Chapter 1

Syntactic Interlude: *Systematic* Counting

In which we shall take full advantage of the fact that the French Revolution decided on March 30, 1791 to create denominators *whose rate of change would match the number of available* digits*, namely* TEN*, the result being a most wonderfully simple* system *in favor of which even the English gave up the so-called English system but that, alas, the U.S. has not yet adopted—other than for* money*, which it did long ago. But then, of course, money* counts*. The other two countries that have not* officially *adopted the metric system are* **Liberia** *(in western Africa) and* **Myanmar** *(also known as Burma, in Southeast Asia.)*

Note. In this interlude chapter, we will focus on what we *write on the board* and leave to the reader the handling on the counter of the collections of objects being counted.

1.1 Systematic Denominators

We will now introduce several ways to create denominators *systematically*. These will be *used* just like we used **Clevelands**, **Franklins**, **Hamiltons**, **Washingtons**, **Dimes** and so we will focus here on how these denominators are *built*.

In the previous chapter, we saw that a very important advantage of the decimal number-phrases system is its flexibility in that the sender can "pick" the (select) denominator and the recipient can change it at will. In

unit (denominator)
prefix
Metric denominators
Exponential denominators
Decimal denominators
English denominators
Decimal fraction denominators
metric notation (system)
metric prefixes

developing ways to create denominators systematically, we want to have a similar flexibility. Essentially, then, what we will do will be to "pick" a **unit (denominator)** and use **prefix** to create the other denominators from the unit (denominator).

The first two notation (systems) are those used by scientists, often interchangeably, and they involve:

1. **Metric denominators**,
2. **Exponential denominators**.

The next three notation (systems) are of a different type in that they use "pieces of numerators" in the prefixes. They are generally *not* used by scientists.

3. **Decimal denominators**,
4. **English denominators**,
5. **Decimal *fraction* denominators**.

1. The **metric notation (system)** of denominators was the third breakthrough, not so much in mathematics but in sciences in general. The idea was simply to match the rate at which succeeding *denominators* are exchanged with the number of digits we use to write *numerators*, namely TEN. Thus, for instance, in the U.S. *money* is metric but *length* and *weight* are not and nowhere is *time* metric.

a. The metric notation (system) is based on the following **metric prefixes**:

Kilo	Hecto	Deka	–	Deci	Centi	Milli

For instance, if we pick **Franklins** as our unit (denominator), then the heading that corresponds to

Clevelands	Franklins	Hamiltons	Washingtons	Dimes

in the metric notation (system) involves the following metric denominators:

Deka Franklins	Franklins	Deci Franklins	Centi Franklins	Milli Franklins

and, for instance,

3.2 **Clevelands** becomes 3.2 **DekaFranklins**,
23.77 **Franklins** remains 23.77 **Franklins**,
7.45 **Hamiltons** becomes 7.45 **DeciFranklins**.

On the other hand, the heading that corresponds to

Clevelands	Franklins	Hamiltons	Washingtons	Dimes

when we pick **Washingtons** as our unit (denominator) involves the following metric denominators:

Kilo Washingtons	Hecto Washingtons	Deka Washingtons	Washingtons	Deci Washingtons

and then

3.2 **Clevelands** is written 3.2 **KiloWashingtons**,
23.77 **Franklins** is written 23.77 **HectoWashingtons**,
7.45 **Hamiltons** is written 7.45 **DekaWashingtons**.

b. The main advantage of the metric system is that when we want to change the (*select*) denominator or the *pointed digit*, there is a simple interplay between the two that is often expressed by some such "rule": "when we move one in one direction, we must move the other in the same direction". For instance, given

3.825 **DekaFranklins**

let us say we wanted the numerator to be 382.5. In other words, we would want to "move the decimal point two places to the right". To obtain the corresponding (*select*) denominator, we would then "take the (*select*) denominator two places to the right". This would give us

382.5 **DeciFranklins**

To see that the two are indeed the same, we use a heading:

Deka Franklins	Franklins	Deci Franklins	Centi Franklins	Milli Franklins
3	8	2	5	

Note that while we changed the (*select*) denominator, the *unit* (denominator) did not change.

