A LAGRANGIAN, INTEGRATED PRECALCULUS – DIFFERENTIAL CALCULUS

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0. Introduction.

There is surely no need to make here the case for the importance of the Differential Calculus, the mathematics of change, for the technological future of this country. On the other hand, the widespread dissatisfaction with it does not seem to have resulted, at least so far, in much change other than a bit of this, a tad of that and mostly more of the same. In fact, there does not even seem to be much of an agreement about the cause of the "crisis".

We believe that, to a large extent, the crisis has been of academia's own doing in that, starting in the thirties[1], it insisted on a particular mathematical conception of the calculus which relies, from the outset and in an essential, profound manner, explicitly or implicitly, on limit considerations beyond most students currently in college, in particular in two-year colleges. As such, it leaves no middle ground between a "rigorous" level, sufficient to consider *mathematical* problems but that excludes most students, and an "intuitive" level, attainable to many students but powerless for doing any mathematics. It is because of this lack of middle ground that the calculus has become "*a watered down, cookbook course in which all (the students) learn are recipes, without even being taught what it is that they are cooking*" (R. G. Douglas, 1986, p. iv). To avoid this dilemma, the trend has been to focus instead on *application* problems but this creates difficulties of its own[2]. Moreover, in our experience, instructors in client disciplines would rather we left their discipline to them and, in any case, this seems hardly the way to give the students a real appreciation of the calculus *per se*.

The calculus is currently very much in the same situation as the internal combustion engine. Like this type of engine, it has been perfected over the past fifty years[3] to the point where it does exceedingly well what it can do but finds it extremely difficult, if not impossible, to accommodate the further demands of the "New Century". On the other hand, engines such as the Sterling engine, based on thermodynamic cycles quite different from that of the internal combustion engine, have not yet benefited from the current technology and have therefore not yet been refined to the point where they can effectively compete with the internal combustion engine. But they are, even now, being developed and *they* will meet the new demands. Similarly, we believe that there is a conception of the calculus, going back to Lagrange, which, while still far from having the polish of the conventional approach, already offers an alternative particularly well suited to those students who are not, at least a priori, favorably inclined towards mathematics[4].

What we are presenting here is an embodiment of Lagrange's approach for a two four-hour-semester sequence, **Differential Calculus I and II**, intended to replace the conventional three semester sequence Precalculus I - *Alge-*

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[1] Textbooks such as J. W. Young & F. M. Morgan (1922) show how much was lost in the process.

[2] We feel that there is a bit of a contradiction here. It is quite natural for an economics instructor who has just defined the cost function to want to use the notion of derivative to define marginal cost. It seems much less natural for a mathematics instructor to use the marginal cost to motivate the definition of the derivative. Presumably, if economics students are still at a point where they don't know what a derivative is, then neither can they be expected to know what a marginal cost is. Which is how one winds up having first to teach economics.

[3] Most if not all books currently on the market indeed appear to be descendents of Thomas.

[4] In the January-February 1988 issue of FOCUS, mention is made of a 1987 MAA Award for Expository Excellence, the George Polya Award for articles in THE COLLEGE MATHEMATICS JOURNAL, to Irl. C. Bivens for "What a Tangent Line is When it isn't a Limit" in the March 1986 issue, pages 133-143. The committee's citation is quoted in part as: "*By defining the tangent line as the best linear approximation to the graph of a function near a point, [Bivens] has narrowed the gap, always treacherous to students, between an intuitive idea and a rigorous definition. The subject of this article is* fundamental to the first two years of college mathematics and should simplify things for students...." (Emphasis added). Lagrange's approach is of course just the extension of this idea to best approximations of any degree. In other words, what we are really dealing with are jets of differentiable functions.

braic Functions, Precalculus II - *Transcendental Functions*, Calculus I - *Differential Calculus* and, therefore, one particularly suited for two-year colleges[1].

