Calculus Anyone?

In case you have not been as attentive as you should have been to the ongoing saga of the calculus crisis, here is an update.

Act III. - Where it begins to look like there never was a crisis.
SCENE 1.

This past January, at the Joint Meetings in Louisville, I went to a "panel discussion" of successful textbook authors. All the millionaires were there. What did they say? What could they possibly say other than, essentially, that all is for the best in the best of all possible worlds. Anton even said something to the effect that the calculus crisis was bullshit. I don't remember the exact words but bullshit was one of them. He reminded us that, out there, there were so many bad teachers that our only hope was for the students to have well crafted textbooks, written by dedicated, experienced, expert authors.
I assume that when he was talking about all these bad teachers he was not talking about the audience but, if you weren't there, I would be worried. On the other hand, with his book, you are safe, or so I thought he said. I am picking on Anton but, really, he should be the last one on whom I should pick since he obviously said aloud what all the others were trying very hard not even to whisper.
Maybe I am just jealous as the one thing that is as sure as taxes is that I will never write a best seller.

SCENE 2.

In the March issue of UME Trends, T. Tucker, Vice President of the MAA and chair of CRAFTY, the CUPM Subcommittee in Calculus Reform and the First Two Years, had a "report from the field" in which he sounded a bit gloomy.
He found that there was a great reliance on computers and the like but, it seemed to me, he didn't appear any more enthusiastic about this than I am.
Another one of his worries is that the projects might, after all, turn out to be "castles in the sand" to use Douglas's expression at the Calculus for a New Century conference. He thinks that there might be too much reliance on NSF if only because of what might happen should support be withdrawn. Here of course I disagree. First, because I did get a grant and, second, because I didn't get a second one. It is not a question of what will happen if support is withdrawn. We all know that. It is a question that this country cannot afford to even consider one minute no to continue to support this kind of work and for a very long haul. Does anybody really, seriously think that five millions a year for a few years will be enough to stem a very well documented tide of tsunami proportions. Castle in the sand indeed! The problem is how to reinforce the castles until the tide ebbs.
An even more serious problem he discusses is that the castles look much the same as the usual housing.

II

I would like to argue here that what we take to be the nature of the calculus is at the heart of the problem or, if you prefer, crisis. But let me first illustrate with an example.

Back in April, I was invited at a state two-year mathematics meeting to talk about limits in connection with Lagrange's approach. As is my wont, I prepared a talk knowing full well I would do something entirely different. This time it was because, during the drive there, it occured to me that I did not know why we were teaching limits in calculus. So I asked the audience and I kept note of the answers. Here they are:

1. To resolve undetermined forms. 2. Because they are in the text. 3. To define derivatives (twice). 4. To define e. 5. To look at limits of sequences. 6. Because it is a mathematically satisfying concept. 7. To be able to define integrals. I also have a note saying 8. e - d versus sequence and series.

I don't know what I expected but this launched me very nicely because I was able to show that we didn't need limits for 1., 3., 4., 7., that, in fact, we can do these things as well if not better than usual, and without any pain, using Lagrange's approach. Now here is the cruncher. Can you believe for one minute that I convinced anyone? I sure didn't believe it for one second. Eppur si muove. But it is 2. that is the real killer.

Just as in this example, I think that if we were to look at the problems which we give on exams, we would find that the real reason, the only reason, that we give them is that they are in the texts (and on standardized exams although I don't know many people who dare give standardized exams.)

We are all aware that the calculus has devolved into a cookbook course. And as someone said, that's insulting to cooks. I have argued before that one reason for this is our reliance on limits. But I believe that even this is a bit of a red herring.

The calculus was invented by people who were curious about certain problems. The calculus as it is currently delivered to the students is absolutely not meant for people with curiosity. Nor is it intended or capable to foster any curiosity. It is tailor made to prevent the students from doing anything but memorize.

Let me use an analogy. A child needs to investigate the world and that entails some risk. To achieve this, you must design some kind of "garden", surrounded by a wall, in which you can let the child loose to try and err. The child will of course incur accidents but, if you have designed the garden well, these should not be dangerous and the child needs these accidents. Another aspect is that there must be a structure to the garden, neither too obvious nor unattainable, if the child is to make sense of what she observes. Finally, as the child discovers the garden and invents ways to deal with its "dangers", the enclosure must be made to encompass progressively new grounds in order for the child to always have some new wilderness to explore .

