \;\&\; 1\,\frac{\textsf{Hamilton}}{\textsf{People}} \\
\cline{2-2}
3\,\textsf{People} & \big)\; 6\,\textsf{Franklins} \;\&\; 3\,\textsf{Hamiltons} \;\&\; 9\,\textsf{Washingtons} \\
 & \; 6\,\textsf{Franklins} \\
\cline{2-2}
 & \qquad\qquad 3\,\textsf{Hamiltons} \;\&\; 9\,\textsf{Washington} \\
 & \qquad\qquad 3\,\textsf{Hamiltons} \\
\cline{2-2}
 & \qquad\qquad 0\,\textsf{Hamilton} \;\;\&\; 9\,\textsf{Washington}
\end{array}
$$

As noted before, we normally don't write 0 **Hamiltons**.

- Dividing 9 **Washingtons** among 3 **People** gives us 3 $\frac{\textsf{Washington}}{\textsf{People}}$ as quotient and multiplying 3 ~~**People**~~ × 3 $\frac{\textsf{Washington}}{\cancel{\textsf{People}}}$ gives us that this uses up 9 **Washingtons**. We write:

$$
\begin{array}{rl}
 & 2\,\frac{\textsf{Franklins}}{\textsf{People}} \;\&\; 1\,\frac{\textsf{Hamilton}}{\textsf{People}} \;\&\; 3\,\frac{\textsf{Washingtons}}{\cancel{\textsf{People}}} \\
\cline{2-2}
3\,\cancel{\textsf{People}} & \big)\; 6\,\textsf{Franklins} \;\&\; 3\,\textsf{Hamiltons} \;\&\; 9\,\textsf{Washingtons} \\
 & \; 6\,\textsf{Franklins} \\
\cline{2-2}
 & \qquad\qquad 3\,\textsf{Hamilton} \;\&\; 9\,\textsf{Washington} \\
 & \qquad\qquad 3\,\textsf{Hamiltons} \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad\quad 9\,\textsf{Washingtons} \\
 & \qquad\qquad\qquad\qquad\quad 9\,\textsf{Washingtons}
\end{array}
$$

- Subtracting 9 **Washingtons** from 9 **Washingtons** leaves us with 0 **Washington** remaining to be divided. We write

$$
\begin{array}{r r}
 & 2\,\frac{\textsf{Franklins}}{\textsf{People}} \;\&\; 1\,\frac{\textsf{Hamilton}}{\textsf{People}} \;\&\; 3\,\frac{\textsf{Washingtons}}{\textsf{People}} \\ \cline{2-2}
3\ \textsf{People}\ \big) & 6\ \textsf{Franklins} \;\&\; 3\ \textsf{Hamiltons} \;\&\; 9\ \textsf{Washingtons} \\
 & 6\ \textsf{Franklins} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \\ \cline{2-2}
 & 3\ \textsf{Hamilton} \;\&\; 9\ \textsf{Washingtons} \\
 & 3\ \textsf{Hamiltons} \qquad\qquad\qquad\quad \\ \cline{2-2}
 & 9\ \textsf{Washingtons} \\
 & 9\ \textsf{Washingtons} \\ \cline{2-2}
 & 0\ \textsf{Washington}
\end{array}
$$

As noted before, we normally don't write 0 **Washington**.

b. We thus have

$$6\ \textsf{Franklins} \;\&\; 3\ \textsf{Hamiltons} \;\&\; 9\ \textsf{Washingtons} \div 3\ \textsf{People} = 2\,\frac{\textsf{Franklins}}{\textsf{People}} \;\&\; 1\,\frac{\textsf{Hamilton}}{\textsf{People}} \;\&\; 3\,\frac{\textsf{Washingtons}}{\textsf{People}}$$

or, by a very, very slight abuse of language,

$$6\ \textsf{Franklins} \;\&\; 3\ \textsf{Hamiltons} \;\&\; 9\ \textsf{Washingtons} \div 3\ \textsf{People} = \frac{2\ \textsf{Franklins}\ \&1\ \textsf{Hamiltons}\ \&\ 3\ \textsf{Washingtons}}{\textsf{People}}$$

so that we can write

$$639\ \textsf{Washingtons} \div 3\ \textsf{People} = \frac{213\ \textsf{Washingtons}}{\textsf{People}}$$

and, finally,

$$639\ \textsf{Washingtons} \div 3\ \textsf{People} = 213\,\frac{\textsf{Washingtons}}{\textsf{People}}$$

c. We now write the division as it appears when we use *number-phrases* instead of a *heading*:

$$
\begin{array}{r r}
 & 213\,\frac{\textsf{Washingtons}}{\textsf{People}} \\ \cline{2-2}
3\ \textsf{People}\ \big) & 639\ \textsf{Washingtons} \\
 & 600\ \textsf{Washingtons} \\ \cline{2-2}
 & 39\ \textsf{Washingtons} \\
 & 30\ \textsf{Washingtons} \\ \cline{2-2}
 & 9\ \textsf{Washingtons} \\
 & 9\ \textsf{Washingtons} \\ \cline{2-2}
 & 0\ \textsf{Washingtons}
\end{array}
$$

Note. Of course, as usual, we do not have to use **Washingtons** as denominator and we could use any other one.

