INTEGRATED PRECALCULUS – DIFFERENTIAL CALCULUS
A LAGRANGIAN APPROACH

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Supported in part by NSF Grant USE-8814000

0. Introduction.

There is surely no need to make here the case for the importance of the Differential Calculus, the mathematics of change, for the technological future of this country. On the other hand, the widespread dissatisfaction with calculus instruction does not seem to have resulted in much change. In fact, there is little agreement about the cause of the "crisis".

We do not believe that the calculus "crisis" is the result of the *methods* and *techniques* used in teaching the course. Nor do we believe that it is the result of the students' lack of motivation. We believe that the 50% nationwide failure rate in first year calculus is mainly due to it having been *made* inherently difficult. To see how the calculus was presented until about the thirties, see for instance J. W. Young & F. M. Morgan (1922). But starting then, the mathematical community insisted on a conception of the calculus based on limit considerations beyond the comprehension of most college students, particularly those in two-year colleges. This approach leaves no middle ground between a "rigorous" treatment and and "intuitive" treatment. It is because of this lack of middle ground that the calculus has become "*a watered down, cookbook course in which all (the students) learn are recipes, without even being taught what it is that they are cooking*" (R. G. Douglas, 1986, p. iv). To avoid this dilemma, the trend has been to focus instead on *application* problems but this creates other difficulties. For example, it is quite natural for an economics instructor who has just defined the cost function to want to use the notion of derivative to define marginal cost. But it seems much less natural for a mathematics instructor to use the marginal cost to motivate the definition of the derivative. Presumably, if students are still at a point where they don't know what a derivative is, then neither can they be expected to know what a marginal cost is. Moreover, in our experience, instructors in client disciplines would rather we left their discipline to them and, in any case, this seems hardly the way to give the students a real appreciation of the calculus *per se*.

The calculus is currently in much the same situation as the internal combustion engine. It has been perfected over the past fifty years to the point where it performs its present function rather well but cannot accommodate the further demands of the "New Century". On the other hand, engines such as the Sterling, based on thermodynamic cycles quite different from that of the internal combustion engine, have not yet benefited from the current technology and thus have not yet been refined to the point where they can effectively compete with the internal combustion engine. However, such engines are being developed and *they* will meet the new demands. In like manner, there is a conception of the calculus, dating back to Lagrange (1797) which, though without, yet, the polish of the conventional approach, offers an alternative particularly well suited to those students who are not, at least a priori, favorably inclined towards mathematics.

In what follows, we present what we think is an alternative to the conventional three semesters Precalculus I - *Algebraic Functions*, Precalculus II - *Transcendental Functions*, Calculus I - *Differential Calculus*. It is an embodiment of Lagrange's approach for two four-hour-courses, **Differential Calculus I and II**. It was designed for students who, in the words of the NSF Request for Proposals for the Development of a Calculus Curriculum, almost all"*choose not to continue their study of mathematics through calculus, thereby closing career options in mathematics, engineering and the sciences*." Many are women, minorities and "returning" adults[1].

Lagrange's approach.

To illustrate the dilemma mentioned in the introduction, consider how we define continuity in the conventional approach:

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[1] In the spring of 1988, out of a Full Time Equivalent total of 6770 at Community College of Philadelphia, 48.3 % were Black, 40.3 % were white, 5.2 % were Hispanic, and 6.2 % were Oriental. Also, 39.2 % were Male and 60.8 % were Female.

$$f \text{ is } \mathbf{continuous} \text{ at } x_0 \quad \text{iff} \quad \forall\varepsilon\, \exists\delta\, [\, 0 < |x - x_0| < \delta \;\Rightarrow\; |f(x) - f(x_0)| < \varepsilon]$$

Even allowing for the use of ε's and δ's, one problem with *basing* the calculus on limits is that there is no procedure for *finding* them using the definition: one can only *check* whether or not a given candidate is the limit. In other words, there is only a *relation* between x_0 and the limit, and it is no more realistic to ask most students to come up with a candidate when the function is *not* continuous than with the requisite Skolem function $\delta = \delta(\varepsilon)$ when it is. Another problem is that it is difficult to prevent the students from identifying $\lim_{x \to x_0} f(x)$ with $f(x_0)$.