c. Another *advantage* of the metric notation (system) is that it works for *goods* just as well as for *money.*
For instance, anywhere on earth, other than just about only in the U. S., the unit (denominator) for *length* is the **Meter**. (A ***meter*** is a bit longer than

systematic prefixes
exponential notation (system)

a ***yard*** which is why American sprinters training for the HUNDRED ***meters*** dash, standard in all international competitions, do so on HUNDRED AND TEN ***yards***.) Using the above heading, we then have, *automatically*, the metric denominators—some of which ought to be familiar:

KILO Meter	HECTO Meter	DEKA Meter	Meter	DECI Meter	CENTI Meter	MILLI Meter

d. One *shortcoming* that remains with the metric notation (system) is that more and more *prefixes* need to be "invented". However, in order to create a notation (system) that keeps the number of new prefixes down, we will need a part of yet *another* notation (system) and so we will deal with this shortcoming *after* we have introduced this other notation (system) that we need a part of.

2. In order not to keep having to "invent" more and more prefixes, scientists use **systematic prefixes**. The first such system is the **exponential notation (system)**.

a. Say we pick **Hamiltons** as our (select) denominator, then the heading that corresponds to

Clevelands	Franklins	Hamiltons	Washingtons	Dimes

in the exponential notation (system) involves the following exponential denominators:

TEN^{+2} Hamiltons	TEN^{+1} Hamiltons	TEN^{0} Hamiltons	TEN^{-1} Hamiltons	TEN^{-2} Hamiltons

and, for instance,

7.82 **Clevelands** is written 7.82 TEN^{+2} **Hamiltons**,
0.081 **Franklins** is written 0.081 TEN^{+1} **Hamiltons**,
27.4 **Washingtons** is written 27.4 TEN^{-1} **Hamiltons**.

b. Really to understand how this notation *works*, we need to return to the problem we encountered almost at the beginning of arithmetic, that is the problem of aggregating collections too large for us to have numerators to count the aggregate collection:

definition
abbreviate

♠ Given the two *collections* of ***dimes***

dime, dime, dime, dime, dime,
and
dime, dime, dime, dime, dime, dime, dime,

we aggregated them

dime, dime, dime, dime, dime, dime, dime, dime, dime, dime, dime, dime,

then we bundled TEN out of the TWELVE ***dimes***

bundle-of-TEN-dimes,
dime, ***dime***.

and then we exchanged the bundle for a ***dollar*** so that we finally had:

dollar,
dime, ***dime***.

❖ The (board) representation of this was:

$$\begin{aligned} 5 \ \textsf{Dimes} + 7 \ \textsf{Dimes} &= \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \\ &\quad \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime} \\ &= (\textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \ \textsf{Dime}, \textsf{Dime}, \\ &\quad \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime}), \textsf{Dime}, \textsf{Dime} \\ &= 1 \ \textsf{Dollar} \ \& \ 2 \ \textsf{Dimes}. \end{aligned}$$

As an alternative to this, namely bundling the *objects* and then exchanging the bundle for another object and then representing that object, we could, in the spirit of children writing, say, 𝍸 𝍸 /// **Dimes**, bypass the bundling stage and agree on a **definition** to the effect that, *on the board*, the denominator **TenDimes** stands for **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**.
In other words, what this definition does is to **abbreviate** on the *board*

Dime, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**, **Dime**

exponent

by

TenDimes

Then, we can write on the board

$$5 \ \textsf{Dimes} \ \& \ 7 \ \textsf{Dimes} = \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, \textsf{Dimes}, = 1 \ \textsf{TenDimes} \ \& \ 2 \ \textsf{Dimes}$$

Of course, we can just as easily define **TenTenDimes** etc.

c. This is *exactly* what the exponential system does except that it is even more economical as it uses

Ten^{+1} **Dimes** instead of **TenDimes**

and

Ten^{+2} **Dimes** instead of **TenTenDimes**

In other words, the small characters at the upper right of **Ten**, called the **exponent**, indicates how many copies of **Ten** are to be writtten. However, we cannot explain *negative* exponents in *this* manner and we will have to wait for a while for *that* explanation.
By the way, note that we could use, for example, **Ten**$^{+4}$ **Washington** for the denominator we were missing at the end of Chapter 2.