And, finally, here is how it would appear in elementary school.

$$\begin{array}{r} 213 \\ 3\,\big)\overline{\,639} \\ \underline{6} \\ 3 \\ \underline{3} \\ 9 \\ \underline{9} \\ 0 \end{array}$$

4. So far, we have been splitting money among fewer than TEN ***people***. Here we begin our investigation of what happens when we split *more* than NINE ***dollars*** among *more* than NINE ***people***. We use $800 \div 20$ as our first example.

♠ Suppose we have EIGHT HUNDRED ***dollars*** on the counter that are in the form of EIGHT ***hundred-dollar-bills*** and that we want to split it among ***twenty people***.

The first thing we do is to aggregate the ***people*** in TWO ***groups-of-TEN-People***.

Then we give FOUR ***hundred-dollar-bills*** to each ***group-of-TEN-people***.

In the absence of changing facilities we cannot go any further.

If we can change the ***hundred-dollar-bills*** for TEN ***ten-dollar-bills*** each, then each ***group-of-TEN-People*** gets FORTY ***ten-dollar-bills*** and so each ***people*** gets FOUR ***ten-dollar-bills***.

❖ The (board) representation of this is quite nice: We will write **TENPeople** to represent on the board a ***group-of-TEN-People***.

$$\frac{\textsf{8 Franklins}}{\textsf{2 TENPeople}} = 4\ \frac{\textsf{Franklins}}{\textsf{TENPeople}} = 4\ \frac{\cancel{\text{TEN}}\ \textsf{Hamiltons}}{\cancel{\text{TEN}}\ \textsf{People}} = 4\ \frac{\textsf{Hamiltons}}{\textsf{People}}$$

Similarly, we have that

$$1\ \frac{\textsf{Franklins}}{\textsf{TENPeople}} = 1\ \frac{\cancel{\text{TEN}}\ \textsf{Hamiltons}}{\cancel{\text{TEN}}\textsf{People}} = 1\ \frac{\textsf{Hamiltons}}{\textsf{People}}$$

5. We now look at an example in which we would like to use but do *not* have *changing facilities*. We use $715 \div 3$ as our example.

♠ Suppose we have SEVEN HUNDRED FIFTEEN ***dollars*** on the counter in the form of SEVEN ***hundred-dollar-bills***, ONE ***ten-dollar-bill*** and FIVE ***one-dollar-bills*** and that we want to split this money among THREE ***people***.

❖ On the board, we set up as follows:

$$3\ \textsf{People}\ \big)\overline{\ 7\ \textsf{Franklins}\ \ \&\ \ 1\ \textsf{Hamiltons}\ \ \&\ \ 5\ \textsf{Washingtons}}$$

♠ To split the SEVEN ***hundred-dollar-bills*** among the THREE ***people***, we give TWO ***hundred-dollar-bill*** to each ***people***. This uses SIX ***hundred-dollar-bills*** and leaves us with ONE ***hundred-dollar-bills*** which is not enough for another round.
Since we cannot change it, the ONE ***hundred-dollar-bills*** remains undivided.
❖ On the board, we write the **Franklin** part of the quotient:

$$\begin{array}{rl} & \quad 2\,\dfrac{\textsf{Franklins}}{\textsf{People}} \\ \cline{2-2} 3\ \textsf{People} & \big)\ 7\ \textsf{Franklins}\ \ \&\ \ 1\ \textsf{Hamiltons}\ \ \&\ \ 5\ \textsf{Washingtons} \end{array}$$

we multiply to find how many **Franklins** have been used up:

$$\begin{array}{rl} & \quad 2\,\dfrac{\textsf{Franklins}}{\cancel{\textsf{People}}} \\ \cline{2-2} 3\ \cancel{\textsf{People}} & \big)\ 7\ \textsf{Franklins}\ \ \&\ \ 1\ \textsf{Hamiltons}\ \ \&\ \ 5\ \textsf{Washingtons} \\ & \quad 6\ \textsf{Franklins} \end{array}$$

we subtract 6 **Franklins** from 7 **Franklins** (really from 7 **Franklins** & 1 **Hamilton** & 5 **Washingtons**) to get the *current* remainder (= the **Franklins** that *cannot* be divided along with the **Hamiltons** and the **Washingtons** that have *not yet* been divided):

$$\begin{array}{rl} & \quad 2\,\dfrac{\textsf{Franklins}}{\textsf{People}} \\ \cline{2-2} 3\ \textsf{People} & \big)\ 7\ \textsf{Franklins}\ \ \&\ \ 1\ \textsf{Hamilton}\ \ \&\ \ 5\ \textsf{Washingtons} \\ & \quad 6\ \textsf{Franklins} \\ \cline{2-2} & \quad 1\ \textsf{Franklin}\ \ \&\ \ 1\ \textsf{Hamilton}\ \ \&\ \ 5\ \textsf{Washingtons} \end{array}$$