If, on the other hand, we decide, as in the usual "intuitive" presentation, to avoid ε's and δ's altogether, the problem then is that we are left without any definition. Continuity and differentiability basically become primitive terms whose understanding depends on whatever everyday connotations, if any, the words have for the students. Hence the gimmicks, electronic or otherwise, to buttress or create this "intuition". Most important, though, is that, even when *shown*, or even *experienced*, these concepts cannot be *used* and, eventually, must remain meaningless.

In the January-February 1988 issue of FOCUS, mention is made of a 1987 MAA Award for Expository Excellence, the George Polya Award for articles in THE COLLEGE MATHEMATICS JOURNAL, to Irl. C. Bivens for "What a Tangent Line is When it isn't a Limit" in the March 1986 issue, pages 133-143. The committee's citation is quoted in part as: "*By defining the tangent line as the best linear approximation to the graph of a function near a point, [Bivens] has narrowed the gap, always treacherous to students, between an intuitive idea and a rigorous definition. The subject of this article is* fundamental to the first two years of college mathematics and should simplify things for students...." (Emphasis added).

Lagrange's approach is just the extension of this idea to best approximations of degree n. In other words, what we are really dealing with are jets of differentiable functions: we systematically approximate the value $f(x_0 + h)$ of a function *near* the given point x_0 by a polynomial function whose degree depends on the problem at hand. Thus, Taylor polynomials are the basis of Lagrange's approach rather than being introduced at the end of the course, almost as an afterthought. We obtain them easily by truncating binomial expansions for polynomial functions and by division of polynomials for rational functions. In "all" other cases, we use the method of undetermined coefficients from the functional equation, algebraic or differential, of which they are the solution. Incidentally, even in the conventional approach, using the differential equation is by far the easier way for introducing transcendental functions. See for instance S. Lang (1976), Sections 4-1, 2, and 3 or R. L. Finney - D. R. Ostbey (1984) Section 4-8 and exercise 3.

We can read off all the usual notions of the differential calculus—continuity, differentiability, concavity, etc.—from the coefficients in the Taylor polynomial approximations. For instance, we have

$$f \text{ is } \mathbf{continuous} \text{ at } x_0 \quad \text{iff} \quad f(x_0 + h) = f(x_0) + \ldots$$

and

$$f \text{ is } \mathbf{differentiable} \text{ at } x_0 \quad \text{iff} \quad f(x_0 + h) = f(x_0) + lh + \ldots \text{ for some } l$$

and, if f is differentiable on an open set, we define $f'(x)$ as the function whose value at x_0 is l.

For example, given $f(x) = x^3$, we have $f(x_0 + h) = x_0^3 + 3x_0^2h + 3x_0h^2 + \ldots$. Then, *by definition*, the value of $f'(x)$ at x_0 is the coefficient of h and that of $f''(x)$ is twice the coefficient of h^2. Note that the equation of the tangent at x_0 is $t_{x_0}(x) = x_0^3 + 3x_0^2(x - x_0)$ and that of the osculating parabola is $q_{x_0}(x) = x_0^3 + 3x_0^2(x - x_0) + 3x_0(x - x_0)^2$. Also, just by looking at the graph of $f(x_0 + h) = x_0^3 + 3x_0^2h$, we can *find* the *sided* limits at x_0. In fact, we can find sided limits this way even in cases which, in the conventional approach, require the use of L'Hôpital's rule.

Thus, while these notions appear to the students as disconnected topics in the conventional approach, they appear here as systematic features intrinsic to the successive Taylor approximations of $f(x_0 + h)$. Moreover, these definitions can be used to *prove* the usual theorems quite simply by substituting the approximations for the functions. For instance, to prove the quotient rule, just divide $f(x_0 + h) = A_0 + A_1h + \ldots$ by $g(x_0 + h) = B_0 + B_1h + \ldots$. The coefficient of the first degree term in the quotient is the value of the derivative of $f(x)/g(x)$ at x_0.

