Chapter 13

Cubic Functions: Local Analysis

Cubic functions are functions specified by a global input-output rule of the form

\[ x \xrightarrow{\text{CUBIC}_{a,b,c,d}} \text{CUBIC}_{a,b,c,d}(x) = ax^3 + bx^2 + cx + d \]

where \( \text{CUBIC}_{a,b,c,d} \) is the name of the cubic function and where \( a, b, c \) and \( d \) stand for the four bounded numbers that are needed to specify the cubic function \( \text{CUBIC}_{a,b,c,d} \).

1. In order to facilitate our investigation of cubic functions, it is necessary to begin by introducing some language special to cubic functions.

Given the cubic function \( \text{CUBIC}_{a,b,c,d} \), that is the function specified by the global input-output rule

\[ x \xrightarrow{\text{CUBIC}_{a,b,c,d}} \text{CUBIC}_{a,b,c,d}(x) = ax^3 + bx^2 + cx + d \]

- The given signed number \( a \) is called the cubing coefficient of the cubic function \( \text{CUBIC}_{a,b,c,d} \) and \( ax^3 \), that is the cubing coefficient multiplied...
by three copies of the input, is called the \textbf{cubing term} of the cubic function $CUBIC_{a,b,c,d}$.

- The given signed number $b$ is called the \textbf{squaring coefficient} of the cubic function $CUBIC_{a,b,c,d}$ and $bx^2$, that is the squaring coefficient multiplied by two copies of the input, is called the \textbf{squaring term} of the cubic function $CUBIC_{a,b,c,d}$.

- The given signed number $c$ is called the \textbf{linear coefficient} of the cubic function $CUBIC_{a,b,c,d}$ and $cx^1$, that is the linear coefficient multiplied by one copy of the input, is called the \textbf{linear term} of the cubic function $CUBIC_{a,b,c,d}$.

- The given signed number $d$ is called the \textbf{constant coefficient} of the cubic function $CUBIC_{a,b,c,d}$ and $dx^0$, the constant coefficient multiplied by no copy of the input, is called the \textbf{constant term} of the cubic function $CUBIC_{a,b,c,d}$.

In other words, the \textit{constant term} is the same as the \textit{constant coefficient} and what words we will use will depend on what viewpoint we will be taking.

**Example 1.** The function $TATA_{-3, +5, -2, +4}$ specified by the input-output rule

$$x \xrightarrow{TATA_{-3, +5, -2, +4}} TATA_{-3, +5, -2, +4}(x) = -3x^3 + 5x^2 - 2x + 4$$

is the cubic function whose

- \textit{cubing term} is $(-3)x^3$, that is the \textit{cubing coefficient} $-3$ multiplied by three copies of the input $x$.

- \textit{squaring term} is $(+5)x^2$, that is the \textit{squaring coefficient} $+5$ multiplied by two copies of the input $x$.

- \textit{linear term} is $(-2)x$, that is the \textit{linear coefficient} $-2$ multiplied by one copy of the input $x$.

- \textit{constant term} is $+4$, that is the \textit{constant coefficient} $+4$ multiplied by no copy of the input $x$. In other words, the \textit{constant term} is the same as the \textit{constant coefficient}.

2. Unless there is a need for the above way of writing things, as there will be, for instance, when we have to deal with more than one function at a time, we will write only whatever information is needed.

\textbf{a.} Instead of writing the complete global input-output rule as above,

$$x \xrightarrow{CUBIC_{a,b,c,d}} CUBIC_{a,b,c,d}(x) = ax^3 + bx^2 + cx + d$$

we may let the coefficients go without saying since they appear in the output and we will often write the simplified global input-output rule

$$x \xrightarrow{CUBIC} CUBIC(x) = ax^3 + bx^2 + cx + d$$
since there is no loss of information in that the coefficients, $a, b, c$, are still given.

**EXAMPLE 2.** Instead of writing

$$x \xrightarrow{TATA_{-3,+.5,-,2,+4}} TATA_{-3,+.5,-,2,+4}(x) = -3x^3 + 5x^2 - 2x + 4$$

we will often write just

$$x \xrightarrow{TATA} TATA(x) = -3x^3 + 5x^2 - 2x + 4$$

because there is no loss of information.

**b.** Actually, since the name of the function, $CUBIC$, occurs in two places, above the arrow, $\xrightarrow{CUBIC}$, and in the name of the output, $CUBIC(x)$, the simplified global input-output rule could be simplified even further and written into either one of two even simpler ultra-simplified global input-output rules. Indeed:

- Some people write the name of the function, $CUBIC$, only once, in the name of the output, $CUBIC(x)$ and do not write the name of the function, $CUBIC$, on top of the arrow, $\xrightarrow{CUBIC}$.

**EXAMPLE 3.** Instead of writing

$$x \xrightarrow{TATA} TATA(x) = -3x^3 + 5x^2 - 2x + 4$$

some people just write

$$x \xrightarrow{TATA} TATA(x) = -3x^3 + 5x^2 - 2x + 4$$

because there is still no loss of information.

- Other people prefer to do the opposite and write the name of the function, $CUBIC$, only on top of the arrow, $\xrightarrow{CUBIC}$ but not in the output.

**EXAMPLE 4.** Instead of writing

$$x \xrightarrow{TATA} TATA(x) = -3x^3 + 5x^2 - 2x + 4$$

other people prefer to write

$$x \xrightarrow{TATA} -3x^3 + 5x^2 - 2x + 4$$

because there is still no loss of information.

In this text, though, we will use the simplified global input-output rule mentioned above, that is we will not write the coefficients in the name of the functions but we will write the name of the function both above the arrow and in the name of the output as:
full name

- it cannot hurt,
- it is not much additional writing and
- it might prevent mistakes

**c.** On the other hand, instead of specifying a cubic function by its complete global input-output rule

\[ x \xrightarrow{\text{CUBIC}_{a,b,c,d}} \text{CUBIC}_{a,b,c,d}(x) = ax^3 + bx^2 + cx + d \]

or even by its simplified global input-output rule

\[ x \xrightarrow{\text{CUBIC}} \text{CUBIC}(x) = ax^3 + bx^2 + cx + d \]

we may use just its **full name**, that is its name including the coefficients.

\[ \text{CUBIC}_{a,b,c,d} \]

because, again, there is no loss of information in that both the name of the function and the coefficients, \(a, b, c, d\), are still given.

**Example 5.** Instead of saying: “Given the cubic function \(\text{TATA}_{-3,3,5,-2,4}\), that is the function specified by the global input-output rule

\[ x \xrightarrow{\text{TATA}_{-3,3,5,-2,4}} \text{TATA}_{-3,3,5,-2,4}(x) = -3x^3 + 5x^2 - 2x + 4 \]

or even instead of saying “Given the cubic function \(\text{TATA}\) specified by the global input-output rule

\[ x \xrightarrow{\text{TATA}_{-3,3,5,-2,4}} -3x^3 + 5x^2 - 2x + 4 \]

we can just as well say, without loss of information,

Given the **cubic** function \(\text{TATA}_{-3,3,5,-2,4}\)

because there is no loss of information.

**3.** There are three ways to look at **cubic functions** and which view we will take will depend on which is the most convenient for us in the mathematical situation we are in.