1. The problems.

The Integrated Precalculus-Calculus sequence was designed for students who, in the words of the NSF Request for Proposals for the Development of a Calculus Curriculum, almost all"*choose not to continue their study of mathematics through calculus, thereby closing career options in mathematics, engineering and the sciences.*" Many are women, minorities and "returning" adults[2].

It is always convenient to blame the victims and explain away the unacceptably low survival rates in calculus courses across the nation, as well as the above-mentioned situation, as resulting from poor students' preparation. Indeed, many students at Community College of Philadelphia are indeed initially characterized by a low reading level and poor algebra preparation and usually start by taking Remedial English and/or Algebra I. However, out of 7216 students who enrolled in Algebra I between 1980 and 1985, 2260 (31%) passed it with an A meaning that they were Precalculus ready. Yet, out of the latter,

421	(19 %)	enrolled in Precalculus I	of whom	338	(80 %)	passed it with at least a C,
140	(6 %)	enrolled in Precalculus II	of whom	124	(89 %)	passed it with at least a C,
37	(1.6 %)	enrolled in Calculus I[3]	of whom	32	(86 %)	passed it with at least a C,
10	(0.4 %)	enrolled in Calculus II	of whom	8	(80 %)	passed it with at least a C.

Thus, when they do enroll in later courses, these students do remarkably well and the problem is why so few do so.

The first of three major difficulties is a linguistic one, both syntactic and semantic (F. Schremmer & A. Schremmer, in press). For instance, at Community College of Philadelphia, even calculus students take a very long time to *assimilate* new terms. This is particularly unfortunate given that they often don't know even the *everyday* meaning of terms such as function, positive/negative, increasing/decreasing, maximum/minimum, etc. It should then not come as a surprise that these students are, by and large, initially incapable of reading textbooks where a term that just appeared in a definition and/or example is then supposed, by the next paragraph, to have been already completely assimilated. Nothing in the students' prior experience has prepared them for that kind of requirement. A related problem is that they are not used to conceptualizing from graphics. For instance, they find it very hard to interpret a line in a coordinate system as determining a function and, even after they do, to see where it is positive/negative, increasing/decreasing, etc. and even then to verbalize what they see. And there are many other linguistic difficulties, such as confusing the problem of finding the value of a function at 3 with that of finding where the function has the value 3.

The second major cause is an attitudinal one. Since just about every permutation of the order of the contents seems to be represented in current textbooks which, to add insult to injury, boast about how flexible their table of contents is, it is difficult to see what remains of the deductive structure of mathematics and how students can ever be expected to make any sense of this madness. Navigating through such a course is like being told to operate in an apparently random high voltage environment after having been given a set of instructions but without having an inkling of what makes the equipment tick (J. Mason & A. Schremmer, 1989). It should then be not too surprising that the students do not wish to invest much time and energy in mathematics. For instance, even when a term keeps reappearing, the students often continue for a long time not to learn what it means. It is as if they hoped to outlast an unfortunate run of recurrences.

The third major cause is pedagogical. Lecturing to introduce information is inappropriate for students who may not be initially able to take usable notes. Just assigning readings in a textbook and relying on regular homework is equally unrealistic for students who often have to work for a living and/or have to raise a family.

It should be stressed that these are by no means insurmountable difficulties. But they must be taken into account and our experience suggests that Lagrange's approach does in fact enable the students to surmount them.

2. Mathematical conception.

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[1] But we are also using this approach in a regular Calculus I and we will be developing a Calculus II.

[2] In the spring of 1988, out of a Full Time Equivalent total of 6770 at Community College of Philadelphia, 48.3 % were Black, 40.3 % were white, 5.2 % were Hispanic, and 6.2 % were Oriental. Also, 39.2 % were Male and 60.8 % were Female.

[3] This represents 4% of the total Calculus I enrollment at Community College of Philadelphia for these years.