Note. The usage in elementary school is to write only 1 **Franklin** & 1 **Hamilton** with 5 **Washingtons** going "without saying" until we get there.
♠ Next, since we cannot split ONE ***ten-dollar*** bill among THREE ***people***, the share (of each people) is NO ***ten-dollar-bill*** with the ONE ***ten-dollar-bill*** now remaining along with the ONE ***hundred-dolar-bill*** that was already remaining. ❖
On the board, we write the **Hamilton** part of the *quotient*:

$$\begin{array}{rl} & \quad 2\,\dfrac{\textsf{Franklins}}{\textsf{People}}\ \ \&\ \ 0\,\dfrac{\textsf{Hamilton}}{\textsf{People}} \\ \cline{2-2} 3\ \textsf{People} & \big)\ 7\ \textsf{Franklins}\ \&\ \ 1\ \textsf{Hamilton}\ \ \&\ \ 5\ \textsf{Washingtons} \\ & \quad 6\ \textsf{Franklins} \\ \cline{2-2} & \quad 1\ \textsf{Franklin}\ \ \&\ \ 1\ \textsf{Hamilton}\ \ \&\ \ 5\ \textsf{Washingtons} \end{array}$$

we multiply:

$$\begin{array}{r|l}
 & 2\,\frac{\textbf{Franklins}}{\textbf{People}} \;\&\; 0\,\frac{\textbf{Hamilton}}{\cancel{\textbf{People}}} \\ \hline
3\,\cancel{\textbf{People}} & 7\,\textbf{Franklins} \;\&\; 1\,\textbf{Hamilton} \;\&\; 5\,\textbf{Washingtons} \\
 & 6\,\textbf{Franklins} \\ \hline
 & 1\,\textbf{Franklin} \;\&\; 1\,\textbf{Hamilton} \;\&\; 5\,\textbf{Washingtons} \\
 & \qquad\quad 0\,\textbf{Hamilton}
\end{array}$$

we subtract 0 **Hamilton** from 1 **Hamilton** (really from 1 **Franklins** & 1 **Hamilton** & 5 **Washingtons**) to get the *current* remainder (= the **Franklins** and the **Hamiltons** that *cannot* be divided along with the **Washingtons** that have *not yet* been divided):

$$\begin{array}{r|l}
 & 2\,\frac{\textbf{Franklins}}{\textbf{People}} \;\&\; 0\,\frac{\textbf{Hamilton}}{\textbf{People}} \\ \hline
3\,\textbf{People} & 7\,\textbf{Franklins} \;\&\; 1\,\textbf{Hamilton} \;\&\; 5\,\textbf{Washingtons} \\
 & 6\,\textbf{Franklins} \\ \hline
 & 1\,\textbf{Franklin} \;\&\; 1\,\textbf{Hamilton} \;\&\; 5\,\textbf{Washingtons} \\
 & \qquad\quad 0\,\textbf{Hamilton} \\ \hline
 & 1\,\textbf{Franklin} \;\&\; 1\,\textbf{Hamilton} \;\&\; 5\,\textbf{Washingtons}
\end{array}$$

♠ Finally, to split the FIVE ***one-dollar-bills*** among the THREE ***people***, we give ONE ***one-dollar-bill*** to each ***people***. This uses THREE ***one-dollar-bills*** and leaves us with TWO ***one-dollar-bills*** which is not enough for another round.

