form
data
specifying-phrase
sentence

Chapter 1

Comparing Collections of Discrete Goods

In order to deal with this kind of problem, though, we must begin by developing some *language*.

1.1 Specifying-phrases

A **form** is a *phrase* in which not all the information has been filled in but with box(es) in which to enter the missing information. A **data** is given information to be entered in the relevant box(es) of the form. So, a *form* together with *data* will produce a **specifying-phrase**.

For instance, "The President of the United States in $\boxed{}$ " is a *form* that, with 1800 as *data*, produces the *specifying-phrase*:

$$\text{The President of the United States in } \boxed{1800}$$

Once we find out that this was

John Adams

we can write the **sentence**:

$$\text{The President of the United States in } \boxed{1800} = \text{John Adams}$$

With the same form, the data 1900 gives the sentence

$$\text{The President of the United States in } \boxed{1900} = \text{William McKinley}$$

specify
brackets
parentheses
:=

Thus, a *specifying-phrase* is a phrase that **specify** somebody/something but without *us* necessarily knowing who/what that body/thing is. Specifying-phrases are much used in mathematics but in a somewhat different *format* that we shall now introduce.

1. We could make *forms* just as we did above, for instance as in the following:

$$4 \cdot \boxed{} - 8$$

where the box is meant to be filled in with a *copy* of the numerator in the *data*. Say the *data* is 5. Then we *write* a copy of 5 within the box:

$$4 \cdot \boxed{5} - 8$$

and then we *compute*

$$4 \cdot \boxed{5} - 8 = 20 - 8 = 12$$

Thus, the *form* $4 \cdot \boxed{5} - 8$ together with the *data* 5 specifies 4.

2. Boxes, though, are quite difficult both to type and to type *in* and so we omit the top and bottom of the boxes and use **brackets** to fake the sides of the boxes or **parentheses** to fake circular boxes. Parentheses are actually much more common.

$$4 \cdot [\quad] - 8 \qquad\qquad 4 \cdot (\quad) - 8$$

Say the *data* is 5. Then we *write* copies of 5 within the parentheses:

$$4 \cdot (5) - 8$$

3. This would still be awkward for our purpose. The idea then is to use, instead of brackets or parentheses, a letter, usually x. So we write the above *form* as

$$4x - 8$$

and the *instruction* to enter the *data* 5 as

$$|_{\text{where } x:=5}$$

where the symbol **:=**, borrowed from a computer language called Pascal, is read as "is to be replaced by".
Altogether, the *specifying-phrase* would be:

$$4x - 8|_{\text{where x:=5}}$$

Now, whereas we *entered* copies of the data *within the parentheses*, we now **substitute** copies of the data *for the letter x* wherever x appears. We thus write substitute

$$4x - 8|_{\text{where x:=5}} = 4 \cdot 5 - 8 = 20 - 8 = 12$$

4. When substituting a *signed*-number, it is usually safer to surround it with parentheses!

For instance, given the above form, if the data is -5, we write

$$4x - 8|_{\text{where x:=-5}} = 4 \cdot (-5) - 8 = -20 - 8 = -28$$

While, here, the danger is not great, we shall see later on that it can be.

5. The letter x alone can be considered as a *form* and, indeed,

$$x|_{\text{when } x:=5} = 5$$

This is often "abbreviated" as $x = 5$ but, as usual when cutting corners, and as we shall see when we deal with *equations*, this is a bit of an abuse of language and creates an *ambiguity*.

6. Moreover, and in order to conform with usual mathematical practice, and even though technically this is not correct, we *shall* use "when $x =$" instead of "where $x :=$". (The word "when" in fact goes usually without saying but *we shall* say it.) For instance, we shall write

$$4x - 8|_{\text{when x=5}} = 4$$

7. Quite often, the instruction to enter the *data*, say $|_{\text{when } x=+5}$, will remain *unstated* for a while, during which time we will think of, and operate with, the *form* as if it were a *numerator*. This can save us a lot or work when we have a lot of *data*.

For instance, given the form $3x - 12 + 5x + 3$, and given the data 2, 3, 4, instead of going the route

$$3x-12+5x+3 \xrightarrow{\textit{enter data}} 3x-12+5x+3|_{\text{when x=2}} \xrightarrow{\textit{arithmetic}} 3\cdot 2-12+5\cdot 2+3 = 6-12+10+3 = 7$$

$$\xrightarrow{\textit{enter data}} 3x-12+5x+3|_{\text{when x=3}} \xrightarrow{\textit{arithmetic}} 3\cdot 3-12+5\cdot 3+3 = 9-12+15+3 = 15$$

$$\xrightarrow{\textit{enter data}} 3x-12+5x+3|_{\text{when x=4}} \xrightarrow{\textit{arithmetic}} 3\cdot 4-12+5\cdot 4+3 = 12-12+20+3 = 23$$

we can go the route:

$$3x-12+5x+3 \ \big\downarrow{\scriptstyle \textit{algebra}}\ 8x-9$$

$$8x-9 \xrightarrow{\textit{enter data}} 8x-9|_{\text{when x=2}} \xrightarrow{\textit{arithmetic}} 8\cdot 2-9=16-9=7$$

$$\xrightarrow{\textit{enter data}} 8x-9|_{\text{when x=3}} \xrightarrow{\textit{arithmetic}} 8\cdot 3-9=24-9=15$$

$$\xrightarrow{\textit{enter data}} 8x-9|_{\text{when x=4}} \xrightarrow{\textit{arithmetic}} 8\cdot 4-9=32-9=23$$

While this might not be exactly an overwhelming example, the algebra does save us a lot on the arithmetic, the more we have to evaluate, the more we will save.

Later we shall see much more convincing ones.

1.2

We begin with the case involving one kind of *goods* and *money*.

♠ For instance, say Jack has THREE ***apples*** and ELEVEN ***cents*** and Jill has TWO ***apples*** and THIRTEEN ***cents*** and we want to know for which unit-values will Jack have the same as Jill.

As we saw in Chapter 2, whether Jack has *more than* Jill, *less than* Jill or *the same as* Jill depends on the unit-values of ***apples***.

❖ On the board, we write

$$\begin{aligned}
\textsf{Jack} \quad & \textsf{Jill} \\
3\ \textsf{Apples}\ \&\ 11\ \textsf{Cents} \quad & 2\ \textsf{Apples}\ \&\ 13\ \textsf{Cents} \\
\textsf{Jack's value} &= \textsf{Jill's value} \\
3\ \textsf{Apples} \times x\ \frac{\textsf{Cents}}{\textsf{Apples}} + 11\ \textsf{Cents} &= 2\ \textsf{Apples} \times x\ \frac{\textsf{Cents}}{\textsf{Apples}} + 13\ \textsf{Cents} \\
3\ \cancel{\textsf{Apples}} \times x\ \frac{\textsf{Cents}}{\cancel{\textsf{Apples}}} + 11\ \textsf{Cents} &= 2\ \cancel{\textsf{Apples}} \times x\ \frac{\textsf{Cents}}{\cancel{\textsf{Apples}}} + 13\ \textsf{Cents} \\
3x\ \textsf{Cents} + 11\ \textsf{Cents} &= 2x\ \textsf{Cents} + 13\ \textsf{Cents} \\
(3x + 11)\ \textsf{Cents} &= (2x + 13)\ \textsf{Cents}
\end{aligned}$$

Since the *denominators* are the same we need only compare the *numerators*:

$$3x + 11 = 2x + 13$$