The stage in the epistemological development of a person where she/he can *speculate*, that is operate *under assumptions*, is, in Piaget terminology, the abstract operational, or hypothetico-deductive stage. It is the stage required by and large by the conventional approach. For instance, a person at the hypothetico-deductive stage would be willing to speculate on the consequences of *assuming* that $|x-x_0|$ is less than ∂ while persons at the concrete operational stage would absolutely refuse to do so on the ground that they don't *know* for a fact that $|x-x_0|$ *is* less than ∂. By contrast, the concrete operational stage is the stage where a person can only *calculate*, that is operate *on existing data*. It is unfortunately the stage at which most students currently in college are (W. Palow, 1986). But, because the algebra prerequisites to Lagrange's approach are only those necessary to get the Taylor polynomials, it "*reduce(s) the demands on students for traditional manipulations of equations*", (NSF, 1987), and thus allows us to serve these students too.

Content architecture.

Given the problems encountered by our students, it seems most important that the degree of difficulty, as perceived by them, not increase exponentially as the course progresses. This is precisely what happens with the conventional precalculus - calculus architecture as it introduces a succession of concepts—limits and continuity, differentiability and derivative, etc.— *illustrated* with functions from a "*dictionary of functions*" presumably learned in an elementary fashion in some pre-calculus course (J. Mason & A. Schremmer, 1989).

By contrast, the deep significance of Lagrange's approach for the first year calculus is that it ties the conceptual development of the calculus to a hierarchy of functions reflected by the Taylor polynomial approximations: locally, continuous functions are viewed as approximately constant, differentiable functions as approximately affine, twice-differentiable functions as approximately quadratic, etc. We are thus led to look at the differential calculus as the study of successive classes of increasingly complicated functions. In Differential Calculus I, we investigate **Affine Functions** ($f(x) = ax + b$), **Quadratic Functions** ($f(x) = ax^2 + bx + c$), **Homographic Functions** ($f(x) = [ax + b] / [cx + d]$). We start Differential Calculus II with **Power Functions** ($f(x) = \pm x^{\pm n}$) and **Binomial Functions**($f(x) = \pm x^{\pm m} \pm x^{\pm n}$) and we continue with **Polynomial Functions**, **Laurent–Polynomial Functions**, **Rational Functions**, **Irrational Functions**, and **Transcendental Functions**. We treat each class completely before we move on to the next one. To achieve both a very lean calculus and a very sharp focus, all is eliminated that is not of *immediate* use in the sequel. Moreover, analytic geometry is almost entirely avoided and what is kept is only what can be recast in functional terms. For instance, rather than talking about the point-slope formula for a straight line, we solve the initial value differential problem $f''(x) = 0$ with $f'(x_0) = y'_0$ and $f(x_0) = y_0$. This Lagrangian architecture has powerful pedagogical advantages.

The first one is that the *level* of difficulty perceived by the students does not increase appreciably as they go on. What changes is the *nature* of the difficulties they encounter. During the first semester, the technical difficulties are very small and it is with the concepts themselves that the students have to cope. But the number of concepts needed initially is fairly small and the concepts themselves elementary. Moreover, the various features of a function near a point, sign, variance, concavity, etc., are visualized and embodied by power functions seen as prototypes and then used as building blocks for "all" other functions. For instance, $p(x) = kx^{2n}$ ($n>0$) is *really* the prototype of a local extreme as the approximation of a function at a local extreme will be of the form $f(x_0 + h) = A_0 + A_{2n}h^{2n} + \dots, (n>0)$. AFFINE FUNCTIONS have–almost always–one and only one zero, have no pole, and essentially only constant approximations. Their instant rate of change is easy to define since the average rate of change is the same between any two points, they have only a first derivative, are monotonic, have no extreme, no curvature, and the differential equations of which they are solutions are almost trivial. Then, QUADRATIC FUNCTIONS still have no pole but can have two kinds of zeros or none at all. They have both constant and affine approximations, the instant rate of change changes linearly, and they have first and second derivatives. They have a turning point and therefore an extreme, they have curvature but are "curling" everywhere, that is have no inflection. The differential equations of which they are solutions are still almost trivial. Finally, HOMOGRAPHIC FUNCTIONS have a pole and two inflection points, one at the pole and one at infinity. They have constant, affine and quadratic approximations, etc. In general, these functions are particularly easy to discuss because they are just translated power functions, *globally* and *exactly*, while "all" other functions are only *locally* and *approximately* so.

