discrete
continuous

# Chapter 1

# Accounting For *Continuous* Goods *on* the Counter (I)

- When we count *money*, what we do each time we have more than TEN of a kind is two things (See Chapter I):
  - We *bundle* TEN of a kind
  - We then *exchange* the bundle of TEN of a kind for 1 of the next kind up—for which we usually already have a denominator.

- When we count *goods*, what we will do will very much depend on the *kind* of goods we are counting.
  - When we count **discrete** *goods* such as, say, ***apples***, and while we will still *bundle* collections of TEN ***objects***, we will usually *not* be able to *exchange* these bundles for other objects, as we do when dealing with *money*, and the *denominators* will usually have to represent *bundles* rather than *objects* as was the case with *money*.
  - When we count **continuous** *goods* such as ***lengths*** or ***liquids***, what we will do will not involve any *bundling* but will involve *changes of denominator*. In most of the world, because of the *metric system*, accounting for this kind of *goods* is thus essentially the same as accounting for *money* and just as easy. However, under the English system, the process, while it remains essentially the same, involves much memorization.

## 1.1 Counting Goods *on* the Counter

In the U. S., accounting for *goods* is usually much more difficult than accounting for *money* because, contrary to what is the case with money, when dealing with goods, we still use English denominators and the English denominators do not change at the rate of TEN to 1.

**1.** Consider for instance the problem presented by, say, the number-phrase 27. **Inches** which corresponds to 2 TEN-**Inches** & 7 **Inches** which however *changes* to 2 **Feet** & 3 **Inches**. There are two ways to look at it.
One way would be to deplore that the English did not match our TEN digits with a denominator for a collection of TEN ***inches***. The other way would be to deplore that we do not have six fingers on each hand because then we would probably be using TWELVE digits which would match the fact that **Foot** is a denominator corresponding to a collection of TWELVE ***inches***.
However, and to make the problem even worse, English denominators do not even all change at the same rate with the result that there is no way that numerators and denominators could ever be matched. For instance, while 1 **Foot** changes for TWELVE **Inches**, 1 **Yard** changes for 3 **Feet**, 1 **Furlong** changes for TWO-HUNDRED AND TWENTY **Yards**, 1 **Mile** changes for 8 **Furlongs**, etc.

**2.** Compare counting money
1 **Dime**, 2 **Dime**, . . . , 9 **Dime**,
1 **Dollar**, 1 **Dollar** & 1 **Dime**, 1 **Dollar** & 2 **Dime**, . . . , 1 **Dollar** & 9 **Dime**,
2 **Dollar**, 2 **Dollar** & 1 **Dime**, 2 **Dollar** & 2 **Dime**, . . . ,
⋮
. . . , 9 **Dollar** & 9 **Dime**,
1 **DekaDollar**, 1 **DekaDollar** & 1 **Dime**, . . . ,
⋮

with counting lengths

1 **Inch**, 2 **Inch**, . . . , 9 **Inch**, TEN **Inch**, ELEVEN **Inch**,
1 **Foot**, 1 **Foot** & 1 **Inch**, 1 **Foot** & 2 **Inch**, . . . , 1 **Foot** & ELEVEN **Inch**,
2 **Foot**, 2 **Foot** & 1 **Inch**, . . . , 2 **Foot** & ELEVEN **Inch**,
1 **Yard**, 1 **Yard** & 1 **Inch**, . . . , 1 **Yard** & ELEVEN **Inch**, 1 **Yard** & 1 **Foot**, 1 **Yard** & 1 **Foot** & 1 **Inch**,
⋮
. . . , TWO-HUNDRED-NINETEEN **Yard** & 2 **Foot** & ELEVEN **Inch**,
1 **Furlong**, 1 **Furlong** & 1 **Inch**, . . .
⋮

## 1.2 Adding Goods *on* the Counter

**1.** Since, regardless of the denominators, we work with number-phrases that are based on TEN digits, this makes addition *very* awkward, even quite tricky.

♠ Say we want to weld the two pipes in Figure **??**.

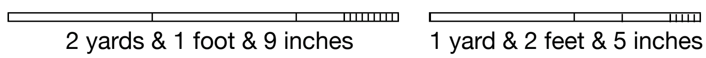


Figure 1.1: A 2 yrd, 1 ft, 9 in pipe and a 1 yrd, 2 ft, 5 in pipe.

When we measure the resulting pipe we find that its length is FOUR ***yards***, ONE ***foot***, TWO ***inches***. (Although we are of course much more likely to say that it is THIRTEEN ***feet***, TWO ***inches*** long.) ❖ On the board, we want to add 2 **Yard** & 1 **Foot** & 9 **Inch** and 1 **Yard** & 2 **Foot** & 5 **Inch** under the heading

| Yards | Feet | Inches |
|---|---|---|

The danger is to proceed with these *goods* just as if we were dealing with *money*:

| Hamiltons | Washingtons | Dimes |
|---|---|---|
| | *1* | |
| 2 | 1 | 9 |
| 1 | 2 | 5 |
| 3 | 4 | 4 |

| Yards | Feet | Inches |
|---|---|---|
| | | |
| 2 | 1 | 9 |
| 1 | 2 | 5 |
| | | |

and to conclude that the result of the addition is 3 **Yard** & 4 **Feet** & 4 **Inches** which of course does *not* represent what we found in the real world. The reason again is that it takes TWELVE **Inches** instead of TEN to get ONE **Foot** and it takes THREE **Feet** instead of TEN to get ONE **Yard**. So, of course, the addition should really proceed with the English rates of exchange, as follows:

| Yards | Feet | Inches |
|---|---|---|
| *1* | *1* | |
| 2 | 1 | 9 |
| 1 | 2 | 5 |
| 4 | 1 | 2 |

which indeed gives 4 **Yard** & 1 **Foot** & 2 **Inches**. This can be confusing particularly if one does not write the denominators.