To illustrate the dilemma mentioned in the introduction, consider how we define continuity in the conventional approach:

$$f \text{ is } \textbf{continuous} \text{ at } x_0 \qquad \text{iff} \qquad \forall\varepsilon\ \exists\delta\ \P\ 0 < |x - x_0| < \delta \ \Rightarrow\ |f(x) - f(x_0)| < \varepsilon^{a}$$

Even allowing for the use of ε's and δ's, the problem with limits is that there is no procedure for *finding* them: one can only *check* whether or not a given candidate is the limit. In other words, there is only a *relation* between x_0 and the limit, and it is no more realistic to ask students to come up with a candidate when the function is *not* continuous than with the requisite Skolem function $\delta = \delta(\varepsilon)$ when it is. Furthermore, since the only way to prevent students from identifying a priori $\lim_{x \to x_0} f(x)$ with $f(x_0)$ is to use *one-sided* limits, things get out of hand very fast.

If, on the other hand, we decide, as in the usual "intuitive" presentation, to avoid ε's and δ's altogether, the problem then is that we are left without any definition. Continuity and differentiability basically become primitive terms whose understanding depends on whatever everyday connotations the words may have, if any, for the students. Hence the gimmicks, electronic or otherwise, to buttress or create this "intuition". Most important, though, is that, even when *shown*, or even *experienced*, these concepts cannot be *used* and, eventually, must remain meaningless.

In Lagrange's approach, and in contradistinction with the conventional approach in which Taylor polynomial approximations for $f(x_0 + h)$ are summarily introduced toward the end, after limits and derivatives, we get the Taylor polynomial approximations *from the outset*. We obtain them easily by truncation of binomial expansions in the case of polynomial functions, by division of polynomials in the case of rational functions and, in "all" other cases, by the method of undetermined coefficients from the functional equation, algebraic or differential, of which they are the solution[1]. We can then read off all the usual notions of the differential calculus—continuity, differentiability, concavity, etc.—from the coefficients in the Taylor polynomial approximations. For instance, we have

$$f \text{ is } \textbf{continuous} \text{ at } x_0 \qquad \text{iff} \qquad f(x_0 + h) = f(x_0) + \ldots$$

Similarly, we have

$$f \text{ is } \textbf{differentiable} \text{ at } x_0 \qquad \text{iff} \qquad f(x_0 + h) = f(x_0) + lh + \ldots \text{ for some } l$$

and, if f is differentiable on an open set, we define $f'(x)$ as the function whose value at x_0 is l. Thus, while these notions appear to the students as disconnected topics in the conventional approach, they appear here as systematic features intrinsic to the successive Taylor approximations of $f(x_0 + h)$.

Moreover, these definitions are operational in the sense that we can use them to easily *prove* "algebraically" theorems such as the chain rule, intuitively and, perhaps surprisingly, correctly. In particular, this enables students to *find* limits and, in fact, *one-sided* limits even in cases which, in the conventional approach, require the use of L'Hôpital's rule! The degree of rigor depends on whether we use $\mathbf{o}[x^n]$ ("little ohs") to evaluate remainders or merely ' ... ' just to retain an audit trail. For details, see F. Schremmer & A. Schremmer (1989 a, b)[2]. In any case, since the algebra prerequisites are only those necessary to get the Taylor polynomials, Lagrange's approach allows us, in the words of the NSF Request For Proposals, to "*reduce the demands on students for traditional manipulations of equations*".

Finally, another consequence of this algebraization of the calculus is that, in Piaget terminology, students need only be at the concrete operational stage, that is the stage in the epistemological development where a person can only *calculate*, that is operate *on existing data*. By contrast, the conventional approach requires that the students be at the stage where they can *speculate*, that is operate *under assumptions*[3].

3. Content architecture.

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[1] Even the rigorous treatment is in fact much simpler that way than the conventional one. See for instance S. Lang (1976), Sections 4-1, 2, and 3 or R. L. Finney - D. R. Ostbey (1984) Section 4-8 and exercise 3.