❖ On the board, we write the **Washington** part of the quotient:

$$\begin{array}{r|l}
 & 2\,\frac{\textbf{Franklins}}{\textbf{People}} \;\&\; 0\,\frac{\textbf{Hamilton}}{\textbf{People}} \;\&\; 1\,\frac{\textbf{Washingtons}}{\textbf{People}} \\ \hline
3\,\textbf{People} & 7\,\textbf{Franklins} \;\&\; 1\,\textbf{Hamilton} \;\&\; 5\,\textbf{Washingtons} \\
 & 6\,\textbf{Franklins} \\ \hline
 & 1\,\textbf{Franklin} \;\&\; 1\,\textbf{Hamilton} \;\&\; 5\,\textbf{Washingtons} \\
 & \qquad\quad 0\,\textbf{Hamilton} \\ \hline
 & 1\,\textbf{Franklin} \;\&\; 1\,\textbf{Hamilton} \;\&\; 5\,\textbf{Washingtons}
\end{array}$$

we multiply:

$$\begin{array}{r|l}
 & 2\,\frac{\textbf{Franklins}}{\textbf{People}} \;\&\; 0\,\frac{\textbf{Hamilton}}{\textbf{People}} \;\&\; 1\,\frac{\textbf{Washingtons}}{\cancel{\textbf{People}}} \\ \hline
3\,\cancel{\textbf{People}} & 7\,\textbf{Franklins} \;\&\; 1\,\textbf{Hamilton} \;\&\; 5\,\textbf{Washingtons} \\
 & 6\,\textbf{Franklins} \\ \hline
 & 1\,\textbf{Franklin} \;\&\; 1\,\textbf{Hamilton} \;\&\; 5\,\textbf{Washingtons} \\
 & \qquad\quad 0\,\textbf{Hamilton} \\ \hline
 & 1\,\textbf{Franklin} \;\&\; 1\,\textbf{Hamilton} \;\&\; 5\,\textbf{Washingtons} \\
 & \qquad\qquad\qquad\qquad 3\,\textbf{Washingtons}
\end{array}$$

we subtract 3 **Washingtons** from 5 **Washingtons** (really from 1 **Franklin** & 1

Hamilton & 5 **Washingtons**) to get the *final* remainder (= the **Franklins**, the **Hamiltons** and the **Washingtons** that *cannot* be divided):

$$
\begin{array}{rllllll}
 & 2\ \frac{\textsf{Franklins}}{\textsf{People}} & \& & 0\ \frac{\textsf{Hamilton}}{\textsf{People}} & \& & 1\ \frac{\textsf{Washingtons}}{\textsf{People}} \\
\cline{2-7}
3\ \textsf{People}\ \Big) & 7\ \textsf{Franklins} & \& & 1\ \textsf{Hamilton} & \& & 5\ \textsf{Washingtons} \\
 & 6\ \textsf{Franklins} & & & & & \\
\cline{2-7}
 & 1\ \textsf{Franklin} & \& & 1\ \textsf{Hamilton} & \& & 5\ \textsf{Washingtons} \\
 & & & 0\ \textsf{Hamilton} & & & \\
\cline{2-7}
 & 1\ \textsf{Franklin} & \& & 1\ \textsf{Hamilton} & \& & 5\ \textsf{Washingtons} \\
 & & & & & & 3\ \textsf{Washingtons} \\
\cline{2-7}
 & 1\ \textsf{Franklin} & \& & 1\ \textsf{Hamilton} & \& & 2\ \textsf{Washington}
\end{array}
$$

Conclusion. In the *absence* of changing facilities,

- the *quotient* is

$$2\ \frac{\textsf{Franklins}}{\textsf{People}}\ \&\ 0\ \frac{\textsf{Hamiltons}}{\textsf{People}}\ \&\ 1\ \frac{\textsf{Washingtons}}{\textsf{People}}$$

- the *remainder* is

1 **Franklin** & 1 **Hamilton** & 2 **Washingtons**

6. We look at the same situation but this time in the case where changing facilities are available. To facilitate the comparison, we use again $715 \div 3$ as our example.

♠ Suppose we have SEVEN HUNDRED FIFTEEN ***dollars*** on the counter in the form of SEVEN ***hundred-dollar-bills***, ONE ***ten-dollar-bill*** and FIVE ***one-dollar-bills*** and that we want to split this money among THREE ***people***.

❖ On the board, we set up as follows:

$$
\begin{array}{rlllll}
\cline{2-6}
3\ \textsf{People}\ \Big) & 7\ \textsf{Franklins} & \& & 1\ \textsf{Hamiltons} & \& \ \ 5\ \textsf{Washingtons}
\end{array}
$$

♠ To split the SEVEN ***hundred-dollar-bills*** among the THREE ***people***, we give TWO ***hundred-dollar-bills*** to each ***people***. This uses SIX ***hundred-dollar-bills*** and leaves us with ONE ***hundred-dollar-bills*** along with the ONE ***ten-dollar-bill*** and the FIVE ***one-dollar-bill*** that we have yet to split.