It is no wonder then that even the English gave up on English denominators! Note that, in the U. S., convenience prevailed over tradition in only a very few cases: money is exchanged TEN to ONE and surveying tapes are marked in ***tenths of a foot*** rather than ***inches***. And of course, systematic denominators prevail in all scientific matters.

**2.** In contrast, here is an example of how *addition* would go in the metric system.

## 1.3 Subtracting Goods *on* the Counter

text

## 1.4 Multiplication Goods *on* the Counter

We now come to *multiplication* which will turn out to be quite a bit more difficult than *addition* to introduce and to discuss.

The problems come from the fact that multiplication occurs in the representation of at least three very different environments which therefore need to be clearly differentiated.

- Multiplication as *additive power* of number-phrases. We saw in Chapter I that when counting the collection of objects we get from unpacking a collection of bundles, the numerator

- Multiplication as *co-multiplication* of number-phrases. In this case, we are multiplying *goods* by they *unit-price* to get their *money* equivalent.

- Multiplication as *multiplication* of number-phrases. While there are a lot of real life situations in which *addition* of number-phrases occurs naturally, there are a lot fewer real life situations in which *multiplication* of number-phrases does. In the case of *addition* of number-phrases, we were able to start from its *meaning*, the aggregation of collections, and there was thus no doubt as to what the *result* was to be. This then allowed us to focus on developing the (board) procedure. By contrast, in the case of the *multiplication* of number-phrases, we must start by *finding* situations in which *multiplying* number-phrases will *mean* something. *If and when* it does mean something, then this will tell us what the resulting number-phrase might be and only then will it make sense to look for a (board) procedure that will give this resulting number-phrase.

### 1.4.1 Can Money Be Multiplied By Money? ×

First, and independently of whether or not *multiplying* counts might or might not mean anything, we introduce the symbol for *multiplication* that we will be using for it when writing on the board. We recall that, when we were dealing with *addition* and *subtraction*, we would write expressions involving two number-phrases with an addition symbol or a subtraction symbol in-between. For instance, we might have written 3 **Dimes** + 2 **Dimes** or 3 **Dimes** − 2 **Dimes**.

Similarly, multiplying counts would have to involve writing expressions involving two number-phrases with the multiplication symbol × in-between. For instance, we might write 3 **Dimes** × 2 **Dimes**.

At this point, though, we must clear up a frequent confusion: an expression like 3 **Dimes** × 2 **Dimes** is absolutely *not* the same as the expression 3 (2 **Dimes**).

Now, we saw in Section **??** that the expression 3 (2 **Dimes**) is nothing more than a number-phrase whose *numerator* is 3 and whose *denominator* (2 **Dimes**) represents a ***collection of two dimes*** so that, when we unpack, we get:

$$\begin{aligned} 3(2\ \textsf{Dimes}) &= (2\ \textsf{Dimes}), (2\ \textsf{Dimes}), (2\ \textsf{Dimes}) \\ &= \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime}, \textsf{Dime} \\ &= 6\ \textsf{Dimes} \end{aligned}$$

However, the fact that an expression on the board such as 3 (2 **Dimes**) makes perfect sense, that is, represents something on the counter, does *not* imply that an expression such as 3 **Dimes** × 2 **Dimes** also makes sense since they are expressions of a *different* kind.

For an expression such as 3 **Dimes** × 2 **Dimes** to make sense it would have to represent the result of doing something with THREE ***dimes*** and TWO ***dimes*** and coming up with a number of ***dimes*** the same way as 3 **Dimes** + 2 **Dimes** represented the result of *aggregating* THREE ***dimes*** and TWO ***dimes*** and the way 3 **Dimes** − 2 **Dimes** represented the result of *removing* TWO ***dimes*** from THREE ***dimes***.

The question then is: what could an expression on the board such as 3 **Dimes** × 2 **Dimes** possibly represent on the counter? The answer is: Absolutely nothing and expressions of the form 3 **Denominator** × 2 **Denominator** are *usually* completely meaningless.

### 1.4.2 Multiplying *Certain* Goods *on* the Counter

length
construct
rectangle
tile

In the case of *certain goods*, though, expressions of the form 2 **Denominator** × 3 **Denominator** *can* represent the result of doing something with the collections represented by 2 **Denominator** and 3 **Denominator**. The objects with which this can be done are quite particular. The example we shall use is that of **length**, as in "a length of material" and we will thus draw our inspiration from "building materials" in which people speak, for instance, of a "four-by-eight" sheet of plywood.