But when we blandly define a function as ... (choose your own definition), or even when we restrict ourselves to continuous functions, we are dishonest because functions in general and even continuous ones do not constitute an explorable universe: just about any conjecture students could make about these functions will be false. After all, it took respected mathematicians long enough to suspect how enormous the universe of functions is. But, worse yet, the students will probably not have the slightest chance to understand the counterexample you might provide them with. Of course you can always tell them to wait until they get in a Real Analysis course. But then why do we insist that we can never be general enough.

Speaking of which, I used to suggest that, for instance, we might limit ourselves to analytic functions and treat them as Howard Levi did in his Polynomials, Power Series, and Calculus. Invariably, there would be a Professor Knowitall in the audience to ask, dripping with heavy sarcasm, whether Professor Schremmer had ever encountered the function e–1/x2.

Eventually, we decided on another way to limit the universe. Start with affine functions (those you probably call linear even though they are not). They do not require much algebra to deal with and they warrant much more attention than most calculus instructors are ready to give them. After a while though, affine functions do get to be a bit boring as they go on and on while the way they change remains constant. But the students should get bored. This should be their only motivation to look for other, more interesting functions.

On the other hand, we can enlarge the universe to include piecewise affine functions which we can even restrict to compact support. We now have a wonderful diversity of behaviours to explore.

Then quadratic functions. They are just complicated enough that it pays to use affine functions to study them locally and approximately. They are still simple enough that we can check our results globally and exactly. And they are the simplest functions with which to introduce the only notions not as yet introduced, namely concavity by way of osculating parabolas.

After that, we can look at incomplete cubics (as used to be the custom in days long gone) or we can go immediately go to Rational Functions and then on to functions that can be locally approximated by polynomials.

In such universes, it is safe for the students to make conjectures because they stand a chance to prove them if they are right and to come up with a counterexample if they aren't. Of course, this is boring stuff. Or is it? If you sat looking at it with a critical eye, you would find a remarkable number of things that are not so obvious. I once had a colleague who was going on and on about how best to introduce the point-slope formula. When I suggested that he introduce it as an initial value problem, he looked completely bewildered. Now, mind you, this is an instructor who also teaches differential equations.

Another aspect of the problem is linguistic in nature. We have all encountered students perfectly capable of finding, say, the maximum xM of a function but, when you ask them to compare f(xM) with, say, f(xM+1), they either look at you uncomprehendingly or, at best, launch into heavy computations. It is extremely difficult to convince students to take definitions seriously. Could it be because we take them for granted?

In an exploratory approach, the language must be developed as needed in order to allow precise discussions, systematizations and generalizations. Ideally of course, the students should evolve the necessary language themselves. At the very least, we should give them a language that is suited to the purpose. This is mostly not the case for the language in conventional texts. For instance, I would like the language to express the easily observable fact that an extreme plays for increasing/decreasing the same role as a zero (of odd order) for positive/negative. Or that zeros are just as critical for the sign as critical points are for optimization. How many students do you suppose have ever been given the opportunity to observe situations where they might discover such facts and how many do you think were given a terminology that would enable them to describe what they saw

and perhaps make some conjectures with any kind of precision and efficiency, not to say elegance. It might just be that if they had, they might have been tempted to prove it.
You are saying: the man is dreaming. I can't even get most of my students to do the little I ask of them. How could they possibly do what is, essentially, what research mathematicians do?
The answer, again, is that you are asking them to work from scratch in much too big a universe. To fiddle around with affine functions is not as stressing as doing from memory those favorite exam problems of ours. You know, for example, find from first definition the derivative of f(x) = x2 + 1. Now, to students used to dealing with affine functions and then to approximating other functions with affine functions, finding from first definition the derivative of f(x) = √x or f(x) = (x2+1)/(x3–1) or finding and proving the quotient rule or the chain rule is just run of the mill stuff.
You don't believe it. You are wrong. But then I don't cover related rate problems
Ha Ha! you will say, but to me, the differential calculus is first and foremost the local study of functions which we can then try to extend to obtain global results. Say we want to go from a to b. We start in a neighborhood of a in which we pick a point x1 on the way to b. Then we take a neighborhood of x1 ... Eventually, we will get a neighborhood that contains b. But, a priori, this may take an infinite number of steps. And this is where compactness comes in!

[Here, presumably, there is an explanation but, courtesy of Word, it is hopelessly garbled. However, I should be able to make that case without too much trouble. Later.]

Another divagation, hopefully irritating, by A. Schremmer