Calculus Anyone?

Is there life without limits? I believe so but Newton, Leibnitz, Lagrange and Robinson must be the only ones to know for sure. In any event, we shouldn't confuse existence theorems and approximation theorems.
But do we really, in fact, need limits? Consider the usual Socratic dialogue:
— We know what a circle is, right?
— I guess.
— You are in bad faith. I am not asking you to define what a circle is but if you know what a circle is.
— (Greatly relieved) All right, I know what a circle is.
— OK, then a circle has an area. Yes?
—(Dubiously) Err, twenty-two-over-seven-times twice-the-radius.
— You mean Piarrskware?
— Yeah ...
— But that's irrelevant. What's important is that circles, just like anything, have an area. Right?
— (Dubiously) Right.
— Then, the problem is to measure this area. That is, there must be a number that is the measure of this area.
— (Hopeful) Twenty two over seven?
— That's only an approximate value.
— Oh! (Disappointed. The student can see that this is destined to go on for a long while)
Follows indeed the standard argument about inscribing and circumscribing polygons on the unit circle. Finally, we reach the climax where it is shown that there is necessarily a number, called Pie, which is the measure of the area of the circle. The instructor is swooning with delight. Then the idiot asks:
— What's the number?
After the instructor has recovered from the shock, he says, sternly:
— It is the limit of the sequence ... (Da Capo)
Note that he is very proud of the fact that he is conveying the idea of limits via sequences rather than via functions. It is so much simpler. The idiot however doesn't appreciate the subtlety and says:
— Yeah, but what's the value of the number?
This isn't working out too well. Nevertheless, if somewhat less triumphantly, the instructor says:
— Pie! You see?
And the student says:
— No.
After the instructor has been revived, he asks
— What's the problem?
— What's the value of Pie?
— Well, we don't know exactly, but for every practical purposes, you can use twenty two over seven or three point thirteen or ... (alarmed) Where are you going?
— I am dropping this course. In fact, I am dropping out of school entirely. In fact ...

I too fail to see the point of this in first year calculus. After all, as the student implies, and as engineers know, the real real numbers are the (finite) decimal numbers. Knowing that there is a number which is the exact measure of the area of the unit circle makes me feel warm all over as much as the next guy but whenever I need that number, I just write π or I use 22/7 or 3.1416 or whatever. Just as you do. So what's the point particularly when the "limit approach" usually doesn't provide an algorithm for finding limits and only verifies whether a candidate is or isn't the limit ? But the discussion with the above instructor usually goes something like:
— But why do you bother at all with the argument?
— Limits are clearly basic to the calculus.
— Really?
— How else would you define the tangent?
— Did you know that Pr. Bivins recently got another prize for his article What a tangent line is when it isn't a limit?
— What is it?
— The graph of the best affine approximation!
— If you don't say that the tangent is just the limit of the secant you just utterly lack in geometric intuition.
— What is un-geometric about the following picture?
— I don't know but limits are clearly basic to the calculus.
— Really?
— How else can you define the derivative?
— As linear coefficient of the best affine approximation, that is as slope of the tangent.
— I still think that limits are clearly basic to the calculus. (Triumphantly) How else would you know what sinx/x is when x equals 0?
— You don't anyway but, near 0, sinx/x is approximately equal to 1.
— (Sneaky) How would you know that?
— From the Taylor expansion of sinx/x.
— (A little bit cautious this time) But how would you get it without knowing the derivatives of sinx and doesn't that in turn depend on the notion of limit?
— (Quite smug). Not at all. Define sinx as the solution of the differential problem f"(x) = – f(x) with the initial conditions f'(0) = 1 and f(0) = 0 and all you have to do to approximate the solution when x is near 0 is to set f(x) = A + Bx + ...
— I still think that ...
Of course, you too think that ... but finding by way of expansion the approximate value of sinx/x when x is near 0 is absolutely correct and you can even interpret it as lim xÆ0 sinx/x. But, from the pedagogical viewpoint, the advantage is that we find 1 as approximate value instead of verifying that it is the limit. Moreover, if we set f(x) = A + Bx +Cx2 +Dx3 + ... we find 1– as approximate value which you can interpret as sided-limit.