We will begin with a very simple case and work our way up. At each stage, we will start with the more familiar *English* denominators and then look at the "same" example with *metric* denominators. The first stage will *not* involve any carryover because, as we already saw in the case of *addition*, English denominators do not lend themselves easily to computation since the English exchange rates are not always the same as is the case in the metric system. Only after we will have figured out what multiplying number-phrases might mean and what the resulting number-phrase then is, will we deal with the technical issue of "carryovers".

**1.** The point of this example is to observe that, contrary to what was the case with *addition* and *subtraction*, where the denominator in the result of the operation was the *same* as the denominator in the number phrases being operated on, in the case of *multiplication*, the denominator in the result is *different* from the denominator in the number phrases being operated on.

**a.** Here it is with English denominators.

♠ Given a TWO ***inch length*** and a THREE ***inch length***,

- We can **construct** on the counter (Figure **??**) a ***two-by-three rectangle***, that is a **rectangle** that is TWO ***inch long*** one way and THREE ***inch long*** the other way:

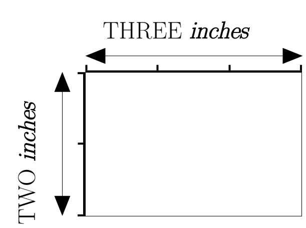


Figure 1.2: A "TWO by THREE" rectangle.

- We may then want to **tile** this rectangle (Figure **??**) with ***one-inch-by-one-inch mosaics***:

multiplication tables

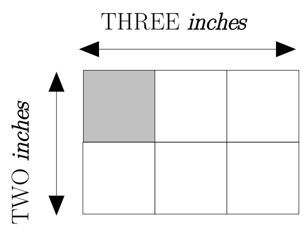


Figure 1.3: The TWO by THREE rectangle tiled with ***mosaics***

Counting the ***mosaics*** shows that we will need SIX ***one-inch-by-one-inch mosaics***.

❖ The expression 2 **Inch** × 3 **Inch** then represents on the board the ***mosaics*** that will be needed to tile the rectangle.

Since, as children, we are usually enjoined to memorize the **multiplication tables**, the (board) procedure for multiplication in this case consists in looking up the relevant multiplication table. We find that

$$2\ \textsf{Inch} \times 3\ \textsf{Inch} = 6\ [\textsf{Inch} \times \textsf{Inch}].$$

where 6 is the numerator and where [**Inch** × **Inch**] is the denominator that represents ***one-inch-by-one-inch mosaics*** on the board.

**b.** We now look at the "same" example but with *metric* denominators.

♠ For instance, given a TWO ***meters*** length and a THREE ***meters*** length, we can construct on the counter (Figure **??**) a ***two-meter-by-three-meter rectangle***, that is a rectangle that is TWO ***meters*** long one way and THREE ***meters*** long the other way and then we can tile it with ***one-meter-by-one-meter tiles*** See Figure ??:

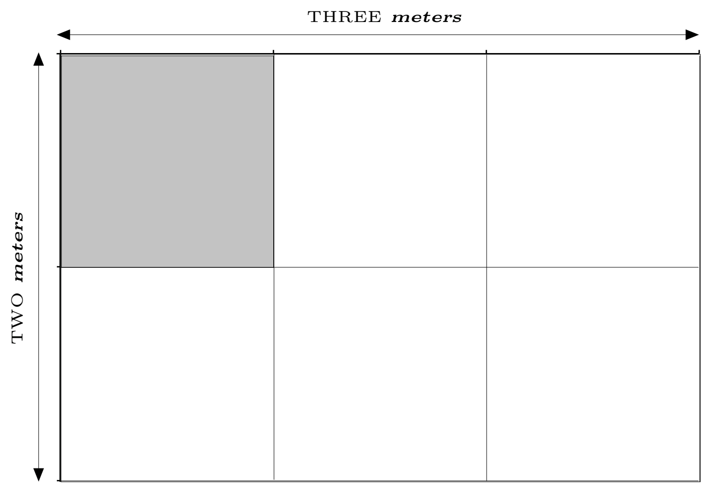


Figure 1.4: A TWO ***meter*** by THREE ***meter*** rectangle tiled with ***one-meter-by-one-meter tiles.***

Counting the ***tiles*** shows that we will need SIX ***one-meter-by-one-meter tiles***.

❖ The expression 2 **Meter** × 3 **Meter** then represents on the board the ***tiles*** that will be needed to tile the rectangle. Looking up the relevant multiplication table gives

$$2\ \textsf{Meter} \times 3\ \textsf{Meter} = 6\ [\textsf{Meter} \times \textsf{Meter}].$$

where 6 is the numerator and where [**Meter** × **Meter**] is the denominator that represents ***one-meter-by-one-meter tiles*** on the board.

**2.** The point of this example is to show that, also contrary to what was the case with *addition*, where the two denominators in the number-phrases being added *had* to be the *same*, in the case of *multiplication*, the denominators of the number-phrases being multiplied *can* be *different*. Indeed, the two sides of a rectangle are often measured with different denominators.