❖ On the board, we divide the **Franklins**:

$$
\begin{array}{rllllll}
 & 2\ \frac{\textsf{Franklins}}{\cancel{\textsf{People}}} & & & & & \\
\cline{2-7}
3\ \cancel{\textsf{People}}\ \Big) & 7\ \textsf{Franklins} & \& & 1\ \textsf{Hamiltons} & \& & 5\ \textsf{Washingtons} \\
 & 6\ \textsf{Franklins} & & & & & \\
\cline{2-7}
 & 1\ \textsf{Franklin} & \& & 1\ \textsf{Hamilton} & \& & 5\ \textsf{Washingtons}
\end{array}
$$

♠ At this point, we change the remaining ONE ***hundred-dollar-bill*** for TEN ***ten-dollar-bills***.

As a result, instead of having to split
ONE ***hundred-dollar-bill*** and ONE ***ten-dollar-bill*** and FIVE ONE ***one-dollar-bill***
among the THREE ***people*** we now have to split the same amount of money but in the form of
NO ***hundred-dollar-bill*** and ELEVEN ***ten-dollar-bills*** and FIVE ONE ***one-dollar-bill***

❖ On the board, this means that instead of the following division

$$
\begin{array}{r|l}
 & 2\ \frac{\textsf{Franklins}}{\textsf{People}} \\ \hline
3\ \textsf{People} & 7\ \textsf{Franklins}\ \&\quad 1\ \textsf{Hamiltons}\quad \&\quad 5\ \textsf{Washingtons} \\
 & 6\ \textsf{Franklins} \\ \hline
 & 1\ \textsf{Franklin}\quad \&\quad 1\ \textsf{Hamilton}\quad \&\quad 5\ \textsf{Washingtons}
\end{array}
$$

we now have the division:

$$
\begin{array}{r|l}
 & 2\ \frac{\textsf{Franklins}}{\textsf{People}} \\ \hline
3\ \textsf{People} & 7\ \textsf{Franklins}\ \&\quad 1\ \textsf{Hamiltons}\quad \&\quad 5\ \textsf{Washingtons} \\
 & 6\ \textsf{Franklins} \\ \hline
 & 0\ \textsf{Franklin}\quad \&\quad \text{ELEVEN}\ \textsf{Hamilton}\quad \&\quad 5\ \textsf{Washingtons}
\end{array}
$$

♠ To split the ELEVEN ***ten-dollar-bills*** among the THREE ***people***, we give THREE ***ten-dollar-bills*** to each ***people***. This uses NINE ***ten-dollar-bills*** and leaves us with TWO ***ten-dollar-bills*** along with the FIVE ***one-dollar-bills*** that we have yet to split.

❖ On the board, we divide the **Hamiltons**:

$$
\begin{array}{r|l}
 & 2\ \frac{\textsf{Franklins}}{\textsf{People}}\quad \&\quad 3\ \frac{\textsf{Hamilton}}{\cancel{\textsf{People}}} \\ \hline
3\ \cancel{\textsf{People}} & 7\ \textsf{Franklins}\ \&\ 1\ \textsf{Hamiltons}\quad \&\quad 5\ \textsf{Washingtons} \\
 & 6\ \textsf{Franklins} \\ \hline
 & \qquad\qquad \text{ELEVEN}\ \textsf{Hamilton}\quad \&\quad 5\ \textsf{Washingtons} \\
 & \qquad\qquad 9\ \textsf{Hamiltons} \\ \hline
 & \qquad\qquad 2\ \textsf{Hamiltons}\quad \&\quad 5\ \textsf{Washingtons}
\end{array}
$$

♠ At this point, we change the remaining TWO ***ten-dollar-bills*** for TWENTY ***one-dollar-bills***.

As a result, instead of having to split
TWO ***ten-dollar-bills*** and FIVE ***one-dollar-bills***
among the THREE ***people*** we now have to split the same amount of money but in the form of
NO ***ten-dollar-bill*** and TWENTYFIVE ***one-dollar-bills***

❖ On the board, this means that instead of the division:

$$
\begin{array}{r|l}
 & 2\,\frac{\text{Franklins}}{\text{People}} \;\&\; 3\,\frac{\text{Hamilton}}{\text{People}} \\ \hline
3\ \text{People} & 7\ \text{Franklins} \;\&\; 1\ \text{Hamiltons} \;\&\; 5\ \text{Washingtons} \\
 & 6\ \text{Franklins} \\ \hline
 & \text{ELEVEN Hamilton} \;\&\; 5\ \text{Washingtons} \\
 & 9\ \text{Hamiltons} \\ \hline
 & 2\ \text{Hamilton} \;\&\; 5\ \text{Washingtons}
\end{array}
$$

we now have the division:

$$
\begin{array}{r|l}
 & 2\,\frac{\text{Franklins}}{\text{People}} \;\&\; 3\,\frac{\text{Hamilton}}{\text{People}} \\ \hline
3\ \text{People} & 7\ \text{Franklins} \;\&\; 1\ \text{Hamiltons} \;\&\; 5\ \text{Washingtons} \\
 & 6\ \text{Franklins} \\ \hline
 & \text{ELEVEN Hamilton} \;\&\; 5\ \text{Washingtons} \\
 & 9\ \text{Hamiltons} \\ \hline
 & 0\ \text{Hamilton} \;\&\; \text{TWENTYFIVE Washingtons}
\end{array}
$$

♠ To split the TWENTYFIVE ***one-dollar-bills*** among the THREE ***people***, we give EIGHT ***one-dollar-bills*** to each ***people***. This uses TWENTYFOUR ***one-dollar-bills*** and leaves us with ONE ***one-dollar-bill***.