Then, as the students familiarize themselves with the concepts, they become progressively able to shift their attention to the technicalities involved in dealing with the more complicated functions of the second semester and to the new concepts that these require. This makes for a reasonably low learning gradient. At the very least, we are in a position where we can, ethically, demand that the students fully document their work (Schremmer, 1989). In the best cases, it also introduces the students to the very basic mathematical temptation of wondering about generalizations such as: Affine functions have at most one zero; is this true of all functions? Affine functions change sign at zeros; is this true of all functions? Affine and quadratic functions do not change sign other than at zeros; is this true of all functions? Etc. The corresponding statement keeps having to be rephrased as the class of functions under investigation is enlarged. This is in marked contrast with the conventional approach where all statements are made from the outset in the most general terms possible, that is way beyond the students' experience and wildest expectations.

Conclusions.

We started developing Lagrange's approach in 1982 with standalone alternatives to the conventional semesters of Precalculus and Differential Calculus (F. Mattei & A. Schremmer, 1983, 1988 a). These were used on a regular basis, if more or less informally, by several of our colleagues at Community College of Philadelphia. Then, in the Summer of 1988, Community College of Philadelphia approved the creation of a new Integrated Precalculus I - II & Differential Calculus two-semester sequence based on F. Schremmer & A. Schremmer (1988 b, c). Shortly afterwards, Community College of Philadelphia received a Calculus Development Grant to run a pilot and Essex Community College in Baltimore agreed to field test the sequence under the grant. While we have solid grounds to believe that our approach does work, at this point, we have made no attempt at statistical research.

The algebra requirements are minimal. Essentially, the students need only to be sufficiently proficient with polynomial algebra that they can obtain the approximations. Anything beyond that is reviewed in algebra review cards: first and second degree equations, simple inequalities. As a result, we have no placement problems. Based on a five year statistical study of the standalones, we believe that the students who come out of our Basic Algebra course with an A, that is about 30%, will be capable of taking Differential Calculus I. To obtain an even more compact sequence, one should design a one semester Integrated Arithmetic-Algebra course. It should bring out the parallel between decimal/ rational arithmetic and polynomial /rational algebra and it should promote a facility with decimal arithmetic and polynomial algebra to help test and appreciate orders of magnitude.

Since the integral calculus is almost entirely disjoint from the differential calculus, transfer of the two semester sequence does not pose any problem. In fact, as soon as we were awarded the grant, we contacted the three schools to which most of our students transfer and all three approved the sequence as equivalent to their Calculus I. Over the years, students who took the Lagrange Differential Calculus standalone have found themselves in our conventional second semester Integral Calculus together with students from the conventional Differential Calculus. The only way we seem to be able to tell them apart is from their reaction to L'Hôpital's rule: students coming from the Lagrange based Differential Calculus fail to see what the problem is with, say, $\lim_{x\text{â}0}[1 - \cos x]/x^2$ since their reaction is, in any case, immediately to approximate $\cos x$ by $1 - x^2/2 + \ldots$!

The only problem is in transferring Differential Calculus I alone. As this is written, the end of the second semester in which we have been offering Differential Calculus I, conversations and negotiations continue to being carried out with neighboring institutions to convince them to try similar courses themselves. Several have already joined us in a new NSF grant proposal. This would obviously solve the problem of Community College of Philadelphia students transferring before they can take Differential Calculus II. But, in any event, students with Differential Calculus I should certainly do quite well in any Precalculus II. In fact, Differential Calculus I has already been accepted by one institution as equivalent to their precalculus. A more delicate problem is to obtain credit for "short calculus", defined as polynomial calculus with a heavy reliance on cookbook manipulations.

NOTE. The systematic use of polynomial approximations is of course well known, in Perturbation Theory, as the method of asymptotic expansions. But other than as such, we do not know of any reference. We invite anyone interested in Lagrange's approach, in our project, and/or in getting copies of our materials to contact us.

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