**a.** We can look at the cubic function \(\text{CUBIC}_{a,b,c,d}\) as a **combination** of the first four **non-negative-exponent power functions**:

\[ x \xrightarrow{\text{UNIT}} \text{UNIT}(x) = x^0 \]
\[ x \xrightarrow{\text{IDENTITY}} \text{IDENTITY}(x) = x^1 \]
\[ x \xrightarrow{\text{SQUARING}} \text{SQUARING}(x) = x^2 \]
\[ x \xrightarrow{\text{CUBING}} \text{CUBING}(x) = x^3 \]

This is the way we will look at polynomial functions in general.
b. We can look at the cubic function \( CUBIC_{a,b,c,d} \) as the result of adding to the quadratic function

\[
x \xrightarrow{\text{QUADRATIC}_{b,c,d}} \text{QUADRATIC}_{b,c,d}(x) = bx^2 + cx + d
\]

to be called the \textbf{quadratic part} of the cubic function \( CUBIC_{a,b,c,d} \), an \( a \)-dilation of the \textit{cubing function}

\[
x \xrightarrow{\text{CUBING}} \text{CUBING}(x) = x^3
\]

that is the function

\[
x \xrightarrow{a \cdot \text{CUBING}} a \cdot \text{CUBING}(x) = ax^3
\]

We will use this view when investigating and discussing local graphs.

c. We can look at the cubic function \( CUBIC_{a,b,c,d} \) as the result of adding to an \( a \)-dilation of the \textit{cubing function}

\[
x \xrightarrow{\text{CUBING}} \text{CUBING}(x) = x^3
\]

that is to the function

\[
x \xrightarrow{a \cdot \text{CUBING}} a \cdot \text{CUBING}(x) = ax^2
\]

the \textit{squaring function}, \( \text{QUADRATIC}_{b,c,d} \), that is a function specified by the global input-output rule

\[
x \xrightarrow{\text{QUADRATIC}_{b,c,d}} \text{QUADRATIC}_{b,c,d}(x) = bx^2 + cx + d
\]

\textbf{Example 6.} Given the \textit{cubic function} \( TATA_{-3,+5,-2,+4} \) specified by the global input-output rule

\[
x \xrightarrow{TATA_{-3,+5,-2,+4}} TATA_{-3,+5,-2,+4}(x) = (-3)x^3 + (+5)x^2 + (-2)x^1 + (+4)x^0
\]

\[
= -3x^3 + 5x^2 - 2x + 4
\]

the function \textit{Quadratic Part} of \( TATA_{-3,+5,-2,+4} \) is the squaring function specified by the global input-output rule

\[
x \xrightarrow{\text{Affine Part of } TATA_{-3,+5,-2,+4}} = (+5)x^2 + (-2)x^1 + (+4)x^0
\]

\[
= +5x^2 - 2x + 4
\]

We will use this view when investigating and discussing the \textit{essential bounded graph}.

\textbf{Pointwise Analysis}
13.1 Input-Output Pairs

As with any function specified by a \textit{global input-output rule}, in order to get the output for a given input we must:

- \textit{i. Read and write} what the global input-output rule says,
- \textit{ii. Replace} \( x \) in the input-output rule by the given input,
- \textit{iii. Identify the resulting specifying phrase.}

1. In the case of a cubic function \( \text{CUBIC}_{a,b,c,d} \), that is of a function specified by the global input-output rule

\[
x \xrightarrow{\text{CUBIC}} \text{CUBIC}(x) = ax^3 + bx^2 + cx + d
\]

and given an input \( x_0 \), in order to get the output, we proceed as follows.

\textbf{i. We read and write} what the input-output rule says:

- The input-output rule \textit{reads}:
  - "The output of \( \text{CUBIC}_{a,b,c,d} \) is obtained by multiplying \( a \) by 3 copies of the input, adding \( b \) multiplied by 2 copies of the input, adding \( c \) multiplied by 1 copy of the input and adding \( d \)."
  - We \textit{write}, or at least \textit{think}:

\[
x \xrightarrow{\text{CUBIC}} \text{CUBIC}(x) = a \cdot x \cdot x \cdot x + b \cdot x \cdot x + c \cdot x + d
\]

\textbf{ii. We indicate that} \( x \) \textit{is about to be replaced} by the given input \( x_0 \)

\[
x \bigg|_{x=x_0} \xrightarrow{\text{CUBIC}} \text{CUBIC}(x) \bigg|_{x=x_0} = ax^3 + bx^2 + cx + d \bigg|_{x=x_0}
\]

which gives us the following specifying-phrase

\[
= ax_0^3 + bx_0^2 + cx_0 + d
\]

\textbf{iii. When we have specific values for} \( a, b, c, d \) \textit{and for} \( x_0 \), \textit{we can then identify} the specifying-phrase \( ax_0^3 + bx_0^2 + cx_0 + d \) \textit{to get the specific value of the output} \( \text{CUBIC}(x_0) \).

\textbf{Example 7.} Given the function \( \text{THORA} \) specified by the global input-output rule

\[
x \xrightarrow{\text{THORA}} \text{THORA}(x) = -42.17x^3 - 32.67x^2 - 52.39x + 71.07
\]

and given the input \(-3\), in order to get the input-output pair, we proceed as follows.
13.1. INPUT-OUTPUT PAIRS

i. We read and write what the input-output rule says:
   - The input-output rule reads:
     “The output of THORA is obtained by multiplying $-32.67$ by 2 copies of the input, adding $-52.39$ multiplied by 1 copy of the input, and adding $+71.07$”
   - We write
     
     \[
     x \xrightarrow{\text{THORA}} \text{THORA}(x) = -42.17x^3 - 32.67x^2 - 52.39x + 71.07
     \]

ii. We indicate that \( x \) is about to be replaced by the given input \(-3\)

\[
\begin{array}{c|c|c}
  x & \text{THORA} & \text{THORA}(x) \\
  \hline
  x & \xrightarrow{\text{THORA}} & \xrightarrow{\text{THORA}(x)} \\
  x & \xrightarrow{\text{THORA}(x)} & \xrightarrow{-3} \\
\end{array}
\]

which gives us the following specifying phrase

\[
= (-42.17)(-3)^3 + (-32.67)(-3)^2 + (-52.39)(-3) + (+71.07)
\]

iii. We identify the specifying phrase

\[
= +1138.59 - 294.03 + 157.17 + 71.07 \\
= +1072.80
\]

2. In practice we usually will not write all of this since there is a lot of redundant information.

i. On the input side of the arrow that represents the function, instead of writing

\[
\begin{array}{c|c|c}
  x & \xrightarrow{\text{THORA}\xrightarrow{\cdot x_0}} & \xrightarrow{\text{THORA}(x)} \\
\end{array}
\]

we will write just the result of replacing \( x \) by \( x_0 \), that is

\[
x_0
\]

ii. On the output side of the arrow that represents the function, instead of writing the full

\[
\text{CUBIC}_{a,b,c,d}(x) \xrightarrow{\cdot x_0}
\]

we will similarly write just the result of replacing \( x \) by \( x_0 \), that is

\[
\text{CUBIC}_{a,b,c,d}(x_0)
\]

iii. But, on the “rule side” will we keep the indication that \( x \) is to be replaced by \( x_0 \)

\[
= ax^3 + bx^2 + cx + c \xrightarrow{\cdot x_0}
\]

because that is where the action will be.
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So, altogether, given an input \( x_0 \), we will write

\[
x_0 \xrightarrow{\text{CUBIC}} CUBIC(x_0) = ax^3 + bx^2 + cx + d \bigg|_{x=x_0}
\]

**Example 8.**

\[
-3 \xrightarrow{\text{THORA}} THORA(-3) = -42.17x^3 - 32.67x^2 - 52.39x + 71.07 \bigg|_{x=-3} \\
= (-42.17)(-3)^3 + (-32.67)(-3)^2 + (-52.39)(-3) + (71.07) \\
= +1138.59 - 294.03 + 157.17 + 71.07 \\
= +1072.80
\]

3. Altogether, and depending on the circumstances, we can then write:

\[
x_0 \xrightarrow{\text{CUBIC}} ax_0^3 + bx_0^2 + cx_0 + d
\]

or

\[
CUBIC(x_0) = ax_0^3 + bx_0^2 + cx_0 + d
\]

or

\((x_0, ax_0^3 + bx_0^2 + cx_0 + d)\) is an input-output pair for the function \( CUBIC \).