d. Note that the denominator **Ten**$^{+1}$**Dimes** conveys a lot of information:

- TEN encodes the exchange rate from one denominator to the next one to the *left*. For instance, 1 **Ten**$^{+1}$ **Hamiltons** = **Ten Hamiltons** just as 1 **Cleveland** = **Ten Hamiltons**, etc.
- The exponents encode the place of the denominator relative to the the unit (denominator): here, + means *left* and − means *right* and the number itself indicates how many places from the unit (denominator). For instance, **Clevelands** represents the same bill as **Ten**$^{+2}$ **Hamiltons** because **Clevelands** is the *second* denominator to the *left* of **Hamiltons** while **Washingtons** represents the same bill as **Ten**$^{-1}$ **Hamiltons** because **Washingtons** is the *first* denominator to the *right* of **Hamiltons**.
- This is one of the very rare instances where we allow ourselves to *write* **Ten**, but observe that we are not doing writing it as a *numerator* but only as *part* of a *denominator*. In fact, we could have used the phrase **Collection − of − Ten** instead of just **Ten** to avoid blurring the distinction between *numerator* and *denominator*.

- As we shall see presently, *none* of this would work for *any* English unit (denominator).

separator
scientific notation (system)

e. In reality, although perfectly good, the exponential notation (system) is not utilized quite as just given. Instead, scientists use a variant in which, instead of **TEN**, they use **x 10** where **x** is a **separator** that is now needed to separate the *numerator* from the *denominator*. (However, as opposed to **TEN**, the prefix **x 10** does *not* encode the exchange rate!)
For instance, picking again **Hamiltons** as our unit (denominator), the heading that corresponds to

Clevelands	**Franklins**	**Hamiltons**	**Washingtons**	**Dimes**

in the exponential notation (system) is the heading

x 10^{+2} Hamiltons	**x 10^{+1} Hamiltons**	**x 10^{0} Hamiltons**	**x 10^{-1} Hamiltons**	**x 10^{-2} Hamiltons**

and, for instance,

7.82 **Clevelands** is written 7.82 **x 10^{+2} Hamiltons**,
0.081 **Franklins** is written 0.081 **x 10^{+1} Hamiltons**,
27.4 **Washingtons** is written 27.4 **x 10^{-1} Hamiltons**.

Note. For rather fascinating illustrations of what this system does, search for "Powers of Ten" on the web. I particularly liked *A Question of Scale: Quarks to Quasars* by Bruce Bryson, an "activation" of *The Powers of Ten*, a book written by Philip and Phyllis Morrison and the Office of Charles and Ray Eames, as well as "activations" by Molecular Expressions and by microcosm.web.cern. But there are others.

f. A *specialization* of the exponential notation (system) is the **scientific notation (system)**. The additional requirement is simply that it should be the first non-zero digit *from the left* that is pointed. (This is often expressed by saying that the numerator should be between 1 and TEN.) Of course, the denominator must be changed accordingly which is best done, at least initially, *under a heading.*
For instance,

4.296 **x 10^{+3} Hamiltons** is already in scientific notation,
0.0081 **x 10^{+1} Franklins** becomes 8.1 **x 10^{-2} Franklins** in scientific notation,
752946.3 **x 10^{-2} Dimes** becomes 7.529463 **x 10^{+3} Dimes** in scientific notation.

decimal notation (system)
English notation (system)

The scientific notation (system) has all but ceased to be used with the advent of computers because its main reason for being was the tables that were used to facilitate computations.

