Calculus anyone?

A sly column by A. Schremmer.

One of the difficulties in trying to disseminate Lagrange's approach to the differential calculus is that few mathematicians make the difference between **limited expansions** (also known as **asymptotic expansions**) and **series expansions**.

Consider first what we mean when we write $\frac{1}{3}$ = 0.3333 ... . We may mean either that the difference between $\frac{1}{3}$ and 0.3333 is negligible for our purpose or that $\frac{1}{3}$ is the limit of the sequence 0.3, 0.33, 0.333, ... where the sequence was to be understood implicitly. The two are vastly different. The first is a relation between two numbers, $\frac{1}{3}$ and 0.333, while the second is a relation between a number, $\frac{1}{3}$, and a sequence of numbers, $S_n = \sum_{i=0}^{i=n} 3\cdot 10^{-i}$, namely that $\frac{1}{3}$ is the limit of S_n as n approaches infinity.

Similarly, when we write $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \dots$, we may either mean that, when x is near 0, the difference between $\cos x$ and the polynomial function $P_2(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$ is small enough to be negligible for our purpose, in which case $P_2(x)$ is what we call a **limited expansion** of $\cos x$, or we may mean that $\cos x$ is the limit, when n approaches infinity, of the sequence of polynomial functions,

$$P_0(x) = 1,\quad P_1(x) = 1 - \frac{x^2}{2},\quad P_2(x) = 1 - \frac{x^2}{2} + -\frac{x^4}{4!},$$

$$P_n(x) = \sum_{i=0}^{i=n} (-1)^i \frac{x^{2i}}{2i!}$$

where the sequence was to be understood implicitly, in which case we are dealing with a **series expansion**.

In order to avoid any ambiguity, we could write $\frac{1}{3}$ = 0.3333 + $R(10^{-4})$ which we would read as saying that $\frac{1}{3}$ is equal to 0.3333 plus a "remainder" (whose order of magnitude is less than 10^{-4}). Similarly, we write $\cos x = 1 - \frac{x^2}{2} + -\frac{x^4}{4!} + R_2(h)$ which we read as saying that $\cos x$ is equal to $1 - \frac{x^2}{2} + \frac{x^4}{4!}$ plus a "remainder" (which is $o[x^4]$). The confusion mentioned earlier arises because, in current calculus textbooks, this limited expansion of $\cos x$ invariably appears as, and only as, Taylor formula with remainder in the middle of the section on Taylor ... series. Limited expansions are used extensively in ODE, Fluid Mechanics, Differential Topology, Number Theory, etc.

Also, perhaps another reason why Lagrange's approach is so little known among calculus educators is that Lagrange used the term *analytic function* by which he meant functions capable of being approximated by limited expansions[1]. Another side to this involves what is known as Peano derivatives and I will pursue it in some future column.

I remarked in an earlier column that: **i.** There is no algorithm to find limits and that: **ii.** Even *sided* limits are not a good elementary tool for analyzing the local behavior of a function when, after all, this is all the differential calculus is about.

On the other hand, an engineer once told me: "Between you and me, the *real* real numbers are the (finite) decimal numbers". Similarly, to deal with a function in Lagrange's approach, we simply substitute a limited expansion for it.

The advantage of using limited expansions in place of limits is that, at least for the functions encountered in elementary calculus, there are very simple algorithms for obtaining the expansions and that, in fact, limited expansions of degree 0, 1 or 2, depending on what we want to do with the function, are usually all we need in order to obtain all we ever wanted to know about a function, including sided limits! For more about Lagrange's approach, see [**5**].

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A while ago, I had promised to laud in detail W. Freed's calculus book [**2**]. I will do this in a later column but, for now, I want to indicate very briefly how hyperreal numbers can be looked at as being determined by *extended* decimal expansions. I learned that from him and you should buy his book.

The Leibnizian idea of an infinitesimal number was that of a number smaller than any real number and yet different from zero. As with the introduction of any new type of number, this was declared meaningless.

Now, to find out "*what a tangent line is when it is not the limit of a secant*" [**1**], we look at the graph with a magnifier. Non-standard analysis consists in building the idea of magnification into the axes, that is, into the number system. All we do is to make an infinite magnification of the real line. A line being continuous, what we see after magnification, finite or infinite, is still a line. But any point you see has to correspond to an infinitesimal number since, having been multiplied by an infinite number, any finite point (except 0) has now gone to infinity, that is out of sight. Thus 0 is both a real number and an infinitesimal but, *under magnification*, it is surrounded only by infinitesimals. For the hyperreals to be a group under addition, in-

[1] This is A. Robinson's reading of Lagrange but, I must say, it is not universally agreed upon.

finitesimals must similarly surround any other real number.