❖ On the board we now divide the **Washingtons**:

$$
\begin{array}{r|l}
 & 2\,\frac{\text{Franklins}}{\text{People}} \;\&\; 3\,\frac{\text{Hamilton}}{\text{People}} \;\&\; 8\,\frac{\text{Washingtons}}{\cancel{\text{People}}} \\ \hline
3\ \cancel{\text{People}} & 7\ \text{Franklins} \;\&\; 1\ \text{Hamilton} \;\&\; 5\ \text{Washingtons} \\
 & 6\ \text{Franklins} \\ \hline
 & \text{ELEVEN Hamiltons} \;\&\; 5\ \text{Washingtons} \\
 & 9\ \text{Hamiltons} \\ \hline
 & \text{TWENTYFIVE Washingtons} \\
 & \text{TWENTYFOUR Washingtons} \\ \hline
 & 1\ \text{Washington}
\end{array}
$$

Conclusion. In the *presence* of changing facilities,

- the *quotient* is

$$
2\,\frac{\text{Franklins}}{\text{People}} \;\&\; 3\,\frac{\text{Hamiltons}}{\text{People}} \;\&\; 8\,\frac{\text{Washingtons}}{\text{People}}
$$

- the *remainder* is

0 **Franklin** & 0 **Hamilton** & 1 **Washingtons**

Finally, we rewrite the above division with *number-phrases* instead of combinations but with the numerators under a “virtual” heading

$$\begin{array}{rr}
 & 238\ \frac{\textbf{Washingtons}}{\textbf{People}} \\ \cline{2-2}
3\ \textbf{People} & \big)\ 715\ \textbf{Washingtons} \\
 & 600\ \textbf{Washingtons} \\ \cline{2-2}
 & 115\ \textbf{Washingtons} \\
 & 90\ \textbf{Washingtons} \\ \cline{2-2}
 & 25\ \textbf{Washingtons} \\
 & 24\ \textbf{Washingtons} \\ \cline{2-2}
 & 1\ \textbf{Washington}
\end{array}$$

and how it would appear in *elementary* school:

$$\begin{array}{rr}
 & 238 \\ \cline{2-2}
3 & \big)\ 715 \\
 & 6 \\ \cline{2-2}
 & 11 \\
 & 9 \\ \cline{2-2}
 & 25 \\
 & 24 \\ \cline{2-2}
 & 1
\end{array}$$

Comparing the procedure given in elementary school with what we did above shows that the latter is nothing but an "abstract" form of the former.

7. We now look at an example which is essentially like the preceding two except that we now have more than NINE ***people*** among whom to split the money. We use 715 ÷ 23 as our example.

♠ Suppose we have SEVEN HUNDRED FIFTEEN ***dollars*** on the counter in the form of SEVEN ***hundred-dollar-bills***, ONE ***ten-dollar-bill*** and FIVE ***one-dollar-bills*** and that we want to split this money among TWENTYTHREE ***people***.

The first thing we do is to aggregate TWENTY of the TWENTYTHREE ***people*** in TWO ***groups-of-TEN-people*** with THREE ***people*** alongside.

❖ Of course, we could, and eventually, will represent on the board TWO ***groups-of-TEN-people*** by the number-phrase 23. **People** or even the number-phrase 2.3 **DEKAPeople**, but first we want to see how things work and so we will represent TWO ***groups-of-TEN-people*** by the combination 2 **TENPeople** & 3 **People**.

We represent this on the board by the combination 2 TEN**People** & 3 **People**. Of course, when we will be using number-phrases, we will write 23 **People**.