3. The first of the three prefix systems that use "pieces of numerator" in the prefixes is the **decimal notation (system)**. Picking again **Hamiltons** as the unit (denominator), the heading that corresponds to

Clevelands	Franklins	Hamiltons	Washingtons	Dimes

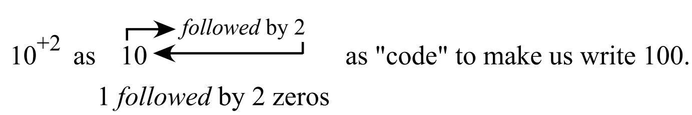
in the decimal notation (system) is the heading

x 100. Hamiltons	x 10. Hamiltons	x 1. Hamiltons	x 0.1 Hamiltons	x 0.01 Hamiltons

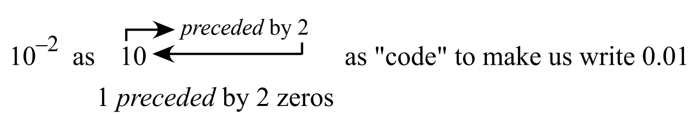
and, for instance

7.82 **Clevelands** is written 7.82 **x 100. Hamiltons**,
0.081 **Franklins** is written 0.081 **x 10. Hamiltons**,
27.4 **Washingtons** is written 27.4 **x 0.1 Hamiltons**.

The decimal notation (system) has the disadvantage of blurring the distinction between *numerator* and *denominator*. For instance, in the (decimal) number-phrase 27.4 **x 0.1 Hamiltons**, 27.4 is the (decimal) *numerator* while **0.1** is part of the (select) *denominator*.
Note. In order to facilitate the translation back and forth between *exponential* and *decimal* denominators, and until we reach "multiplicative powers", it will be convenient to read:



and



Note that, in both cases, the decimal point points to the *last* digit that was written.

4. The second of the three prefix systems that use "pieces of numerator" in the prefixes is the **English notation (system)**. Picking again **Hamiltons** as the unit (denominator), the heading that corresponds to

Clevelands	Franklins	Hamiltons	Washingtons	Dimes

decimal-fraction notation (system)

in the English notation (system) is the heading

x HUNDRED Hamiltons	x TEN Hamiltons	x ONE Hamiltons	x TENTH Hamiltons	x HUNDREDTH Hamiltons

and, for instance

7.82 **Clevelands** is written 7.82 **x HUNDRED Hamiltons**,
0.081 **Franklins** is written 0.081 **x TEN Hamiltons**,
27.4 **Washingtons** is written 27.4 **x TENTH Hamiltons**.

Note. When we say, for instance, "Seven Hundred Dollar Bills" or "Thirty Four Hundredth Of An Inch", we are essentially using the English notation (system).

One disadvantage of the English notation (system) is that it blurs the distinction between *numerator* and *denominator*. For instance, when we say "Seven Hundred Dollar Bills", of which collection of bills are we talking about?

Are we talking about SEVEN ***Hundred-Dollar-Bills*** or SEVEN-HUNDRED ***Dollar-Bills***? That, of course, these are the same amount of *money* is irrelevant here: the *objects* are different. One collection fits easily in a billfold, the other one doesn't. And, in the first case we would write 7. **Franklins** while in the second case we would write 700. **Washingtons**.

The English notation (system) has the further disadvantage of blurring the distinction between what we *write* and what we only allow ourselves to *say*, as in 1. **x TEN Hamiltons** where **TEN** is part of the *denominator* versus **TEN Hamiltons** where **TEN** is a *numerator* that we can *say* but that we cannot *write* on the board because we have no digit for it.

However, after we will have dealt with multiplication, we will see that these prefixes have the advantage of encoding the rate of exchange of each denominator with the unit (denominator).

5. And, finally, the third of the three prefix systems that use "pieces of numerator" in the prefixes is the **decimal-fraction notation (system)** which is just a variant of the English notation (system). For instance, picking **Hamiltons** as the unit (denominator), the heading that corresponds to

Clevelands	Franklins	Hamiltons	Washingtons	Dimes

in the decimal-*fraction* notation (system) involves the following decimal denominators:

compound prefix

x $\frac{100}{1}$ Hamiltons	x $\frac{10}{1}$ Hamiltons	x $\frac{1}{1}$ Hamiltons	x $\frac{1}{10}$ Hamiltons	x $\frac{1}{100}$ Hamiltons

and, for instance

7.82 **Clevelands** is written 7.82 x $\frac{100}{1}$ **Hamiltons**,
0.081 **Franklins** is written 0.081 x $\frac{10}{1}$ **Hamiltons**,
27.4 **Washingtons** is written 27.4 x $\frac{1}{10}$ **Hamiltons**.

