Calculus anyone?

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This column is on the list of endangered species. It is not that I have lost my interest in calculus, but rather that nobody in academia seems really interested in being much more than a delivery system for commercial textbooks. The irony is that the advent of friendly computer algebra systems may make us obsolete before we can retire, even as delivery systems.

I have often said that the main reason that the students have a hard time with calculus is not that calculus is too hard but that, thanks to Thomas, Anton et al, the students are not in a position to look at calculus *mathematically*. By insisting on limits as the underlying concept, these texts effectively bar the students from operating like mathematicians. This is particularly unfortunate as limits need have only a very indirect role in calculus.

In a nutshell, Lagrange's calculus pertains to functions that can be locally approximated by polynomial functions with an error of a given magnitude. Please note immediately that we are not talking about *series* expansion but about *polynomial* expansions. In other words, we are dealing with functions such that, *near* any given point x_0, we can **localize** them, that is we can find a polynomial in h such that

$$f(x_0+h) = A_0 + A_1h + A_2h^2 + \ldots + A_nh^n + (\ldots)$$

where all we know about the remainder $(\ldots)$ is that it is small. It is worth noting that all the functions that most students are likely to ever encounter are of this type.

It is only when we want to investigate the order of magnitude of $(\ldots)$ that limits are needed. But as I would like to show here, quite a lot of good mathematical work can be done without worrying about the nature of $(\ldots)$.

Consider, for instance, the way the exponential function is usually studied. First, introduce integration. Then integrate $f(x) = \frac{1}{x}$ and then take the inverse function. If nothing else, this does not make much for intuition. Here is how it appears in our treatment of calculus.

Starting at a level where students just came out of a basic algebra course in which they learned polynomial algebra, we show that every polynomial can be localized; just replace x by x_0+h and expand. Then we define the derivatives of f as the functions that deliver at x_0 the coefficients A_i modulo the factorial required for $f^{(n+1)}(x)$ to be the first derivative of $f^{(n)}(x)$. In other words, *by definition*,

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f^{(3)}(x_0)}{3!}h^3 + \ldots + \frac{f^{(n)}(x_0)}{n!}h^n + (\ldots)$$

After we have proven the usual rules for differentiation by just looking for the coefficient of the h-term in $[f*g](x_0+h) = f(x_0+h)*g(x_0+h)$, we can introduce the notion of **differential problem**. For instance, we can solve

$$\begin{cases} \text{Differential equation:} & f'(x) = \frac{1}{2}x + 1 \\ \text{Initial condition:} & f(-7) = +6 \end{cases}$$

by Euler's method. To compute the value *near* –7 we need first to compute the derivative *at* –7:

$$f'(-7) = \frac{1}{2}(-7) + 1 = -\frac{5}{2}$$

$$\text{Then, } f(-7+h) = f(-7) + f'(-7)h + \ldots = +6 - \frac{5}{2}h + \ldots$$

We take $h = 2$ for the sake of simplicity even though it is not small. We get:

$$f(-7+2) = +6 - \frac{5}{2}\bullet 2$$
$$f(-5) = +6 - 5 = +1$$

Now we start again from the point –5:

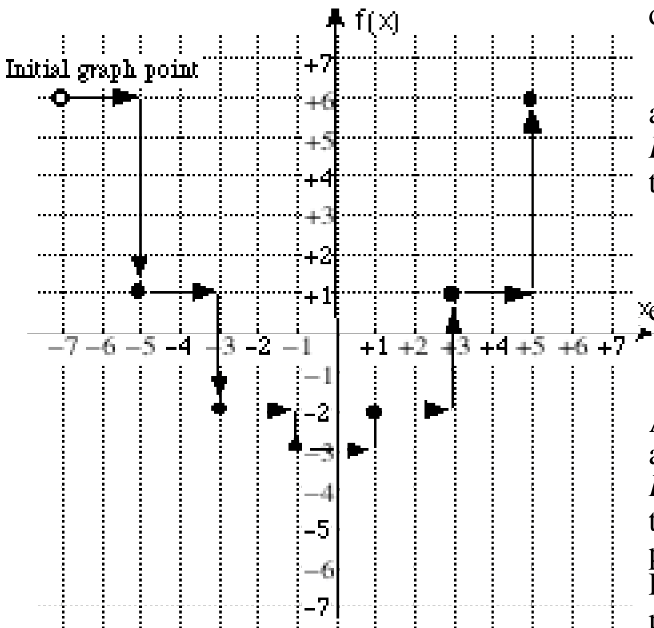
$$f'(-5) = \frac{1}{2}(-5) + 1 = -\frac{3}{2}$$