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

A FAIRE

v We set up as follows:

2 **TENPeople** 3 People 7 **Franklins** 1 Hamilton 5 **Washingtons** ? Splitting SEVEN

hundred-dollar-bills among TWO ***groups-of-TEN-people*** would let each group have THREE ***hundred-dollar-bills*** with one hundred-dollar-bill remaining undivided.
v On the board, we have 3 **Franklins**-per-TENPeople 2 **TENPeople** 3 People 7 **Franklins** 1 Hamilton 5 **Washingtons**
? However, if we want to give THREE ***hundred-dollar-bills*** to each ***group-of-TEN-people***, and since there are also THREE People alongside the TWO ***groups-of-TEN-people***, then in order to be fair, we must also give THREE ***ten-dollar-bills*** to each People. So, in order to treat all TWENTY-THREE People fairly, we need SIX ***hundred-dollar-bills*** and NINE ***ten-dollar-bills***.
v The (board) procedure is an exact representation of what we do with the bills: = = & = & = & = & = & And the situation is now as follows: 3 **Franklins**-per-TENPeople 2 **TENPeople** 3 People 7 **Franklins** 1 Hamilton 5 **Washingtons** 6 **Franklins** 9 **Hamiltons**
? Since we had SEVEN ***hundred-dollar-bills***,ONE ten-dollar-bill and FIVE ***one-dollar-bills***, we can do it but only if there are changing facilities. Altogether, we find that this leaves us with *two* ***ten-dollar-bills*** and *five* ***one-dollar-bills***. v The (board) procedure is an exact representation of what we do with the bills: 3 **Franklins**-per-**TENPeople** 2 **TENPeople** 3 People 7 **Franklins** 1 Hamilton 5 **Washingtons** 6 **Franklins** 9 **Hamiltons** 1 Franklin
? Then, changing the SEVEN hundred-dollar-bill for TEN ***ten-dollar-bills***, v The board is 3 **Franklins** – **per** – **TENPeople** 2 TENPeople 3 People 7 **Franklins** ELEVEN **Hamiltons** 5 **Washingtons** 6 **Franklins** 9 **Hamiltons** 0 Franklin 2 **Hamiltons** 5 **Washingtons** ? which, in view of the fact that there are THREE People along the TWO ***groups-of-TEN-people***, v we prefer to write as 3 **Hamiltons**-per-People 2 TENPeople 3 People 7 **Franklins** ELEVEN Hamilton 5 **Washingtons** 6 **Franklins** 9 **Hamiltons** 0 Franklin 2 **Hamiltons** 5 **Washingtons** ? Distributing fairly TWO ***ten-dollar-bills*** between TWO ***groups-of-TEN-people*** would let each ***group-of-TEN-people*** have ONE ten-dollar-bill. But since there are also THREE People alongside the TWO ***groups-of-TEN-people***, then in order to be fair, we must also give ONE ***one-dollar-bills*** to each People. So, in order to treat all TWENTY-THREE People fairly, we need TWO ***ten-dollar-bills*** and THREE ***one-dollar-bill***. v The (board) procedure is an exact representation of what we do with the bills: = = & = & = & = & = & On the board, the situation is as follows: 3 **Hamiltons**-per-People 3 **Hamiltons**-per-TENPeople 2 TENPeople 3 Chil-dren 7 **Franklins** 1 Hamilton 5 **Washingtons** 6 **Franklins** 9 **Hamiltons** 0 Franklin 2 **Hamiltons** 5 **Washingtons** 2 **Hamiltons** 3 **Washingtons** ? Since we had TWO ***ten-dollar-bills*** and FIVE ***one-dollar-bill***, we can do it. Alto-gether, we find that

this leaves us with NO ***ten-dollar-bills*** and ONE ***one-dollar-bills***. v The (board) procedure is an exact representation of what we do with the bills: 3 **Hamiltons**-per-People 3 **Hamiltons**-per-TENPeople 2 TENPeople 3 Chil-dren 7 **Franklins** 1 Hamilton 5 **Washingtons** 6 **Franklins** 9 **Hamiltons** 0 Franklin 2 **Hamiltons** 5 **Washingtons** 2 **Hamiltons** 3 **Washingtons** 0 Hamilton 2 **Washingtons** ? which, in view of the fact that there are THREE People along the TWO ***groups-of-TEN-people***, v we prefer to write as 3 **Hamiltons**-per-People 3 **Washingtons**-per-People 2 TENPeople 3 People 7 **Franklins** 1 Hamilton 5 **Washingtons** 6 **Franklins** 9 **Hamiltons** 0 Franklin 2 **Hamiltons** 5 **Washingtons** 2 **Hamiltons** 3 **Washingtons** 0 Hamilton 2 **Washingtons** Note that in order to carry out this division we needed to have changing facilities.

5.2 Distributing Money To People

If we have 12 dollars and we want to give 3 dollars-*per*-person, then to how many persons can we give money?

4 persons.