[2] Copies available on request.

[3] For instance, a person at Piaget's hypothetico-deductive stage would be willing to speculate on the consequences of *assuming* that $|x-x_0|$ is less than ∂ while persons at the concrete operational stage would absolutely refuse to do so on the ground that they don't *know* that $|x-x_0|$ *is* less than ∂.

Given the problems encountered by our students, it is most important that the degree of difficulty, as perceived by them, not increase exponentially as the course progresses. This is precisely what happens with the conventional content architecture as it introduces a succession of concepts—limits and continuity, differentiability and derivative, etc.— *illustrated* with functions from a "*dictionary of functions*" presumably learned in an elementary fashion in some pre-calculus course (J. Mason & A. Schremmer, 1989). As already mentioned, an additional difficulty with this architecture is that it is not perceived as such by the students!

By contrast, the deep significance of Lagrange's approach for the First Year Calculus is that it ties the conceptual development of the calculus to a hierarchy of functions reflected by the Taylor polynomial approximations: locally, continuous functions are viewed as approximately constant, differentiable functions as approximately affine, twice-differentiable functions as approximately quadratic, etc. We are thus led to look at the differential calculus as that of successive classes of increasingly complicated functions. During the first semester, we investigate **Affine Functions** ($f(x) = ax + b$), **Quadratic Functions** ($f(x) = ax^2 + bx + c$), **Homographic Functions** ($f(x) = [ax + b] / [cx + d]$). We start the second semester with **Power Functions** ($f(x) = \pm x^{\pm n}$) and **Binomial Functions**($f(x) = \pm x^{\pm m} \pm x^{\pm n}$) and continue with **Polynomial Functions**, **Laurent–Polynomial Functions**, **Rational Functions**, **Irrational Functions**, and **Transcendental Functions**[1]. We treat each class completely before we move on to the next one. To achieve both a very lean calculus and a very sharp focus, all is eliminated that is not of *immediate* use in the sequel. Moreover, analytic geometry is almost entirely avoided[2] and what is kept is only what can be recast in functional terms. For instance, rather than talking about the point-slope formula for a straight line, we solve the initial value differential problem $f''(x) = 0$ with $f'(x_0) = y'_0$ and $f(x_0) = y_0$. This Lagrangian architecture has powerful pedagogical advantages.

The first one is that the *level* of difficulty perceived by the students does not increase appreciably as they go on. What changes is the *nature* of the difficulties they encounter. During the first semester, the technical difficulties are very small and it is with the concepts themselves that the students have to cope. But the number of concepts needed initially is fairly small and the concepts themselves elementary. Moreover, the various features of a function near a point, sign, variance, concavity, etc., are visualized and embodied by power functions seen as prototypes[3] and then used as building blocks for "all" other functions. AFFINE FUNCTIONS have–almost always–one and only one zero, have no pole, and essentially only constant approximations. Their instant rate of change is easy to define since the average rate of change is the same between any two points, they have only a first derivative, are monotonic, have no extreme, no curvature, and the differential equations of which they are solutions are almost trivial. Then, QUADRATIC FUNCTIONS still have no pole but can have two kinds of zeros or none at all. They have both constant and affine approximations, the instant rate of change changes linearly, and they have first and second derivatives. They have a turning point and therefore an extreme, they have curvature but are "curling" everywhere, that is have no inflection. The differential equations of which they are solutions are still almost trivial. Finally, HOMOGRAPHIC FUNCTIONS have a pole and two inflection points, one at the pole and one at infinity. They have constant, affine and quadratic approximations, etc. In general, these functions are particularly easy to discuss because they are just translated power functions, *globally* and *exactly*, while "all" other functions are only *locally* and *approximately* so.