**a.** We begin with an example involving the more familiar English denominators.

♠ For instance, given a THREE ***inch*** length and a TWO ***foot*** length, we can cut on the counter a ***three-inch-by-two-foot plank***, that is a rectangle that is THREE ***inches*** long one way and TWO ***feet*** long the other way. We may then want to tile this plank with ***one-inch-by-one-foot strips***

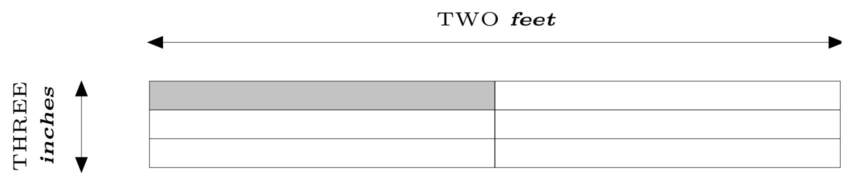


Figure 1.5: A THREE ***inch*** by TWO ***foot*** rectangle tiled with ***one-inch-by-one-foot strips***

Counting the ***strips*** shows that we will need SIX ***one-inch-by-one-foot strips***.

❖ The expression 3 **Inch** × 2 **Foot** then represents on the board the ***strips*** that will be needed to tile the rectangle on the counter. We find that

$$3\ \textsf{Inch} \times 2\ \textsf{Foot} = 6\ [\textsf{Inch} \times \textsf{Foot}].$$

where 6 is the numerator and where [**Inch** × **Foot**] is the denominator that represents ***one-inch-by-one-foot strips*** on the board.

**b.** We now look at the "same" example but with metric denominators

♠ For instance, given a THREE ***meters*** length and a TWO ***dekameters*** length, we can cut on the counter a ***three-meters-by-two-dekameters rectangle***, that is a rectangle that is THREE ***meters*** long one way and TWO ***dekameters*** long the other way. We may then want to tile this rectangle with ***one-meter-by-one-dekameters strips*** (Figure **??**)

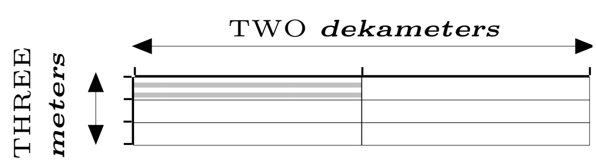


Figure 1.6: A THREE ***meters*** by TWO ***dekameters*** rectangle tiled with ONE ***meters*** by ONE ***dekameters*** strips

❖ The expression 3 **Meter** × 2 **DekaMeter** then represents on the board the ***strips*** that will be needed to tile the rectangle on the counter. We find that

$$3\ \textsf{Meter} \times 2\ \textsf{DekaMeter} = 6\ [\textsf{Meter}{\times}\textsf{DekaMeter}].$$

where 6 is the numerator and where [**Inch** × **Foot**] is the denominator that representsONE ***dekameters*** strips on the board.

**3.** The point of this example is to show that, essentially in the same manner, we can multiply *combinations* of lengths.

**a.** We begin with English denominators.

square
rectangular

♠ For instance, given a TWO ***foot***, TWO ***inch*** length and a THREE ***foot***, ONE ***inch*** length, we can construct and tile the rectangle as in Figure ??)

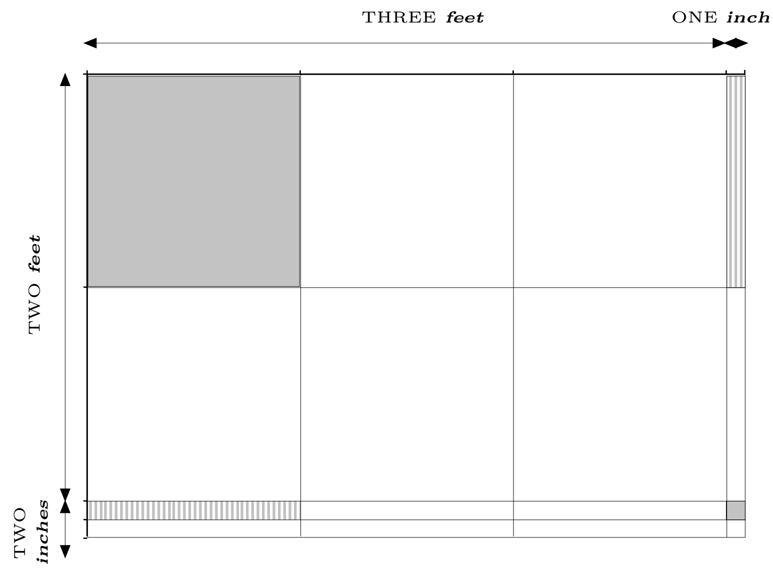


Figure 1.7: The ***three-foot, one-inch by two-foot, three-inch rectangle*** tiled with four different kinds of tiles