**Example 9.** Altogether, and depending on the circumstances, we can then write:

\[
-3 \xrightarrow{\text{THORA}} +1072.80
\]

or

\[
THORA(-3) = +1072.80
\]

or

\((-3, +1072.80)\) is an input-output pair for the function \( THORA \).

**13.2 Normalized Input-Output Rule**

Because, in the case of given cubic functions, we will almost always be able to carry out our investigations all the way to the quantitative stage, qualitative investigations might seem as if they will not make our life that much simpler. Still, they will allow us make “general” statements about cubic functions and that will be well worth the time.
1. In the case of cubic functions, the feature of the input-output rule that will be important to us will be:

- the sign of the cubic coefficient

In fact, though, there is another feature of the input-output rule that will turn out to be also very important to sort out cubic functions and that is the sign of the discriminant of a function tightly associated with the given function but we will let that discriminant come up naturally in the course of our investigations, that is, in the next chapter.

In other words, at this point, we will carry out our “general” investigations without concerning ourselves with the size of the cubic coefficient $a$ and with either the sign or the size of $b$, $c$ and $d$. Only the sign of $a$ will matter.

2. From the qualitative viewpoint, there will therefore be two types of cubic functions:

<table>
<thead>
<tr>
<th>Sign cubic coefficient</th>
<th>TYPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+$</td>
<td>$P$</td>
</tr>
<tr>
<td>$-$</td>
<td>$N$</td>
</tr>
</tbody>
</table>

**Example 10.** The function $MYLE$ specified by the global input-output rule

$$x \xrightarrow{MYLE} MYLE(x) = -16.93x^3 + 46.03x^2 - 56.18x + 73.92$$

is a cubic function whose input-output rule has the following feature

- The cubic coefficient is Negative,

So, the cubic function $MYLE$ is of type $N$.

3. Given an cubic function, we will normalize it by just normalizing the cubic coefficient as follows:

<table>
<thead>
<tr>
<th>If the cubic coefficient is:</th>
<th>we will normalize it to:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive</td>
<td>+1</td>
</tr>
<tr>
<td>Negative</td>
<td>-1</td>
</tr>
</tbody>
</table>

**Example 11.** The function $LOLA$ specified by the global input-output rule

$$x \xrightarrow{LOLA} LOLA(x) = -92.48x^3 + 24.48x^2 - 23.39x - 29.03$$

will be normalized to

$$x \xrightarrow{LOLA^*} LOLA^*(x) = -1x^3 + 24.48x^2 - 23.39x - 29.03$$

which, however, we will usually write

$$= -x^3 + 24.48x^2 - 23.39x - 29.03$$
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4. So, the two types of cubic functions have the following normalized input-output rules:

<table>
<thead>
<tr>
<th>TYPE</th>
<th>Normalized input-output rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>$x \xrightarrow{P} P(x) = +x^3 + bx^2 + cx + d$</td>
</tr>
<tr>
<td>N</td>
<td>$x \xrightarrow{N} N(x) = -x^3 + bx^2 + cx + d$</td>
</tr>
</tbody>
</table>

**NOTE.** It is worth keeping in mind that the $-$ in front of $x^3$ is better read as “opposite of”

LOCAL ANALYSIS NEAR INFINITY

13.3 Local Input-Output Rule Near Infinity

As with all functions, past, present and future, our first move in the investigation of cubic functions will usually be to get the local graph near $\infty$.

As was already the case with affine functions and quadratic functions and will be the case with all functions, given any cubic function $CUBIC$, it will be convenient to begin by specifying a new function whose local graph near $\infty$ will turn out to be the local graph near $\infty$ of $CUBIC$.

1. Given a cubic function $CUBIC_{a,b,c,d}$ that is a function specified by the global input-output rule

$$x \xrightarrow{CUBIC_{a,b,c,d}} CUBIC_{a,b,c,d}(x) = ax^3 + bx^2 + cx + d$$

the global input-output rule says:

i. multiply the cubic coefficient $a$ by three copies of the input $x$,

ii. add the quadratic term, that is the quadratic coefficient $b$ multiplied by two copies of the input $x$,

iii. add the linear term, that is the linear coefficient $c$ multiplied by one copy of the input $x$,

iv. add the constant term $d$, that is the constant coefficient $d$ multiplied by no copy of the input $x$.

We will now be using:

- The fact that a bounded number multiplied by three copies of a large-in-size number gives a result that is large-in-size,
- The fact that a bounded number multiplied by two copies of a large-in-size number gives a result that is large-in-size but not as large-in-size as
the preceding one so that, regardless of its sign, adding this number to the preceding one will not change the size of the result which will still be large-in-size,

- The fact that a bounded number multiplied by one copy of a large-in-size number gives a result that is large-in-size but not as large-in-size as the preceding one so that, regardless of its sign, adding this number to the preceding one will not change the size of the result which will still be large-in-size,

- The fact that, regardless of its sign, adding a bounded number to a large-in-size number will not change the size of the result which will still be large-in-size.

So, when the input $x$ is large-in-size, after we have multiplied the cubic coefficient $a$ by three copies of the input $x$, adding the quadratic term $bx^2$, the linear term $cx$ and the constant term $d$, regardless of their sign, will not change the size of the output.

In other words, in the case of cubic functions, multiplying the cubing coefficient by three copies of the input results in the principal term near $\infty$ of the output, that is the cubic term $ax^3$ makes up “most of the output” when the input is near $\infty$ and the quadratic part of the output, $bx^2 + cx + d$, can be safely ignored when the input is near $\infty$.

2. So, given a cubic function $CUBIC_{a,b,c,d}$, the new function that we will introduce,

$$PRINCIPAL\ \TERM\ \\ of\ \ CUBIC_{a,b,c,d}$$

will be the function whose output is the cubing term of $CUBIC_{a,b,c,d}$, that is the function specified by the global input-output rule

$$x \xrightarrow{PPL.\ \ TERM.\ \ \ of\ \ CUBIC_{a,b,c,d}} PPL.\ \ TERM.\ \ \ of\ \ CUBIC_{a,b,c,d}(x) = ax^3$$

3. However, even for large inputs, the output of $CUBIC$ is not quite equal to the output of $PPL\ \ TERM\ \ of\ \ CUBIC$ because, even for large inputs, while the quadratic part of $CUBIC$ is “too small to matter here”, it is not 0.

**Example 12.** $0.33$ is not equal to $\frac{1}{3}$ because $3 \times 0.33 = 0.99$ while $3 \times \frac{1}{3} = 1$ and so we can only write $\frac{1}{3} = 0.33 + [...]$

But, in mathematics, we do want to write equalities if only because they are easier to work with. So, in order to be able to write an equality, we will use again [...] as a shorthand for “something too small to matter here”.
We will then be able to write, completely truthfully,

\[ x \big|_{x \text{ near } \infty} \xrightarrow{\text{CUBIC}_{a,b,c,d}} \text{CUBIC}_{a,b,c,d}(x) \big|_{x \text{ near } \infty} = ax^3 + [...] \]

which we will of course call the approximate local input-output rule near \( \infty \) of the cubic function \( \text{CUBIC}_{a,b,c,d} \).

**Example 13.** Given the cubic function \( \text{MOAN}_{+93.37,-21.74,+31.59,-71.09} \), that is the function specified by the global input-output rule

\[ x \xrightarrow{\text{MOAN}} \text{MOAN}(x) = +93.37x^3 - 21.74x^2 + 31.59x - 71.09 \]

the function \( \text{PPL. TERM. of MOAN} \) is the function specified by the global input-output rule

\[ x \xrightarrow{\text{PPL. TERM. of MOAN}} \text{PPL. TERM. of MOAN}(x) = +93.37x^3 \]

and we can write truthfully

\[ x \big|_{x \text{ near } \infty} \xrightarrow{\text{MOAN}} \text{MOAN}(x) \big|_{x \text{ near } \infty} = +93.37x^3 + [...] \]

In other words, we have:

**Theorem 1 (Approximation Near \( \infty \)).** Near \( \infty \) the output of a cubic function is approximately the same as the output of its principal term.