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I wish to return briefly to the subject of infinitesimals. The infinitesimal calculus is lean and lively. So, the way you react to infinitesimals puzzles me. Why is it that you insist that we cannot teach infinitesimals because they are too difficult to present "correctly".

When we first learn to count, we do not use Peano's axioms or equipotence in Zermelo set theory. When we learn the integers or the rationals, we do not do so by way of the Grothendieck construction (of a group from a commutative cancellation semi-group). Thus insisting on "rigorous" infinitesimals in a first course is a blatant case of double standard if ever there was one. Moreover, I do not see that you are doing a more rigorous job with limits than Leibniz, L'Hôpital, Euler, Lagrange, etc did with infinitesimals? Once again, I do not see what is wrong, given for instance f(x) = x2 so that f(x0 + h) = x02 + 2x0h + h2, with saying that, when h is small, h2 is infinitesimal so that the best affine approximation of f near x0 is tx0(h) = x02 + 2x0h and the tangent, by definition, the graph of tx0. Then, we can say that the value of the derivative of f at x0 is 2x0. As a matter of fact, I remember seeing a picture like that in Hille [4].
Now we all agree that, for whatever reason, limits are "hard to see", at least for many people. So, why do we persist? I think at first it was snobism: the infinitesimal first-year calculus had become the domain of "teaching" mathematicians while research mathematicians would teach advanced calculus necessarily by way of limits since, at the time, there was no rigorous treatment of infinitesimals. But we soon felt that this garden variety calculus was somehow second rate and we decided that we too could use limits. Then it became complacency and now, I think, it is just plain intellectual laziness. I agree that Keisler [3] might be a bit much in two-year colleges but how come you haven't adopted, or at least read, Freed [2]? This is the first calculus book that I have enjoyed reading in a very very long time. I will report about it next time but don't wait.

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One of the nice things about doing this column is that my friends give me things to read, things I should have read. For instance, I was directed to two very interesting articles in the Monthly. Perhaps by coincidence, they are both "reviews", that is, they use reviewing books as pretexts for their disquisitions. Dudley [1] makes the case, among other things, against applications. Hamming [3] makes the case for what the calculus is, or, at least, was or should be.
I also read recently preliminary materials by W. Rosenthal who is writing a text to implement a humanistic approach to the calculus. What I have read is really very good. and I wish I were as daring as he is and wrote as well as he does. If he would only stay away from limits, he would be perfect or, rather, so much more ... humane. I still hope to convince him one day. I shall discuss his work in more details in a future column but interested readers should contact him at Ursinus College in Collegeville, PA, 19426.
I am reading Lagrange and I am fascinated! I will tell you about it in a future column.

Alain Schremmer.

P. S. The other day, I was talking with the Marketing Representative of one of the big publishers. He was showing me a blurb describing their "big risk" project, that is their Lean and Lively Calculus for the Next Century. Of course the blurb took great pains to describe the text to come as a mainstream calculus and, indeed, the table of contents was completely undistinguishable from that of any textbook you are likely to be using. By the way, the blurb insisted that lean didn't mean less but more. As the French are fond of saying, the more things change and the more they stay the same.

So, Dear Colleagues, there is nothing to fear: the Calculus for the New Century is already here and you have already seen it. You will thus be able to continue complaining about the students.

P. P. S. I forgot to mention the icing on the cake which is that the book will be sold with an HP 28S. So, the students won't have to read this book any more than any other book but, if they can punch on the HP the questions on the final, they will be able to pass the course with at least a B.

P3. S. What exactly will have been accomplished thereby?

1. U. Dudley. Review of "Calculus with Analytic Geometry" by George F. Simmons. The American Mathematical Monthly. 95(9) (1988) 888-892.
2. W. Freed. "Infinitesimal Calculus." 1986 Concordia College. Edmonton.
3. H. J. Keisler. "Elementary Calculus." 1976 Prindle, Weber & Schmidt. Boston.
4. R. W. Hamming. Review of 'Toward a Lean and Lively Calculus'. The American Mathematical Monthly. 95(5) (1988) 466-471.
5. E. Hille. "Analysis." Blaisdell Books in Pure and Applied Mathematics. Springer ed. 1964 Blaisdell. New York.