Counting the ***tiles*** shows that we will need:
the following **square** tiles

- SIX ***one-foot-by-one-foot tiles***,
- TWO ***one-inch-by-one-inch tiles***,

and the following **rectangular** tiles

- TWO ***one-foot-by-one-inch tiles***,
- SIX ***one-inch-by-one-foot tiles***,

The ***one-inch-by-one-foot tiles*** and the ***one-foot-by-one-inch tiles*** are counted *separately* if only because of the different ways they are striped.
❖ The expression [3 **Foot** & 1 **Inch**] × [2 **Foot** & 3 **Inch**] then represents on the board the ***tiles*** that we will need to tile it.
The (board) procedure for multiplication in this case is a bit more complicated. First we set up:

| | | | |
|---|---|---|---|
| | 3 **Foot** | & | 1 **Inch** |
| × | 2 **Foot** | & | 2 **Inch** |

The next step is to get the different kinds of tiles using the appropriate *multiplication tables*. Observe that we are handling **Inch**×**Foot** and **Foot**×**Inch** separately:

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | 3 **Foot** | & | 1 **Inch** |
| | | | × | 2 **Foot** | & | 2 **Inch** |
| | | | | 6 **Inch**×**Foot** | & | 2 **Inch**×**Inch** |
| 6 **Foot**×**Foot** | & | 2 **Foot**×**Inch** | | | | |

Altogether, we thus find:

$$
\begin{aligned}
3[3\ \mathsf{Foot}\ \&\ 1\ \mathsf{Inch}] \times [2\ \mathsf{Foot}\ \&\ 2\ \mathsf{Inch}] = {} & 6\ \mathsf{Foot}{\times}\mathsf{Foot}, \\
& \&\ 2\ \mathsf{Foot}{\times}\mathsf{Inch} \\
& \&\ 6\ \mathsf{Inch}{\times}\mathsf{Foot} \\
& \&\ 2\ \mathsf{Inch}{\times}\mathsf{Inch}
\end{aligned}
$$

which is the (board) representation of the above.

**b.** We now look at the “same” example but with metric denominators ♠ Given a TWO ***dekameter***, TWO ***meter*** length and a THREE ***dekameter***, ONE ***meter*** length , we can construct and tile a rectangle as in Figure **??**)

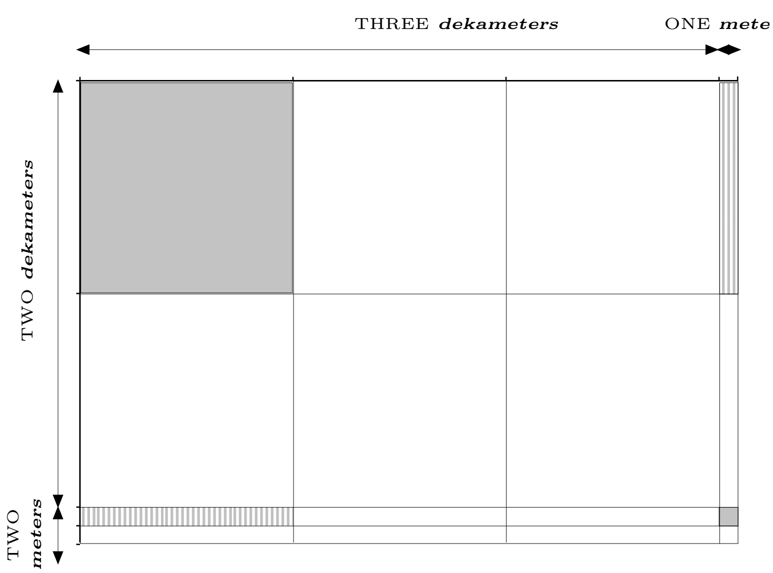


Figure 1.8: The ***two-dekameter, two-meter by three-dekameter, one-meter rectangle*** tiled with four different kinds of tiles

Counting the ***tiles*** shows that we will need:
The following *square* tiles

- SIX ***one-dekameter-by-one-dekameter tiles***,
- TWO ***one-meter-by-one-meter tiles***,

The following *rectangular* tiles

- TWO ***one-dekameter-by-one-meter tiles***,
- SIX ***one-meter-by-one-dekameter tiles***,

The rectangular tiles, ***one-meter-by-one-dekameter tiles*** and ***one-dekameter-by-one-meter tiles***, are counted *separately* if only because of the different ways they are striped.
❖ Exceptionally, for reasons of space, here we abbreviate **Meter** as **M** and **DekaMeter** as **DekaM**.
The expression [3 **DekaM** & 1 **M**] × [2 **DekaM** & 3 **M**] then represents on the board the ***tiles*** that we will need to tile the rectangle. We get:

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | 3 **DekaM** | & | 1 **M** |
| | | | × | 2 **DekaM** | & | 2 **M** |
| | | | | 6 **M**×**DekaM** | & | 2 **M**×**M** |
| 6 **DekaM**×**DekaM** | & | 2 **DekaM**×**M** | | | | |

Altogether, we thus find:

$$
\begin{aligned}
[3\ \textsf{DekaM}\ \&\ 1\ \textsf{M}] \times [2\ \textsf{DekaM}\ \&\ 2\ \textsf{M}] = {} & 6\ \textsf{DekaM}\times\textsf{DekaM}, \\
& \&\ 2\ \textsf{DekaM}\times\textsf{M} \\
& \&\ 6\ \textsf{M}\times\textsf{DekaM} \\
& \&\ 2\ \textsf{M}\times\textsf{M}
\end{aligned}
$$

which is the (board) representation of the above.