### 13.4 Local Graph Near Infinity

Since we already investigated the local graph of the cubing function, the Approximation Near \( \infty \) Theorem then gives us a procedure by reduction, that is a procedure that reduces the current problem to a problem we have already been able to solve, for finding the local graph near \( \infty \) of affine functions:

- **i.** Get the approximate local input-output rule near \( \infty \)
- **ii.** Normalize it,
- **iii.** Draw its local graph near \( \infty \).

**Example 14.** Given the cubic function \( \text{ALMA}_{-38.38,-21.36,-45.78,+53.20} \), find its local graph near \( \infty \).

\( \text{ALBA}_{-21.36,-45.78,+53.20} \) is specified by the global input-output rule:

\[ x \xrightarrow{\text{ALMA}} \text{ALMA}(x) = -38.38x^3 - 21.36x^2 - 45.78x + 53.20 \]
i. The approximate local input-output rule near \( \infty \) is:

\[
x|_{x \text{ near } \infty}^{\text{ALMA}} \rightarrow \text{ALMA}(x)|_{x \text{ near } \infty} = -38.38x^3 + [...]
\]

ii. We normalize ALMA to ALMA*:

\[
x|_{x \text{ near } \infty}^{\text{ALMA}^*} \rightarrow \text{ALMA}^*(x)|_{x \text{ near } \infty} = -x^3 + [...]
\]

iii. The local graph near \( \infty \) of ALMA* is:

\[\text{Output Ruler} \quad \text{Input Ruler} \quad +\infty -\infty\]

\[\text{Screen} \quad \text{Output Ruler} \quad \text{Input Ruler} \quad +\infty -\infty\]

13.5 Types of Local Graphs Near Infinity

Since the local graph near \( \infty \) of a cubic function is approximately the graph of its principal term near \( \infty \), we get that the local graph near \( \infty \) of a cubic function is approximately either

- the function CUBING

\[
x \xrightarrow{\text{CUBING}} \text{CUBING}(x) = +x^3
\]

or

- the function OPP-CUBING

\[
x \xrightarrow{\text{OPP-CUBING}} \text{OPP-CUBING}(x) = -x^3
\]

They are shown in the table below as seen from two viewpoints:

i. As seen from “not too far”, that is we see the screen and only the part of the local graph near \( \infty \) that is near the transition, that is for inputs that are large but not “that” large so that we can still see the slope.

ii. As seen from “faraway”. Indeed, in order to see really large inputs, we need to be “faraway” and the parts of the local graph near \( \infty \) that we see are still straight and vertical.
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standard notation

<table>
<thead>
<tr>
<th>Input-output rule</th>
<th>From “not too far”</th>
<th>From “faraway”</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \xrightarrow{\text{CUBING}} \text{CUBING}(x) = +x^3$</td>
<td><img src="image1.png" alt="Graph" /></td>
<td><img src="image2.png" alt="Graph" /></td>
</tr>
<tr>
<td>$x \xrightarrow{\text{OPP-CUBING}} \text{OPP-CUBING}(x) = -x^3$</td>
<td><img src="image3.png" alt="Graph" /></td>
<td><img src="image4.png" alt="Graph" /></td>
</tr>
</tbody>
</table>

13.6 Local Features Near Infinity

As we saw in Chapter 3, Section 7, we can read the local features near $\infty$ off the local graph near $\infty$.

1. In order to state Local Feature Theorems as simply as possible, it will be convenient to use the standard notation in which we think of

\[ \cup \quad \text{and} \quad \cap \quad \text{as being negative, that is as} \]

\[ + \]

\[ - \]

2. Then, based on the local graphs in the previous section, we have:

**THEOREM 2 (Local Features Near $\infty$).** Given a cubic function $\text{CUBIC}_{a,b,c,d}$, the local features near $\infty$ are:

- Height-sign $\text{CUBIC} \mid_{x \nearrow \infty} = (\text{Sign } a, -\text{Sign } a,)$
- Slope-sign $\text{CUBIC} \mid_{x \nearrow \infty} = (\text{Sign } a, \text{Sign } a,)$
- Concavity-sign $\text{CUBIC} \mid_{x \nearrow \infty} = (\text{Sign } a, -\text{Sign } a,)$

**EXAMPLE 15.** Given the cubic function $\text{MAON}_{-38.82,+34.54,-40.38,-94.21}$, that is the function specified by the global input-output rule

\[ x \xrightarrow{\text{MAON}} \text{MAON}(x) = -38.82x^3 + 34.54x^2 - 40.38x - 94.21 \]

we find its local features near $\infty$ as follows.

i. The local features near $\infty$ according to the Local Features Theorem are:
13.6. **LOCAL FEATURES NEAR** $\infty$

- Height-sign $|_x$ near $\infty = (-, +)$
- Slope-sign $|_x$ near $\infty = (-, -)$
- Concavity-sign $|_x$ near $\infty = (-, +)$

which, translated back into our language, gives

- Height-sign $|_x$ near $\infty = (-, +)$
- Slope-sign $|_x$ near $\infty = (\backslash, \backslash)$
- Concavity-sign $|_x$ near $\infty = (\cup, \cap)$

### ii. Should we want to check that the **Local Features Theorem** gave us the correct information, we would get the normalized approximate local input-output rule near $\infty$

$$x|_x \text{ near } \infty \xrightarrow{\text{MAON}^*} \text{MAON}^*(x)|_x \text{ near } \infty = -x^3 + [...]$$

and then the local graph near $\infty$

![Local Features Graph](image)

from which we get the following local features

![Local Features Graph](image)

which are the same as those that were given by the **Local Features Theorem** keeping in mind that the local graph near $\infty$ is viewed here by **Mercator** rather than by **Magellan**.

**LOCAL ANALYSIS NEAR A FINITE INPUT**
13.7 Addition Formula For Cubes

In order to get the local input-output rule of a given cubic function near a given finite input \( x_0 \), we need first to get an addition formula to give us \((x_0 + h)^3\) in terms of \( x_0 \) and \( h \) and which we will call the addition formula for cubes. We prefer to establish it here, ahead of time, rather than in the midst of the developing the local input-output rule. There are two approaches.

1. The computational approach to the addition formula for cubes is to multiply three copies of \((x_0 + h)\):
   
   a. We begin by multiplying two copies of \((x_0 + h)\):
      
      \[
      \begin{array}{c}
      x_0 + h \\
      x_0 + h \\
      \hline
      x_0h + h^2 \\
      x_0^2 + x_0h \\
      x_0^2 + 2x_0h + h^2
      \end{array}
      \]

   b. We multiply \(x_0^2 + 2x_0h + h^2\), the result of the multiplication of two copies of \((x_0 + h)\), by a third copy of \((x_0 + h)\):
      
      \[
      \begin{array}{c}
      x_0 + h \\
      x_0 + h \\
      \hline
      x_0^2h + 2x_0h^2 + h^3 \\
      x_0^2 + 2x_0^2h + x_0h^2 \\
      x_0^3 + 3x_0^2h + 3x_0h^2 + h^3
      \end{array}
      \]

2. The graphic approach to the addition formula for cubes is to go back to the real world and to the definition of double multiplication in terms of the volume of a rectangular box so that \((x_0 + h)^3\) is the volume of a \(x_0 + h\) by \(x_0 + h\) by \(x_0 + h\) cube:

What we will do is to start with an \(x_0\) by \(x_0\) by \(x_0\) cube, the initial cube,
and then see how enlarging the sides of the cube square by \( h \) enlarges the volume of the cube. For the sake of clarity, we will enlarge the initial cube one step at a time:

i. The three sides of the initial cube are equal to \( x_0 \) and the volume of the initial cube is therefore \( x_0^3 \):

\[
\begin{align*}
\text{initial cube} & : x_0 \\
total volume & = x_0^3
\end{align*}
\]

ii. We now enlarge the initial cube with three \((x_0 + h)\) by \((x_0 + h)\) by \( h \) slabs:

\[
\begin{align*}
\text{enlarged cube} & : (x_0 + h) \\
total volume & = (x_0 + h)^3
\end{align*}
\]

We glue the three slabs to the initial cube which, however, leaves three groves:

iii. We fill the three groves with three \( x_0 \) by \( h \) by \( h \) rods:
We glue the three rods in the three groves which leaves an indentation in the corner:

iv. We finish the enlargement of the initial cube by adding one \( h \) by \( h \) by \( h \) cube to fill the indentation in the corner:

v. Altogether, the volume of the enlarged cube, whose side is \( x_0 + h \),
3. Contrary to what one might think at first, the **graphic approach** is greatly preferable because it has three major advantages over the **computational approach**:  

i. The terms in the sum automatically come in order of *diminishing sizes*. Indeed, since $x_0$ is *finite* and $h$ is *small*,

- all three dimensions of the “initial cube” are *finite*, say “in the ones”, so $x_0^3$ is also *finite* or “in the ones”,
- two dimensions of the “slabs” are *finite*, say “in the ones”, but the third dimension is *small*, say “in the tenths”, so the volume of the slabs is *small*: since $x_0$ is “in the ones” and $h$ is “in the tenths”, then $3x_0^2h$ will also be “in the tenths”,
- one dimension of the “rods” is finite, say “in the ones”, but the other two dimensions are *small*, say “in the tenths”, so the volume of the rods is *smaller*: since $x_0$ is “in the ones” and $h$ is “in the tenths”, then $3x_0h^2$ will be “in the hunfredths”,
- all three dimensions of the “little cube” are *small* so that the volume of the little cube is even smaller than the volume of the slabs: if $h$ is “in the tenths”, then $h^3$ will be “in the thousandths”.

ii. If all we need is only a particular one of the terms, and this will very often be the case, we can get it straight from the picture without having to go through the whole multiplication.

iii. Later on, when we shall need formulas for $(x_0 + h)^4$, $(x_0 + h)^5$, etc, not only will the length of the **computational approach** get very rapidly out of hand but, as we shall see, since we will never need more than the first two or three terms of the result, the computational approach will also become
more and more inefficient. On the other hand, even though we will not be able to draw pictures as we have been able to do so far, one can extend the patterns we have found so far in the graphic approach and the graphic approach will thus survive.

In any case, we have

**THEOREM 3 (Addition Formula For Cubes).**

\[(x_0 + h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3\]

### 13.8 Local Input-Output Rule Near A Finite Input

When we investigated power functions near 0, we replaced \(x\) by *small* in the global input-output rule and *computed* the size of the output using the Multiplication Theorem. Here, we will proceed as follows

i. We will replace \(x\) by \(x_0 + h\) (where \(h\) is *small*) in the global input-output rule

ii. We will use the Addition Formula For Squares and the Addition Formula for Cubes

iii. We will use the Multiplication Theorem to compute with \(h\)

In other words, even though we will want to be able to investigate cubic functions near *any* finite input \(x_0\), this will involve only one extra step, namely the use of addition formulas.

1. More precisely, given the cubic function \(CUBIC_{a,b,c,d}\), that is a function specified by the global input-output rule

\[x \xrightarrow{CUBIC} CUBIC(x) = ax^3 + bx^2 + cx + d\]

and given a finite input \(x_0\), we find the local input-output rule of \(CUBIC\) near \(x_0\) just as in the case of quadratic functions:

i. We *localize* the function \(CUBIC\) at the given input \(x_0\), that is we count inputs from the given input \(x_0\) instead of from the origin of the input ruler. In other words, we use the *location* of \(x\) in relation to the given input \(x_0\), that is we replace in the global input-output rule \(x\) by \(x_0 + h\), (where \(h\) is *small*).

\[x|_{x=x_0+h} \xrightarrow{CUBIC_{a,b,c,d}} CUBIC_{a,b,c,d}(x)|_{x=x_0+h} = ax^3 + bx^2 + cx + d\big|_{x=x_0+h} = a[x_0 + h]^3 + b[x_0 + h]^2 + c[x_0 + h] + d\]
13.8. LOCAL I-O RULE NEAR $X_0$ 393

ii. using the Addition Formula Theorem, we get

$$
= a \left[ x_0^3 + 3x_0^2 h + 3x_0h^2 + h^3 \right] + b \left[ x_0^2 + 2x_0h + h^2 \right] + c \left[ x_0 + h \right] + d \\
= ax_0^3 + 3ax_0^2h + 3ax_0h^2 + ah^3 + bx_0^2 + 2bx_0h + bh^2 + cx_0 + ch + d
$$

iii. Collecting “like terms”,

$$
= \left[ ax_0^3 + bx_0^2 + cx_0 + d \right] + \left[ 3ax_0^2 + 2bx_0 + c \right] h + \left[ 3ax_0 + b \right] h^2 + \left[ a \right] h^3
$$

2. Since $x_0$ is given for the duration of the local investigation and since it is therefore $h$ that is the actual input, we will often want to think in terms of the local function $CUBIC_{(x_0)}$ that is the function which returns for $h$ the same output that the global function $CUBIC$ would return for $x = x_0 + h$. The computation we just did gives us the local input-output rule that specifies the local function $CUBIC_{(x_0)}$:

**THEOREM 4 (Local Input-Output Rule).** Given the cubic function $CUBIC$ specified by the global input output rule

$$
x \xrightarrow{CUBIC} CUBIC(x) = ax^3 + bx^2 + cx + d
$$

the local function $CUBIC_{(x_0)}$ is specified by the local input-output rule

$$
h \xrightarrow{CUBIC_{(x_0)}} CUBIC_{x_0}(h) = \left[ ax_0^3 + bx_0^2 + cx_0 + d \right] + \left[ 3ax_0^2 + 2bx_0 + c \right] h + \left[ 3ax_0 + b \right] h^2 + \left[ a \right] h^3
$$

**EXAMPLE 16.** Given the cubic function $x \xrightarrow{BEN} BEN(x) = -2x^3 + 3x^2 - 5x + 7$, and given an input, say $x_0 = -4$, we get the local input-output rule near $x_0 = -4$ by localizing at $-4$ and then “just” do the computations as follows:

$$
-4 + h \xrightarrow{BEN} BEN(-4 + h) = -2(-4)^3 + 3(-4)^2 - 5(-4) + 7\bigg|_{x=-4+h}
= -2\left[ -4 + h \right]^3 + 3\left[ -4 + h \right]^2 - 5\left[ -4 + h \right] + 7
$$

and, using the Addition Formula Theorem with $x_0 = -4$,

$$
= -2\left[ (-4)^3 + 3(-4)^2h + 3(-4)h^2 + h^3 \right] + 3\left[ (-4)^2 + 2(-4)h + h^2 \right] - 5\left[ -4 + h \right] + 7 \\
= -2\left[ -64 + 48h - 12h^2 + h^3 \right] + 3\left[ 16 - 8h + h^2 \right] - 5\left[ -4 + h \right] + 7 \\
= +128 - 96h + 24h^2 - 2h^3 + 48 - 24h + 3h^2 + 20 - 5h + 7
$$
and, collecting "like" terms,
\[
= \left[ +203 \right] + \left[ -125 \right] h + \left[ +27 \right] h^2 + \left[ -2 \right] h^3
\]

So, the localization of \(BEN\) when \(x = -4\) is the function specified by the local input-ouput rule:
\[
h \xrightarrow{ZEN(-4)} ZEN_{-4}(h) = +203 - 125h + 27h^2 - 2h^3
\]

### 13.9 Local Coefficients Near A Finite Input

When looking only for one of the local features, instead of computing the whole local input-ouput rule, we will only compute the single term of the local input-output rule that controls the local feature. At first, getting just this one single term rather than the whole local input-output rule will look more difficult because it cannot be done in a humdrum manner but that will not last.

