Let's face it.

A model column by Alain Schremmer.
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Sometimes in September, we were advised that the Federal Register had published new regulations which specified that Pell grants could not be awarded to students taking courses at the *elementary* level. This of course immediately put in question the viability of the basic remedial courses that is, in our case, a course in, mostly elementary, arithmetic.

What followed was interesting, if entirely predictable. The administration ordered the department to promptly deal with the new situation since the regulations are to take effect in January. There seemed to be a bare hint that we should simply certify our arithmetic course as secondary. And, indeed, a few in the department invoked the fact that such a course was being taught by the Philadelphia school system at the secondary level … Reason however prevailed and the department voted to revive a logic course which had been offered in the past with a fair measure of success and to offer *For All practical Purposes*, the COMAP course cum videos. The administration was not interested: it claimed that a "*considerable disruption*" would be caused to the students and that we should just study the situation to death. This was already a bit surprising but the conclusion was startling: "*Since* [the arithmetic course] *has carried credit toward graduation* [for over 10 years], *it qualifies as collegiate* [sic] *level for the purposes of financial aid eligibility.*"

In the September 1991 issue of FOCUS, there was an editorial in response to an editorial attacking the study of algebra which had appeared in the *Washington Post*. There was also a reprint of "*Algebra Is Not Just Math, It's the Language of Science*", a response by Wayne Roberts of Macalester College. While I would argue that the real language of science is the differential calculus and that its study does not require the customary pre-calculus paraphernalia, it is certainly true that mathematics is often identified with arithmetic by the public at large, and particularly by administrators coming, as they most do, from the Humanities. A recent survey of the school-wide faculty's perception of our department's performance indeed showed that everybody would be happy if students could just compute without fail percentages and areas and volumes of simple figures (they called this geometry). Needless to say, the students couldn't and the general faculty didn't think highly of us.

But this raises the far more interesting question: since this type of students is usually deficient in more than just arithmetic, why the exclusive insistence on why Johnny can't add?

For my part, I would be a lot less worried about Johnny if she were able to do other things in a systematic, thoughtful and critical manner. In fact, I would argue, and I shall do so at some other time, that given the way we teach arithmetic, most students would be better off without it. But I would like to devote the rest of this column to the course which we wanted to replace arithmetic with, and which, in fact, had itself been replaced ten years ago, by administrative fiat, with the current arithmetic course.

The course dealt with elementary mathematical logic. Those courses in logic which were usually offered at the time mostly consisted of basic exercises in truth table manipulations and, as such, were about as useful as arithmetic. The course we designed was model theoretic based. But rather than talk *about* it, let me present it, however briefly.

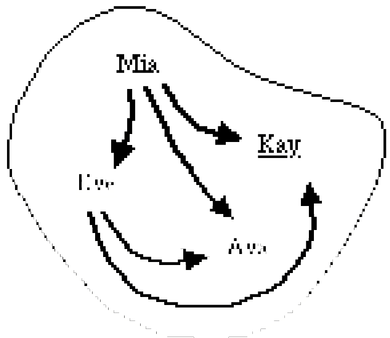
We start with a small given **situation**, meant to act as a substitute for the world, say with attribute blocks or with the students themselves. For instance, we may start with

	Tim	**Mia**	**Kay**	**Jim**	**Abe**	**Eve**	**Ava**
Weight	152	121	113	115	125	105	121
Years	21	19	23	23	17	28	16
Sex	M	F	F	M	M	F	F
Height	5'6"	6'1"	5'6"	5'9"	5'4"	5'8"	5'6"

We then should decide on what aspect we want to concentrate. For instance, we might want to deal with the following **structure**:

OBJECTS: The women
RELATIONS: __ is over 21
__ is taller than __

which can be represented by the following diagram:



To talk about this structure, we progressively introduce a first order predicate **language**:

<u>Semantic Vocabulary</u>

NAMES: *a*, *b*, *c*, *d*
VERBS: __ *P*
__ *R* __

and as soon as we fix the

<u>Denotation</u>:

a	denotes	Mia
b	denotes	Kay
c	denotes	Eve
d	denotes	Ava
__*P*	denotes	__ is over 21
__*R*__	denotes	__ is taller than __

we can start talking and even decide whether what we are saying is **T** or **F**:

For instance, *aRb* is **T** but *dP* is **F**

However, to increase our expressive power, we are almost immediately led to introduce a

<u>Logical Vocabulary</u>

QUANTIFIERS: ∀, ∃
PLACE HOLDERS: *x*, *y*, *z*.
CONNECTORS: ∧, ∨, ∅, ~

which allows us to make statements such as

$$\forall x\ [aRx] \text{ which is } \mathbf{F},$$
$$\exists x\ \exists y\ [xRy] \text{ which is } \mathbf{T},$$

and more interesting statements such as

$$\forall x\ [xP \varnothing aRx] \text{ which is } \mathbf{T},$$
$$\exists x\ \forall y\ [yP \varnothing xRy] \text{ which is } \mathbf{T}.$$

The truth or falsehood of a sentence depends on the **interpretation**, that is, on the structure and the denotation being used to give **meaning** to the sentence. An interpretation under which a sentence is true is a **model** of that sentence.

The next step then consists in looking at the truth of a sentence under an **ontology**, that is, a (small) set Ω of interpretations: In particular, if a sentence is true under the class[1] of *all* interpretations, we shall say that it is **logically true**. More useful is the notion of **entailment**, a (meta) relation among sentences: a sentence S_1 **entails** a sentence S_2 (with respect to Ω) if and only if every model of S_1 is also a model of S_2. In other words, to say that S_1 entails S_2 is to say that S_2 is a **consequence** of S_1 (with respect to Ω).The *Reduction Lemma*, S_1 entails S_2 with respect to Ω if and only if the sentence $S_1 \varnothing S_2$ is true under Ω, clarifies the confusion between entailment and ∅ : The relationship is the same as that between > and – namely: $x > y$ iff $x–y$ is positive. As with truth, we extend the notion of entailment to that of **logical entailment**.

The last step consists in introducing syntactic rules to create a syntactic (meta) relation among sentences: a sentence S_1 **yields** a sentence S_2 if and only if there exists a finite number of applications of the syntactic rules that get us from S_1 to S_2. In other words, to say that S_1 yields S_2 is to say that we can **prove** S_2 from S_1.

The intention is for the syntactic relation "to yield" to reflect the semantic relation "to entail". Gödel's Completeness Theorem asserts that a certain set of rules does so *satisfactorily*, that is that S_1 yields S_2 if and only if S_1 entails S_2.

What is the point for the students? As a reviewer once said of our text: "*All this in one semester? In a two-year college? Come, now!*" I shall comment in a later column.

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[1] I will be damned whether I say "class"or "set".