**4.** The point now is to see what happens in the preceding example if we don't care about the way the tiles are striped or if they are striped the same way.

**a.** We begin with English denominators

♠ First, observe (Figure **??**) that a ***one-inch-by-one-foot rectangle*** and a ***one-foot-by-one-inch rectangle*** can both be tiled with TWELVE ***one-inch-by-inch-inch mosaics***.

common denominator

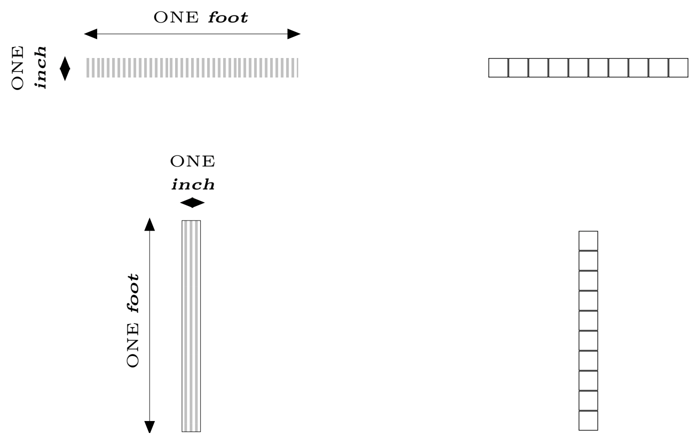


Figure 1.9: Both a ***one-inch-by-one-foot rectangle*** and a ***one-foot-by-one-inch rectangle*** can be tiled with TWELVE ***one-inch-by-inch-inch mosaics***

Thus, from that viewpoint, the SIX ***one-foot-by-one-inch tiles*** and the TWO ***one-inch-by-one-foot tiles*** in Figure **??** are of the same kind and we can *aggregate* them.

❖ We set up in the way we learned in elementary school because it will make it easier to *add* **Inch**×**Foot** and **Foot**×**Inch**.

Which *denominator* to use, **Inch**×**Foot** or **Foot**×**Inch**, is up to us but we need to agree on it.

| | | | | |
|---|---|---|---|---|
| | | 3 foot | & | 1 inch |
| | × | 2 foot | & | 2 inch |
| | | 6 Inch×Foot | & | 2 Inch×Inch |
| 6 Foot×Foot | & | 2 Foot×Inch | | |
| 6 Foot×Foot | & | 8 Foot×Inch | & | 2 inch × Inch |

if we agree on **Foot**×**Inch** as **common denominator**, or

| | | | | |
|---|---|---|---|---|
| 6 Foot×Foot | & | 8 Inch×Foot | & | 2 inch×Inch |

if we agree on **Inch**×**Foot** as common denominator.

**b.** We look at the "same" example but with metric denominators.

♠ First, observe (Figure **??**) that a ***one-meter-by-one-dekameter rectangle*** and a ***one-dekameter-by-one-meter rectangle*** can both be tiled with TEN ***one-meter-by-inch-meter tiles***.

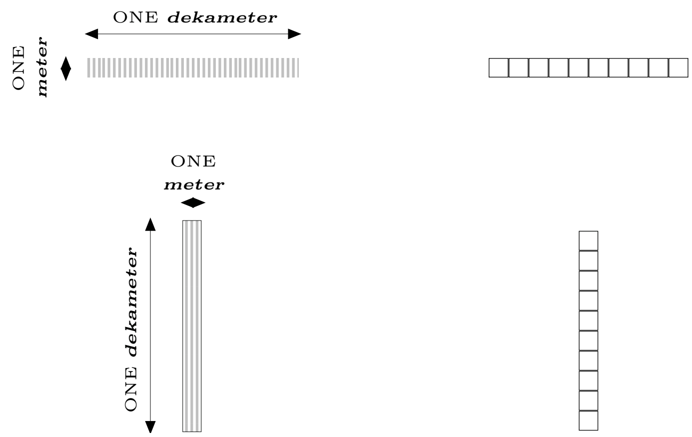


Figure 1.10: Both a ***one-meter-by-one-dekameter rectangle*** and a ***one-dekameter-by-one-meter rectangle*** can be tiled with TEN ***one-meter-by-one-meter tiles***

Thus, from that viewpoint, the SIX ***one-dekameter-by-one-meter tiles*** and the TWO ***one-meter-by-one-dekameter tiles*** in Figure **??** are of the same kind and we can *aggregate* them.

❖ We set up again in the way we learned in elementary school because it will make it easier to *add* **Meter×DekaMeter** and **DekaMeter×Meter**.

| | | | | |
|---|---|---|---|---|
| | | 3 **DekaMeter** | & | 1 **Meter** |
| | × | 2 **DekaMeter** | & | 2 **Meter** |
| | | 6 **Meter×DekaMeter** | & | 2 **Meter×Meter** |
| 6 **DekaMeter×DekaMeter** | & | 2 **DekaMeter×Meter** | | |
| 6 **DekaMeter×DekaMeter** | & | 8 **DekaMeter×Meter** | & | 6 **Meter×Meter** |

if we agree on **DekaMeter×Meter** as common denominator, or

| | | | | |
|---|---|---|---|---|
| 6 **DekaMeter×DekaMeter** | & | 8 **Meter×DekaMeter** | & | 2 **Meter×Meter** |

if we agree on **Meter×DekaMeter** as common denominator.