More precisely, given a function \(CUBIC\) specified by the global input-output rule:
\[
x \xrightarrow{CUBIC} CUBIC(x) = ax^3 + bx^2 + cx + d
\]

and whose local input-output rule near \(x_0\) is therefore
\[
h \xrightarrow{CUBIC(x_0)} CUBIC(x_0)|_{x=x_0+h} = \left[ ax_0^3 + bx_0^2 + cx_0 + d \right] + \left[ 3ax_0^2 + 2bx_0 + c \right] h + \left[ 3ax_0 + b \right] h^2 + \left[ a \right] h^3
\]

we will say that
- \(\left[ ax_0^3 + bx_0^2 + cx_0 + d \right]\) is the **local constant coefficient**,  
- \(\left[ 3ax_0^2 + 2bx_0 + c \right]\) is the **local linear coefficient**,  
- \(\left[ 3ax_0 + b \right]\) is the **local quadratic coefficient**,  
- \(\left[ a \right]\) is the **local cubic coefficient**.

and we will now investigate how to get just a single one of the local coefficients without getting the whole local input-output rule. At first, getting just this one single term rather than the whole local input-output rule will
13.9. **LOCAL COEFFICIENTS NEAR \(X_0\)**

look more difficult than getting the whole local input-output rule but, with a little bit of practice, writing less and less each time, this will soon get easy.

We begin by writing

\[
x|_{x=x_0+h} \xrightarrow{CUBIC_{a,b,c,d}} \left. CUBIC_{a,b,c,d}(x) \right|_{x=x_0+h} = \left. \frac{\partial}{\partial x} \right|_{x=x_0+h} \left( ax^3 + bx^2 + cx + d \right)
\]

\[
= a[x_0 + h]^3 + b[x_0 + h]^2 + c[x_0 + h] + d
\]

1. A first approach is to work from the *addition formulas* but its drawback will be readily apparent in that it requires that the addition formulas be readily available, one way or the other.

a. To find the local constant coefficient, we proceed as follows:

i. Since the **Addition Formula for Cubes** is:

\[
(x_0 + h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3
\]

the contribution of \(a[x_0 + h]^3\) to the local constant term will be \(a \cdot x_0^3h^0\)

ii. Since the **Addition Formula for Squares** is:

\[
(x_0 + h)^2 = x_0^2 + 2x_0h + h^2
\]

the contribution of \(b[x_0 + h]^2\) to the local constant term will be \(b \cdot x_0^2h^0\)

iii. The contribution of \(c[x_0 + h]\) to the constant term will be \(c \cdot x_0h^0\)

iv. The contribution of \(d\) to the local constant term will be \(d \cdot h^0\)

Altogether, the local constant term adds up to:

\[
a \cdot x_0^3h^0 + b \cdot x_0^2h^0 + c \cdot x_0h^0 + d \cdot h^0 = a x_0^3 h^0 + b x_0^2 h^0 + c x_0 h^0 + d h^0
\]

and therefore the local constant coefficient is:

\[
ax_0^3 + bx_0^2 + cx_0 + d
\]

b. To find the local linear coefficient, we proceed as follows

i. Since the **Addition Formula for Cubes** is:

\[
(x_0 + h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3
\]

the contribution of \(a[x_0 + h]^3\) to the local linear term will be \(a \cdot 3x_0^2h\)

ii. Since the **Addition Formula for Squares** is:

\[
(x_0 + h)^2 = x_0^2 + 2x_0h + h^2
\]

the contribution of \(b[x_0 + h]^2\) to the local linear term will be \(b \cdot 2x_0h\)
local cubing coefficient

iii. The contribution of \( c[x_0 + h] \) to the local linear term will be \( c \cdot h \)

iv. The contribution of \( d \) to the local linear term will be nothing.

Altogether, the linear term adds up to:

\[
a \cdot 3x_0^2h + b \cdot 2x_0h + c \cdot h = 3ax_0h^2 + 2bx_0h + ch
\]

\[
= \left[ 3ax_0^2 + 2bx_0 + c \right] h
\]

and therefore the local linear coefficient is:

\[
3ax_0^2 + 2bx_0 + c
\]

c. To find the local squaring coefficient, we proceed as follows:

i. Since the Addition Formula for Cubes is:

\[
(x_0 + h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3
\]

the contribution of \( a[x_0 + h]^3 \) to the local squaring term will be \( a \cdot 3x_0h^2 \)

ii. Since the Addition Formula for Squares is:

\[
(x_0 + h)^2 = x_0^2 + 2x_0h + h^2
\]

the contribution of \( b[x_0 + h]^2 \) to the local squaring term will be \( b \cdot h^2 \)

iii. The contribution of \( c[x_0 + h] \) to the local squaring term will be nothing

iv. The contribution of \( d \) to the local squaring term will be nothing.

Altogether, the squaring term adds up to:

\[
a \cdot 3x_0^2h^2 + b \cdot h^2 = 3ax_0h^2 + bh^2
\]

\[
= \left[ 3ax_0 + b \right] h^2
\]

and therefore the local squaring coefficient is:

\[
3ax_0 + b
\]

d. To find the local cubing coefficient we proceed as follows:

i. Since the Addition Formula for Cubes is:

\[
(x_0 + h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3
\]

the contribution of \( a[x_0 + h]^3 \) to the local cubing term will be \( a \cdot h^3 \)

ii. Since the Addition Formula for Squares is:

\[
(x_0 + h)^2 = x_0^2 + 2x_0h + h^2
\]

the contribution of \( b[x_0 + h]^2 \) to the local cubing term will be nothing
iii. The contribution of \( c[x_0 + h] \) to the local cubing term will be nothing.

iv. The contribution of \( d \) to the local cubing term will be nothing.

Altogether, the cubing term “adds up” to:

\[
a \cdot h^3 = ah^3 = \left[ a \right] h^3
\]

and therefore the local cubing coefficient is:

\[ a \]

2. Keeping in mind the pictures that are at the basis of the addition formulas, namely the pictures of the enlarging of a square and of a cube, allows us to compute the local coefficients a lot more efficiently by dispensing with writing the whole addition formulas and, instead, by focusing on the appropriate term by visualizing the corresponding piece(s) used to enlarge the base square or base cube.

a. To find the local constant coefficient, we proceed as follows:

i. We start with the \( h^0 \) term of an enlarged cube which is the measure of the original cube that is the volume of:

that is \( x_0^3 \) which we must multiply by the coefficient \( a \)

ii. To which we must add the \( h^0 \) term of an enlarged square which is the measure of the original square that is the area of:
that is $x_0^2$ which we must multiply by the coefficient $v$

**iii.** To which we must add the $h^0$ term of an *enlarged segment* which is the *measure* of the original segment that is the *length* of:

```
  x_0
  |   |
  |   |
  |   |
  x_0
```

that is $x_0^1$ which we must multiply by the coefficient $c$

**iv.** To which, finally, we must add $x_0^0 h^0$ which we must multiply by the coefficient $d$

Altogether, the $h^0$ terms add up to:

$$a \cdot x_0^3 h^0 + b \cdot x_0^2 h^0 + c \cdot x_0 h^0 + d \cdot h^0 = ax_0^3 h^0 + bx_0^2 h^0 + cx_0 h^0 + dh^0$$

and therefore the *local constant coefficient* is:

$$ax_0^3 + bx_0^2 + cx_0 + d$$

**b.** To find the *local linear coefficient*, we proceed as follows

**i.** We start with the $h^1$ term of an enlarged cube which is the measure of the three add-on slabs

```
  x_0
  |   |
  |   |
  |   |
  x_0
```

that is $3x_0^2 h$ which we must multiply by the coefficient $a$

**ii.** To which we must add the $h^1$ term of an enlarged square which is the measure of the two add-on rectangles
that is $2x_0^3h$ which we must multiply by the coefficient $b$