Then, as the students familiarize themselves with the concepts, they become progressively able to shift their attention to the technicalities involved in dealing with the more complicated functions of the second semester and to the new concepts that these require. This makes for a reasonably low learning gradient. At the very least, we are in a position where we can, ethically, demand that the students fully document their work (Schremmer, 1989). In the best cases, it also introduces the students to the very basic mathematical temptation of wondering about generalizations such as: Affine functions have at most one zero; is this true of all functions? Affine functions change sign at zeros; is this true of all functions? Affine and quadratic functions do not change sign other than at zeros; is this true of all functions? Etc. The corresponding statement keeps having to be rephrased as the class of functions under investigation is enlarged. This is in marked contrast with the conventional approach where all statements are made from the outset in the most general terms possible, that is way beyond the students' experience and wildest expectations.

4. Instructional format.

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[1] This architecture seems to have been abandoned at the same time as the current approach became fashionable. See again J. W. Young and F. M. Morgan for an example.

[2] This is not the result of any bias: one of us actually wrote a "geometric introduction to calculus".

[3] For instance, $p(x) = kx^{2n}$ ($n>0$) is *really* the prototype of a local extreme as the approximation of a function at a local extreme will be of the form $f(x_0 + h) = A_0 + A_{2n}h^{2n} + \ldots$, ($n>0$).

Pedagogically, we feel that there is a very great need to turn from a *teaching* process, that is one where the students are essentially passive, to a *learning* process, that is one where the students are active.

Already a long time ago, Z. P. Dienes (1960, 1963) observed that playing children followed a cycle which he proposed as a basis for *learning*. First children play with objects according to *given* rules which they follow rigidly while they observe the effect of the rules on the objects. Then, they start changing the rules randomly with the emphasis remaining on the objects. But, after a while, their attention progressively turns to the rules themselves which then become the new objects of the game. And the cycle starts again with super rules according to which the rules can be changed so that the level of abstraction goes up one level. An important aspect of this cycle is that the children oscillate between their need for the reassurance given by hard and fast rules and their need for experimenting with these rules.

Our primary goal was to allow and foster a similar cycle in the academic context. We wanted to create a controlled environment where learning would occur through the students' active participation in the development of the calculus. This implied the creation of an instructional medium readily adaptable to a variety of situations and pedagogies, individualized learning, small group learning, etc. It was thus very important that, because it is defined entirely in *mathematical* terms, Lagrange's approach is *pedagogy* independent.

We developed a format (F. Mattei & A. Schremmer, 1983, 1988 a, b, c)[1] whereby the materials are apprehended through a succession of tasks[2]. It acknowledges the students' initial need for hard and fast rules. There also seems to be a definite psychological advantage for the students to go through a book, cover to cover, without having to skip anything and the taskbook also serve as preformatted notebook which eventually reflects all of the student's work for the semester. The textual part is reduced to what is absolutely necessary to do the task *at hand* and so the students are not reinforced in the idea that texts are something redundant and better avoided. The leanness of the textual materials makes this format particularly well suited for students starting with very low verbal skills. From a psychological point of view, it conforms to the initial expectations of students who insist on being told "what they are supposed to do" and allows those so inclined to confine themselves to the minimal mechanics. This will of course usually produce only C grades but should *not* be disparaged. While we do want to modify these attitudes, we see no advantage in confronting the students head on from the outset.

However, the fine structure of the contents is designed to give the students a basis from which to explore well delineated mathematical situations without the risk of getting bogged down in premature complexities. The task format facilitates rapprochements, comparisons, etc, invites generalizations, conjectures, etc. and it does this *indirectly* so as to leave the students the pleasure of discoveries commensurate with their newly acquired conceptual and technical means[3]. And, once they attain a certain level of familiarity with the materials, given the opportunity, many students do indeed try to investigate, to question, even to generalize. But the task confronting the students at any given time remains bounded and even the broader view can be apprehended progressively.