**5.** To see how multiplication works when we have "carryovers", we will only use metric denominators because, as we already saw in Section xxx, English denominators do not lend themselves easily to computation since the English exchange rates are not always the same. (For instance, 1 **Foot** = TWELVE **Inch** while 1 **Yard** = 3 **Feet**.)

**a.** First we look at an example where the carryover will occur in the *addition*.

♠ Given a THIRTY-TWO ***meter*** length and a TWENTY-THREE ***meter*** length, we look at them as being made-up as follows:

- We look at the THIRTY-TWO ***meter*** length as being made up of THREE ***dekameters*** and TWO ***meters***
- We look at the TWENTY-THREE ***meter*** length as being made up of TWO ***dekameters*** and THREE ***meters***

We then construct a THIRTY-TWO ***meter*** by TWENTY-THREE ***meter*** rectangle, that is a rectangle that is THREE ***dekameters*** and TWO ***meters*** long one way and TWO ***dekameters*** and THREE ***meters*** long the other way.

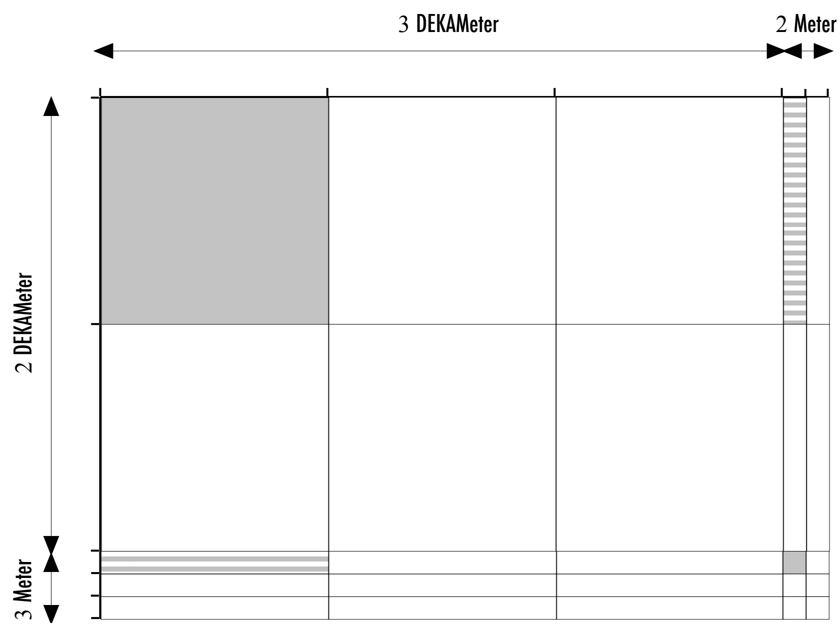


Figure 1.11: The tiling of a THIRTY-TWO ***meter*** by TWENTY-THREE ***meter*** rectangle.

Counting the ***tiles*** shows that we will need:

- SIX ***one-dekameter-by-one-dekameter tiles***
- FOUR ***one-dekameter-by-one-meter tiles***
- NINE ***one-meter-by-one-dekameter tiles***
- SIX ***one-meter-by-one-meter tiles***

Since we don't distinguish the ***one-dekameter-by-one-meter tiles*** from the ***one-meter-by-one-dekameter tiles***, we can aggregate them and we get

THIRTEEN ***one-dekameter-by-one-meter tiles***

or

THIRTEEN ***one-meter-by-one-dekameter tiles***

depending on how we want to see them.

However, since we are not going to be able to write THIRTEEN, we must *change* THIRTEEN ***one-dekameter-by-one-meter tiles*** (or THIRTEEN ***one-meter-by-one-dekameter tiles***) and the question is for what?

Figure **??** shows that TEN ***one-meter-by-one-dekameter tiles*** tile ONE ***one-dekameter-by-one-dekameter tiles***

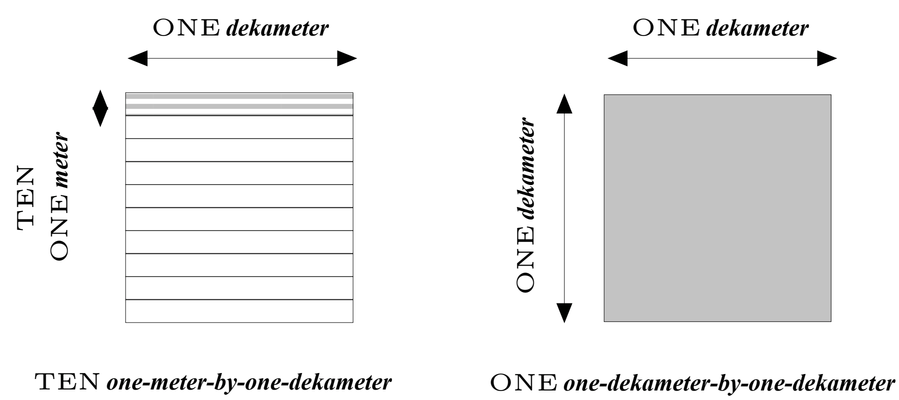


Figure 1.12: Changing TEN ***one-meter-by-one-dekameter tiles***

❖ Here again, the (board) procedure reflects what we just did.