**iii.** To which we must add the $h^1$ term of an enlarged segment which is the measure of the add-on

that is $x_0^2h$ which we must multiply by the coefficient $c$

Altogether, the $h^1$ terms add up to:

$$a \cdot 3x_0^2h + b \cdot 2x_0h + c \cdot h = 3ax_0h^2 + 2bx_0h + ch$$

$$= \left[3ax_0^2 + 2bx_0 + c\right]h$$

and therefore the **local linear coefficient** is:

$$3ax_0^2 + 2bx_0 + c$$

**c.** To find the **local squaring coefficient**, we proceed as follows

**i.** We start with the $h^2$ term of the enlarged cube which is the measure of the next three add-ons

that is $3x_0h^2$ which we must multiply by the coefficient $a$
ii. To which we must add the $h^2$ term of the enlarged square which is the measure of the add-on square

that is $h^2$ which we must multiply by the coefficient $b$
Altogether, the $h^2$ terms add up to:

$$a \cdot 3x_0h^2 + b \cdot h^2 = 3ax_0h^2 + bh^2$$

and therefore the local squaring coefficient is:

$$3ax_0 + b$$

**Example 17.** Given the cubic function $x \rightarrow \text{BEN}(x) = -2x^3 + 3x^2 - 5x + 7$, and given the input $-4$, find the local squaring term.
We start as usual:

$$-4 + h \rightarrow \text{BEN}(-4 + h) = -2x^3 + 3x^2 - 5x + 7|_{x=-4+h}$$

$$= -2[-4 + h]^3 + 3[-4 + h]^2 - 5[-4 + h] + 7$$

now, instead of using the full Addition Formula Theorems, we just pick the $h^2$ terms in each addition formula:

$$= -2[\cdots + 3(-4)h^2 + \cdots] + 3[\cdots + h^2] - 5[\cdots] + \cdots$$

where $3(-4)h^2$ are the three rods in the enlarged cube and $+h^2$ is the little square in the enlarged square. So,

$$= -2[\cdots - 12h^2 \cdots] + 3[\cdots + h^2] - 5[\cdots] + \cdots$$

$$= \cdots + 24h^2 \cdots + \cdots + 3h^2 + \cdots + \cdots$$

and, collecting "like" terms,

$$= \left[ \cdots \right] + \left[ \cdots \right]h + \left[ +27 \right]h^2 + \left[ \cdots \right]h^3$$

So, the quadratic term in the local input-output rule of $\text{BEN}$ when $x$ is near $-4$ is:

$$+27h^2$$
d. To find the local cubing coefficient, we proceed as follows
i. We start with the $h^3$ term of the enlarged cube which is the measure of
the little cube add-on

To which there is nothing to add.
Altogether, the cubing term “adds up” to:

$$a \cdot h^3 = ah^3$$

and therefore the local cubing coefficient is:

$$a$$

13.10 Derivative Functions

We saw just above that the local constant coefficient for a cubic function $CUBIC(a, b, c, d)$ is:

$$ax_0^3 + bx_0^2 + cx_0 + d$$

It so happens, of course, that $ax_0^3 + bx_0^2 + cx_0 + d$ is the output returned by $CUBIC$ for the input $x_0$:

$$x_0 \xrightarrow{CUBIC} CUBIC(x) \mid_{x=x_0} = ax^3 + bx^2 + cx + d \mid_{x=x_0}$$

$$= ax_0^3 + bx_0^2 + cx_0 + d$$

In order for the other local coefficients also to be outputs of functions, we will automatically associate with $CUBIC$ several functions whose purpose will indeed be, given an input $x_0$, to output the local coefficients. We will call these functions the Derivative functions of the function $CUBIC$.

More precisely, since the local input-output rule of $CUBIC$ near $x_0$ is:

$$h \xrightarrow{CUBIC(x_0)} CUBIC(x_0) \mid_{x=x_0+h} = \left[ ax_0^3 + bx_0^2 + cx_0 + d \right] + \left[ 3ax_0^2 + 2bx_0 + c \right] h + \left[ 3ax_0 + b \right] h^2 + \left[ a \right] h^3$$

we will say that:
• The 1st Derivative of CUBIC is the function, which we will call CUBIC', that outputs the local linear coefficient, that is the coefficient of \( h^1 \) in the local input-output rule of CUBIC near \( x_0 \).

In other words, the function 1st Derivative CUBIC\(_{a,b,c,d}\) is specified by the global input-output rule

\[
x \rightarrow \text{1st Derivative CUBIC} \rightarrow \text{1st Derivative CUBIC}(x) = 3ax^2 + 2bx + c
\]

• 2nd Derivative CUBIC\(_{a,b,c,d}\) is the function, which we will call CUBIC'', that outputs twice the local squaring coefficient, that is twice the coefficient of \( h^2 \) in the local input-output rule of CUBIC near \( x_0 \).

In other words, the function 2nd Derivative CUBIC\(_{a,b,c,d}\) is specified by the global input-output rule

\[
x \rightarrow \text{2nd Derivative CUBIC} \rightarrow \text{2nd Derivative CUBIC}(x) = 2 \cdot (3ax + b)
\]

**NOTE.** We will explain below the reason for the weird multiplier twice.

• 3rd Derivative CUBIC\(_{a,b,c,d}\) is the function, which we will call CUBIC''', that outputs thrice twice the local cubing coefficient, that is thrice twice the coefficient of \( h^3 \) in the local input-output rule of CUBIC near \( x_0 \).

In other words, the function 3rd Derivative CUBIC\(_{a,b,c,d}\) is specified by the global input-output rule

\[
x \rightarrow \text{3rd Derivative CUBIC} \rightarrow \text{3rd Derivative CUBIC}(x) = 3 \cdot 2 \cdot a
\]

In fact, we will think of the original function CUBIC as being its own 0th Derivative because it completes the pattern inasmuch as it outputs the coefficient of \( h^0 \)

\[
x \rightarrow \text{0th Derivative CUBIC} \rightarrow \text{0th Derivative CUBIC}(x) = ax^3 + bx^2 + cx + d
\]

**NOTE.** Later on, Derivative functions will appear as the central concept in Differential Calculus and the reason for the weird multipliers, twice and thrice twice is because there it will make the computation of the derivative recursive:

• 2nd Derivative of CUBIC = 1st Derivative of 1st Derivative of CUBIC

• 3rd Derivative of CUBIC = 1st Derivative of 1st Derivative of 1st Derivative of CUBIC

• Etc
which will make it easier to compute derivative functions since there it involves a single procedure that needs only be repeated.

In this text, though, the main use of the derivative functions will be to facilitate writing a number of similar theorems and so the weird multipliers, twice and thrice twice, will in fact complicate our life but, fortunately, not too much.

13.11 Local Graph Near A Finite Input

The local graph of a given cubic function $CUBIC_{a,b,c,d}$ near a given input $x_0$ is the graph of the local function $CUBIC(x_0)$, that is the graph of the function specified by the local input-output rule of $CUBIC_{a,b,c,d}$ near $x_0$.

We will proceed essentially in the same manner as we did with quadratic functions except for one difference: here we will usually not have to all the way up to the cubic term.

More precisely, we will usually use only the quadratic part and we will not add-on the cubic term. The reason is quite simple: the cubic term is so small compared to the quadratic part that it cannot change the local graph qualitatively.