$$\text{and } f(-5+h) = f(-5) + f'(-5)h + \ldots = +1 - \frac{3}{2}h + \ldots$$

We take again $h = 2$:

$$f(-5+3) = +1 - \frac{3}{2}\bullet 2$$
$$f(-2) = +1 - 3 = -2$$

Altogether, we obtain the following:



Algebraically, we get:

$$f(x) = \frac{1}{2} \bullet \frac{x^2}{2} + x + C$$

To get the constant C, we evaluate f at -7:

$$\begin{aligned} f(-7) &= \frac{1}{2} \bullet \frac{(-7)^2}{2} + (-7) + C \\ &= \frac{49}{4} - 7 \bullet \frac{4}{4} + C = \frac{21}{4} + C \end{aligned}$$

From the initial condition we get $\frac{21}{4} + C = +6$ and so $C = \frac{3}{4}$. The algebraic solution is thus:

$$f(x) = \frac{1}{4} x^2 + x + \frac{3}{4}$$

the graph of which it is illuminating to compare with the above plot.

We do this and much much more including rational functions, during the first semester.

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At the beginning of the second semester, we can then begin with the following problem:

$$\begin{cases} f'(x) = f(x) \\ f(0) = 1 \end{cases}$$

Degree considerations lead us very quickly to decide that no polynomial or rational function can be a solution. On the other hand, since we are given the value of f *at* 0, it is natural to try to find an approximate polynomial solution *near* 0, for instance of degree 4. We set:

$$EXP_{(4)}(h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + A_4 h^4$$

where, by the initial condition, we must have $A_0 = 1$. To obtain the other coefficients, we take the derivative:

$$EXP'_{(4)}(h) = A_1 + 2A_2 h + 3A_3 h^2 + 4A_4 h^3$$

and from the differential equation $EXP'_{(4)}(h) = EXP_{(4)}(h) + (\ldots)$ we get from the identification of the coefficients the approximate solution:

$$EXP_{(4)}(h) = 1 + h + \frac{h^2}{2} + \frac{h^3}{3!} + \frac{h^4}{4!}$$

easily shown to satisfy the *addition formula*:

$$EXP_{(4)}(h_1 + h_2) = EXP_{(4)}(h_1) \bullet EXP_{(4)}(h_2) + (\ldots)$$

At this point a leap of faith is necessary: We must accept that there does exist an exact global solution $EXP(x)$, that $EXP_{(n)}(h)$ approximates it near 0 and that the addition formula remains true when h is replaced by x. But this can be made quite plausible: Euler's method applied to the above differential problem with $h = 1$ gives $EXP(n) = 2^n$ and, if we set $e = EXP_{(4)}(1)$, the addition formula gives $EXP_{(4)}(n) = e^n$. In particular, we can postulate from these results the behaviour of EXP at $\pm\infty$. To show how local information can be extended to global information is always a very difficult thing, but here is a good opportunity for introducing the notion of compactness – – really.

A more advanced treatment, but still quite easy, can give all the properties of the exact solution directly from the differential problem.

EXERCISES. Using local approximations,

1. Find the following *sided* limits: $\lim_{x \to 0^+} \frac{x^2 + x + 1}{e^x}$, $\lim_{x \to +\infty} \frac{(x+1)^2}{e^x}$, $\lim_{x \to -\infty} \frac{e^x}{x^2 - 1}$.

2. Graph $f(x) = \frac{e^x}{x^2 - 4}$.

The COS and SIN functions are introduced and studied in exactly the same manner from:

$$\begin{cases} f''(x) = -f(x) \\ f(0) = 1 \\ f'(0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} f''(x) = -f(x) \\ f(0) = 0 \\ f'(0) = 1 \end{cases}$$

It is interesting, at least to me, that the main reason that all this appeals to the students is that it is *coherent*.