Note. The decimal fraction notation system is rarely if ever used.

1.2 Creating Ever "Larger" Denominators

We are now in a position to discuss the way the metric system is extended far to the *left* of the prefixes we already have, that is to the left of **Kilo**. (We shall deal with the extension to the *right* after we have dealt with division.)

1. As we saw, starting from an the unit (denominator), and going from right to left, larger denominator are formed in the following manner:

Kilo Units	Hecto Units	Deka Units	Units

In other words, starting from the unit (denominator) and going from right to left, we keep having new prefixes.
However, in order to avoid having to invent so many new prefixes, it was agreed agreed that, from **Kilo** on, *new* prefixes would be introduced only every *third* place. In the two places in-between, one imitates the English notation (system) with the prefix introduced last. Here is how it works:

a. Using the denominator **Units** for whatever unit (denominator) we may happen to be dealing with, we begin with **Units** and then

- left of **Units** we have **Deka Units**
- left of **Deka Units** we have **Hecto Units**
- left of **Hecto Units** we have **Kilo Units**

But then:

- left of **Kilo Units**, instead of a *new* prefix, we use the **compound prefix** **Ten Kilo** and we have **Ten Kilo Units**,
- left of **Ten Kilo Units** we use the compound prefix **Hundred Kilo** and we have **Hundred Kilo Units**

But then:

- left of **Hundred Kilo**, we introduce the *new* prefix **Mega** instead of using what would be the *compound* prefix **Thousand Kilo** and we have **Mega Units**

In other words, after **Kilo** we use *new* prefixes only for every *third* denominator:

Mega Units	Hundred Kilo Units	Ten Kilo Units	Kilo Units	Hecto Units	Deka Units	Units

Note. The following heading would be a lot more natural than the above one in that it would not be mixing languages.

Mega Units	Hecto Kilo Units	Deka Kilo Units	Kilo Units	Hecto Units	Deka Units	Units

Moreover, we could introduce **Mega** as a shorthand for **Kilo Kilo**. That we don't is probably due to the parallel with the prefixes in the English (Notation) System:

Million Units	Hundred Thousand Units	Ten Thousand Units	Thousand Units	Hundred Units	Ten Units	Units

b. To the left of **Mega**, we start anew and we have **Ten Mega** and **Hundred Mega**. And then, to the left of **Hundred Mega** we have **Giga** instead of **Thousand Mega**. Then we go on with **Ten Giga** and **Hundred Giga** etc.

Hundred Giga Units	Ten Giga Units	Giga Units	Hundred Mega Units	Ten Mega Units	Mega Units

Again, note the parallel with the prefixes in the English (Notation) System:

Hundred Billion Units	Ten Billion Units	Billion Units	Hundred Million Units	Ten Million Units	Million Units

Note. While using *compound* prefixes saves on the trouble of inventing too many *new* prefixes, it raises the same kind of issue we already encountered in the case of the decimal notation (system) when we noted that it blurs the distinction between numerator and denominator.

I AM MAKING A CHANGE HERE

For instance, just as when we *say* (rather than *write*) "Three Hundred Million Bucks" it is not clear what is the unit (denominator) and what is the numerator, whether we mean 300 **MillionBucks** or 3 **HundredMillionBucks**. Similarly, when we say "Three Hundred Mega Bucks", it is no clear whether we mean 300 **MegaBucks** or 3 **HundredMegaBucks**.

2. Left of GIGA, reading *from right to left*, we have the prefixes in Figure

Figure 1.1: Larger and larger prefixes.

By the time you read this, though, more new prefixes will likely have been invented. The odds, though, are very high that you will never even have to use most of the above.

3. The new prefixes correspond to exponential prefixes in the following manner:

	TERA			GIGA			MEGA			KILO
x 10^{+13}	x 10^{+12}	x 10^{+11}	x 10^{+10}	x 10^{+9}	x 10^{+8}	x 10^{+7}	x 10^{+6}	x 10^{+5}	x 10^{+4}	x 10^{+3}

1.3 creating ever smaller denominators