5.3 Ever-Smaller Denominators

DIVISION FORCES US TO CONSIDER EVER-SMALLER DENOMINATORS 1. Note that when we were adding the usual need to carry-over forced us to introduce ever-larger de-nominators. Similarly, when we were subtracting, the usual need to borrow also forced us to proceed from small denominators to larger denominators. And, inasmuch as multiplication involved an addition, there too we had to proceed from small denominators to larger denominators. As opposed to the above, in the case of division the need to change remaining large denominators for smaller denominators forced us to proceed from large denominators to smaller denominators until, even in the presence of exchange facilities, we could no longer divide and we were stuck with a remainder. In the above examples, what stopped the division was the fact that we could not exchange one-dollar-bills for anything smaller. To be sure, we did encounter dimes earlier on and so we could have exchanged the remaining one-dollar-bills for dimes but since we are now going downward to ever-smaller denominators we need to figure out how to introduce smaller denominators just as before we figured out how to introduce larger denominators. So, we will pretend that we do not know what a dime is and we will introduce it officially. 2. In order to explain what a dime is in terms of a one-dollar-bill we cannot say that ONE dime equals a tenth of a one-dollar-bill because, so far, we do not know what a tenth means. We cannot say ei-ther that ONE dime equals TEN cents because, at this point we still do not know what a cent is. On the other hand, all we need to know about what a dime is is that it takes ten of them to change for a one-dollar-bill. So, on the board, we write the dual of 1 Dime = TEN Cents that is 1 Dime = 1 of-which-TEN-will-exchange-for-1-Washington which we abbreviate as 1 Dime = TEN1 Washington. The phrase of-which-TEN-will-exchange-for-1- Washington is of course a denominator and the phrase of-which-TEN-will-exchange-for-1 is a prefix. We define similarly1 Cent = TEN1 Dime which we read as 1 Cent = 1 of-which-TEN-will-exchange-for-1-Dime and, while we are at it, we define 1 Mill = TEN1 Cent which we read as 1 Mill = 1 of-which-TEN-will-exchange-for-1-Cent. So, Mill is the denominator for the mysterious little 9 at the right of gasoline prices-per-gallon. Altogether, we thus have:

Clevelands Franklins Hamiltons Washingtons Dimes Cents Mills 3. In the metric (denomination) system, and choosing Washingtons as origin, this gives

KILO Washingtons HECTO Washingtons DEKA Washingtons Washingtons DECI Washingtons CENTI Washingtons MILLI Washingtons In the exponential notation system, this becomes

TEN+3 Washingtons TEN+2 Washingtons TEN+1 Washingtons TEN0 Washingtons TEN1 Washingtons TEN2 Washingtons TEN3 Washingtons and in the scientific variant

X 10+3 Washingtons X 10+2 Washingtons X 10+1 Washingtons X 100 Washingtons X 101 Washingtons X 102 Washingtons X 103 Washingtons In the decimal notation system, this becomes

X 1000. Washingtons X 100. Washingtons X 10. Washingtons X 1. Washingtons X 0. 1 Washingtons X 00. 1 Washingtons X 000. 1 Washingtons In the English notation system, this becomes

X THOUSAND Washingtons X HUNDRED Washingtons X TEN Washingtons X ONE Washingtons X TENTH Washingtons X HUNDREDTH Washingtons X THOUSANDTH Washingtons In the decimal fraction notation system, this becomes

X Washingtons X Washingtons X Washingtons X Washingtons X Washingtons X Washingtons X Washingtons 4. We are now in a position to discuss the way the metric system is extended to the right. a. As we saw, starting from an original denominator, and going from left to right, smaller denomi-nator are formed using the following prefixes:

DECI CENTI MILLI In other words, going from left to righ, we keep having new prefixes that we change ONE to TEN. After MILLI however, the new prefixes are introduced only every third place and in between we use the English notation system! Thus, to the right of MILLI there is no new prefix and so we use TENTH-OF-A-MILLI instead. To the right of TENTH-OF-A-MILLI there is HUNDREDTH-OF-A-MILLI and to the right of that we do have the new prefix MICRO. To the right of that we have TENTH-OF-A-MICRO and HUNDREDTH-OF-A-MICRO and then we have the new prefix, NANO.

MILLI TENTH-OF-A- MILLI HUNDREDTH-OF-A-MILLI MICRO TENTH-OF-A- MICRO HUNDREDTH-OF-A-MICRO NANO In other words, new prefixes keep being introduced every third place. Note the parallel with

THOUSANDTH TENTH-OF-A- THOUSANDTH HUNDREDTH-OF-A- THOUSANDTH MILLIONTH TENTH-OF-A- MILLIONTH HUNDREDTH-OF-A- MILLIONTH BILIONTH b. Right of Nano, reading from left to right, we have

By the time you read this, though, more prefixes might have been invented but the odds are very high that you will never have to use even most of the above. c. The new prefixes correspond to exponential prefixes in the following manner:

MILLI MICRO NANO PICO FENTO X 103 X 104 X 105 X 106 X 107 X 108 X 109 X 1010 X 1011 X 1012 X 1013 X 1014 X 1015 X 1016