| | | | | |
|---|---|---|---|---|
| | | 3 DekaM | & | 2 M |
| | × | 2 DekaM | & | 3 M |
| | | 9 M×DekaM | & | 6 M×M |
| 6 DekaM×DekaM | & | 4 DekaM×M | | |
| 6 DekaM×DekaM | & | THIRTEEN DekaM×M | & | 6 M×M |

or

| | | | | |
|---|---|---|---|---|
| 6 DekaM×DekaM | & | THIRTEEN M×DekaM | & | 6 M×M |

And we complete the (board) procedure as follows

| | | | | |
|---|---|---|---|---|
| | | 3 DekaM | & | 2 M |
| | × | 2 DekaM | & | 3 M |
| *1* DekaM×DekaM | | | | |
| | | 9 M×DekaM | & | 6 M×M |
| 6 DekaM×DekaM | & | 4 DekaM×M | | |
| 7 DekaM×DekaM | & | 3 DekaM×M | & | 6 M×M |

or

| | | | | |
|---|---|---|---|---|
| 7 DekaM×DekaM | & | 3 M×DekaM | & | 6 M×M |

**b.** Now we look at an example where the carryover will occur in one of the *multiplications*

♠

❖ First we proceed as we did before:

| | | | | |
|---|---|---|---|---|
| | | 1 DekaM | & | 5 M |
| | × | 1 DekaM | & | 3 M |
| | | 3 M×DekaM | & | FIFTEEN M×M |
| 1 DekaM×DekaM | & | 5 DekaM×M | | |
| 1 DekaM×DekaM | & | 8 DekaM×M | & | FIFTEEN M×M |

if we agree on DekaM×M as common denominator, or

| | | | | |
|---|---|---|---|---|
| 1 DekaM×DekaM | & | 8 M×DekaM | & | FIFTEEN M×M |

if we agree on M×DekaM as common denominator.
Now we must change FIFTEEN M×M for 1 DekaM & 5 M×M:

| | | | | |
|---|---|---|---|---|
| | | 1 DekaM | & | 5 M |
| | × | 1 DekaM | & | 3 M |
| | | 1 M×DekaM | | |
| | | 3 M×DekaM | & | 5 M×M |
| 1 DekaM×DekaM | & | 5 DekaM×M | | |
| 1 DekaM×DekaM | & | 9 DekaM×M | & | 5 M×M |

if we agree on DekaM×M as common denominator, or

| | | | | |
|---|---|---|---|---|
| 1 DekaM×DekaM | & | 9 M×DekaM | & | 5 M×M |

if we agree on M×DekaM as common denominator.
Of course, we shouldn't wait and we should do the change *immediately* rather than write FIFTEEN M×M

**6.** We shall now see how the above multiplication looks under a *heading*.

square denominator

**a.** First, we recall that the metric heading for lengths is:

| | | | | | | |
|---|---|---|---|---|---|---|
| Kilo | Hecto | Deka | | Deci | Centi | Milli |
| Meter | Meter | Meter | Meter | Meter | Meter | Meter |

and that the rate of change is TEN for 1.

**b.** Corresponding to each of these denominators, we have the corresponding **square denominator**:

- **MilliMeter×MilliMeter** also called **Square MilliMeter**
- **CentiMeter×CentiMeter** also called **Square CentiMeter**
- **DeciMeter×DeciMeter** also called **Square DeciMeter**
- **Meter×Meter** also called **Square Meter**
- **DekaMeter×DekaMeter** also called **Square DekaMeter**
- **HectoMeter×HectoMeter** also called **Square HectoMeter**
- **KiloMeter×KiloMeter** also called **Square KiloMeter**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Square | | Square | | Square | | Square | | Square | | Square | | Square |
| Kilo | | Hecto | | Deka | | | | Deci | | Centi | | Milli |
| Meter | | Meter | | Meter | | Meter | | Meter | | Meter | | Meter |

Note that the rate of change from one denominator to the next is still TEN to 1 and that the *empty* spaces correspond to the *non-square* denominators. For instance, the empty space between **Square Meter** and **Square DekaMeter** is for

**DekaMeter×Meter**

or

**Meter×DekaMeter**

Note that the rate of change from one square denominator to the next is HUNDRED to 1.


Figure 1.13: Changing HUNDRED ***one-meter-by-one-meter tiles***

**c.** We now write the above multiplication under the heading:

| Square DekaMeter | | Square Meter |
|---|---|---|
| *1* | | |
| | 3<br>2 | 2<br>3 |
| <br>6 | 9<br>4 | 6 |
| 7 | 3 | 6 |