Finally, another advantage of the task format is that it is compatible with any pedagogical method. For instance, in one approach, the students go through the taskcards *in class*, doing every single exercise, according to a predetermined schedule that includes exams. Both students and faculty know where each stand in relation to the schedule. Students with difficulties can immediately be attended to and/or referred to the learning center where tutors know exactly what is expected from the students and can therefore provide optimal assistance. In most cases, the schedule does not require the students to do any homework, unless they want to get ahead, or have to catch up, or want to undertake side explorations. In another approach, both the individual, supervised work of the students and classwide confrontation of learning experiences are encouraged. Other instructors prefer to lecture and the taskbook then serve as combination "student manual" and notebook.

6. Conclusions.

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[1] Copies available on request.

[2] A taskcard is a page consisting of: a PROBLEM, a couple of lines of INFORMATION, a PROCEDURE whereby the problem is worked out step by step and three similar EXERCISES with answers and *space to do the work* so that the taskbook contains all of the student's work for the semester. Each taskcard deals with a single concept and each step in the procedure either invokes a previous taskcard as subroutine or, if not, is extremely simple. Each concept is introduced by a taskcard where the function is given by a graph and then, after an Algebra Review card whenever necessary, by a taskcard where the function is given by a rule $f(x) = ...$

[3] For instance, while there is no taskcard composing a function with its inverse, the sequencing of the taskcards dealing with inverses and composition invites the more adventurous students to try it.

We started developing Lagrange's approach in 1982 with standalone alternatives to the conventional semesters of Precalculus and Differential Calculus (F. Mattei & A. Schremmer, 1983, 1988 a)[1]. These were used on a regular basis, if more or less informally, by several of our colleagues at Community College of Philadelphia. Then, in the Summer of 1988, Community College of Philadelphia approved the creation of a new Integrated Precalculus I - II & Differential Calculus two-semester sequence based on F. Schremmer & A. Schremmer (1988 b, c). Shortly afterwards, Community College of Philadelphia received a Calculus Development Grant to run a pilot and Essex Community College in Baltimore agreed to field test the sequence under the grant. While we have solid grounds to believe that our approach does work, at this point, we have made no attempt at statistical research.

Given that the algebra requirements are minimal, that there are no conceptual requirements, and that Lagrange's approach requires only that students be at Piaget's concrete operational level, there are no placement problems. Students out of any Basic Algebra course with just a mastery of polynomial algebra should be able to pass Differential Calculus I. As a preparatory course, then, it should be sufficient for the students to take a one semester Integrated Arithmetic-Algebra course consisting of decimal[2] and rational arithmetic and polynomial and rational algebra.

Since the integral calculus is almost entirely disjoint from the differential calculus, transfer of the two semester sequence does not pose any problem. In fact, as soon as we were awarded the grant, we contacted the three schools to which most of our students transfer and all three approved the sequence as equivalent to their Calculus I.

Over the years, students having taken the Lagrange Differential Calculus have found themselves in our conventional second semester Integral Calculus together with students from the conventional Differential Calculus. The only way we seem to be able to tell them apart is from their reaction to L'Hôpital's rule: students coming from the Lagrange based Differential Calculus fail to see what the problem is with, say, $\lim_{x\to 0}[1 - \cos x]/x^2$ since their reaction is, in any case, to approximate immediately $\cos x$ by $1 - x^2/2 + \ldots$!

The only problem is in transferring Differential Calculus I alone. As this is written, the end of the first semester in which we have been offering Differential Calculus I, conversations and negotiations are being carried out with neighboring institutions to convince them to offer similar courses. This would obviously solve the problem of Community College of Philadelphia students transferring before they can take Differential Calculus II. But, in any event, students with Differential Calculus I should certainly do quite well in any Precalculus II. In fact, Differential Calculus I has already been accepted by one institution as equivalent to their precalculus. A more delicate problem is to obtain credit for "short calculus", defined as polynomial calculus with a heavy reliance on cookbook manipulations.

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[1] Copies available on request.

[2] A facility with decimal arithmetic is especially important to test and appreciate orders of magnitude.

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