Since real numbers can be written as decimals, so should infinitesimals. Since 0 can be written as 0.000..., (see above) and since a small real number is going to be of the form 0.0...0*d*... where *d* is a non zero digit written after a *finite* number of zeros, to create an infinitesimal we just write 0.0... *d*... where *d* is a non zero digit written after an *infinite* number of zeros! Now all you have to do is to check that this is compatible with addition (how do you add a real number, say 0.333... , to an infinitesimal, say 0.000... 1... ?) and do the right thing to make it work under multiplication. (Hint: while all infinitesimals are small, some are smaller than others. For instance, just as the square of a small real number is smaller by an order of magnitude than the small real number, the square of an infinitesimal is an order of magnitude smaller than the infinitesimal.) The **hyperreals** include the reals, the infinitesimals and the infinities which are reciprocals of infinitesimals. Note that a rectangle whose width is infinitesimal and whose length is correspondingly infinite has area 1.To **round off** a hyperreal means to get the real number around which the hyperreal was sitting. We do this by dropping the decimals after ...

Unless you (and your students) are adept at differential forms, you probably have trouble distinguishing between Δx and dx. *Here*, Δx is a (small) real number while dx is an infinitesimal, that is dx rounds off to 0!

Given a function f, what does $f(x)$ mean when x is hyperreal? Essentially, we have to **extend** the real-valued function f of real variable to a corresponding hyperreal-valued function of hyperreal variable but the distinction is really only a formal one.

To conclude this impossibly brief summary, a function is defined to be **continuous** iff, when you round off $f(x+dx)$ (which is a hyper real since dx is infinitesimal), you obtain $f(x)$. For example, to show that $f(x) = x^2$ is continuous at any x, we compute:

$$\begin{aligned} f(x+dx) &= (x+dx)^2 = x^2 + 2x dx + dx^2 \approx x^2 + 0 + 0 \\ &= x^2 = f(x) . \end{aligned}$$

The **derivative** is defined as the rounded off value of $\frac{f(x+dx) - f(x)}{dx}$ provided this rounds off to the same real for every infinitesimal dx.

So, once again, why do we inflict limits on our students? After all, all we need to use are the *real* real numbers, namely the hyperreal numbers!

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I said before that I was trying to refrain from arguing with other authors in this Newsletter. Nevertheless, as I read *The Concept of Function* in Research Clips of the Winter issue of this Newsletter, my argumentative nature took over but, instead of writing to the Editor, I decided to use my own space.

The second paragraph bemoans the fact that "*given a particular value for one variable* [...], *students view functions as the rule that allows them to determine the value of another variable*". I do not see what is wrong with that. [I don't either.-ed] In fact, I will discuss this aspect of a function at some future time. If you don't want to wait, see [**3**] and [**4**].

The third paragraph deplores that "*students are strongly 'attracted to linearity'*". Again, what is wrong with that? Given most application problems, I mean real ones, don't we necessarily begin by linearizing it? So instead of crying, why not, as the committee responsible for this gem failed to do, capitalize on this non fatal attraction and use it to make the point made in [**1**] that the tangent is the graph of the affine approximation and that the reason we use it instead of the function itself is precisely that it is "linear"? In any case, why pick on the students? Whose fault is it?

The fourth paragraph is the one that initially attracted my attention (i.e., incensed me). It is scandalized that "*students agreed that two functions with identical formulas but different specified domains were the same function.*" I can see Prof. Meany telling his/her student: "Gotcha". What I want to know is what does Prof. Meany do for his/her students to come indeed to the inescapable conclusion that the difference has merit? (Answer: S/he gives the students the definition and, by Bourbaki, this better be enough.) Besides, when we say "restrict the function f to the interval I and graph it", doesn't the pronoun 'it' refer to THE function f? And if you want to invert $\sin x$, then instead of talking about domain and range, just say that, if $\sin x$ is not *globally* invertible (which few functions are anyway), it is *locally* invertible almost everywhere. What's the Inverse Function Theorem for otherwise?

I agree with most of the fifth paragraph, namely that the students "*have difficulty interpreting the function as a whole scheme.*" But I leave it to you, as an exercise, to find out how many howlers are implied in the concluding sentence: "*This may be related to the difficulty that calculus students have in looking at secant lines 'approaching' the tangent line to a curve at a point, or determining the behavior of a function over an interval*".

References

1. I. C. Bivins. What a Tangent Line is When it isn't a Limit. *The College Mathematics Journal*. **17**(2) (1986) 133-143.

2. W. Freed. "Infinitesimal Calculus." 1986 Concordia College. Edmonton.

3. M. E. Munroe. "Bringing calculus up-to-date." Selected Papers on Calculus. Apostol, Chrestenson, Ogilvy, Richmond and Schoonmaker ed. 1969 The Mathematical Association of America. Washington, D.C.

4. M. E. Munroe. "Calculus." 1970 W.B. Saunders. Philadelphia.

5. F. Schremmer and A. Schremmer. An introduction to Lagrangian differential calculus. *AMATYC Review*. **11**(2) (1990) 16-26.