The exception is only when, for one reason or the other, the squaring term of the local input-output rule happens to vanish, that is the squaring coefficient turns out to be 0. In that case, as little as the cubic term contributes to concavity, since that is all there is that accounts for concavity, we must include the cubic term in order not to lose concavity as it would therefore get us a local graph that would be qualitatively different.

Observe that we need to use the cubic term only when it is the squaring term that vanishes. When it is the constant term or the linear term that vanishes, that does not change the local qualitatively because, as well as contributing to concavity, the squaring term also contributes both to the height and to the slope even though the contributions are small.

1. When the quadratic part is complete, in order to construct the graph of $CUBIC_{x_0}$,
   i. We will get the local graph of the constant term of $CUBIC_{x_0}$,
   ii. We get will the local graph of the linear term of $CUBIC_{x_0}$,
   iii. We will construct the local graph of the affine part of $CUBIC_{x_0}$ by adding-on the local graph of the linear term to the local graph of the constant term
   iv. We will get the local graph of the squaring term of $CUBIC_{x_0}$
   v. We will construct the local graph of $CUBIC_{x_0}$ by adding-on the local graph of the squaring term to the local graph of the affine part.
**Example 18.** Given a cubic function \( SHIP \) and given that the local input-output rule of \( SHIP \) near the input \(-4\) is

\[
 h \xrightarrow{SHIP(-4)} SHIP(-4)(h) = +75 - 29h + 3h^2 - 4h^3
\]

construct the local graph of \( SHIP \) near \(-4\), that is the graph of \( SHIP(-4) \).

i. We get the local graph of the constant term of \( SHIP(-4) \):

\[
 h \longrightarrow +75
\]

that is

![Constant Term Graph]

ii. We get the local graph of the linear term of \( SHIP(-4) \):

\[
 h \longrightarrow -29h
\]

that is

![Linear Term Graph]

iii. We get the affine part of \( SHIP(-4) \) by adding-on the linear term to the constant term:

\[
 h \longrightarrow +75 - 29h
\]

that is

![Affine Part Graph]
iv. We get the local graph of the squaring term of \( SHIP_{-4} \):

\[
\begin{align*}
    h & \rightarrow +3h^2 \\
\end{align*}
\]

that is

v. We get the full local graph of \( SHIP_{-4} \) by adding-on the squaring term to the affine part:

\[
\begin{align*}
    h & \rightarrow +75 - 29h + 3h^2 \\
\end{align*}
\]

that is

vi. And since the graph has concavity, we are done!

2. In the case when the squaring term happens to vanish, we will of course

i. get the local graph of the constant term of \( CUBIC_{x_0} \),

ii. get the local graph of the linear term of \( CUBIC_{x_0} \),
iii. construct the local graph of the affine part of $CUBIC_{x_0}$ by adding-on the local graph of the linear term to the local graph of the constant term
iv. get the local graph of the cubing term of $CUBIC_{x_0}$
v. construct the local graph of $CUBIC_{x_0}$ by adding-on the local graph of the cubing term to the local graph of the affine part.

3. In the case when it is the constant term and/or the linear term that vanishes we will of course
i. get the local graph of whatever term there is, if any,
ii. get the local graph of the squaring term of $CUBIC_{x_0}$
iii. construct the local graph of $CUBIC_{x_0}$ by adding-on the local graph of the squaring term to the local graph of whatever term, if any, has not vanished.

13.12 Local Features Near A Finite Input

Once we have the local graph of the function $CUBIC$ near $x_0$, we can “read” the local features off the local graph as discussed in Chapter 3. This is indeed a good way to proceed initially because it is “visual”.

However, the local graph acts only as an intermediary between the local input-output rule and the local features and it is really the coefficients in the local input-output rule that control the local features so that, if all we want is only one particular local feature, it is very inefficient to have to compute the whole local input-output rule to get the local graph. Indeed,

- Why compute coefficients that control features we don’t need?
- As we investigate more functions, computing the whole local input-output rule is rapidly turning out to be more and more labor-intensive, if only because the addition formulas get to be longer and longer and more and more complicated but for other reasons as well, as we will see with rational functions.

More precisely, given the cubic function $CUBIC_{a,b,c,d}$, that is a function specified by the global input-output rule

$$x \xrightarrow{CUBIC} CUBIC(x) = ax^3 + bx^2 + cx + d$$

what controls the local features of $CUBIC$ near $x_0$ is the quadratic part of the local input-output rule near $x_0$

$$h \xrightarrow{CUBIC(x_0)} CUBIC(x_0)|_{x=x_0+h}$$

$$= \left[ ax_0^3 + bx_0^2 + cx_0 + d \right] + \left[ 3ax_0^2 + 2bx_0 + c \right] h + \left[ 3ax_0 + b \right] h^2 + [...]$$

This is the local graph of the quadratic term near $x_0$.
and therefore it is the coefficients of the local quadratic part, that is the three lowest exponent terms that each control a particular local feature:

Since the local coefficients are the output of the derivative functions of CUBIC (up to the weird multipliers but since we will only be interested in the sign of the derivative functions, the weird multipliers will not matter), we will state the resulting theorems in terms of the Derivative functions of CUBIC so as to display the overall pattern.

1. The local constant term

\[ \left[ ax_0^3 + bx_0^2 + cx_0 + d \right] \]

gives most of the local height. This is because, as long as \( h \) remains small, that is as long as \( x \) remains near \( x_0 \), the other three terms of the local input-output rule, namely the linear term, the squaring term and the cubing term contribute very little height to the total local height and certainly not enough to change the sign of the total local height.

The local constant coefficient is therefore what determines the height-sign near any finite input \( x_0 \) which will therefore be determined by the sign of \( \text{CUBIC} (x_0) \) that is by the sign of 0th Derivative \( \text{CUBIC} (x_0) \):

**THEOREM 5 (Height-sign Near \( x_0 \)).** For any quadratic function \( \text{CUBIC}_{a,b,c} \):

- When \( \text{CUBIC}(x_0) = +, \) Height-sign \( \text{CUBIC} \mid_{\text{near } x_0} = (+, +) \)
- When \( \text{CUBIC}(x_0) = -, \) Height-sign \( \text{CUBIC} \mid_{\text{near } x_0} = (-, -) \)
- When \( \text{CUBIC}(x_0) = 0, \) Height-sign \( \text{CUBIC} \mid_{\text{near } x_0} \) is given by the sign of 1st Derivative \( \text{CUBIC}(x_0) \)

In other words:

- When 0th Derivative \( \text{CUBIC}(x_0) = +, \) Height-sign \( \text{CUBIC} \mid_{\text{near } x_0} = (+, +) \)
- When 0th Derivative \( \text{CUBIC}(x_0) = -, \) Height-sign \( \text{CUBIC} \mid_{\text{near } x_0} = (-, -) \)
- When 0th Derivative \( \text{CUBIC}(x_0) = 0, \) Height-sign \( \text{CUBIC} \mid_{\text{near } x_0} \) is given by the sign of 1st Derivative \( \text{CUBIC}(x_0) \)

2. The local linear term

\[ \left[ 3ax_0^2 + 2bx_0 + c \right] h \]
is what controls the local slope. This is because the constant term has no slope and both the squaring term and the cubing term contribute very little to the total slope—as long as \( h \) remains small—and certainly not enough to change the sign of the total slope.

The local linear coefficient is therefore what determines the slope-sign near any finite input \( x_0 \) which will be determined by the sign of 1st Derivative \( \text{CUBIC}(x_0) \):

**THEOREM 6 (Slope-sign Near \( x_0 \)).** For any cubic function \( \text{CUBIC}_{a,b,c,d} \):

- **When** 1st Derivative \( \text{CUBIC}(x_0) = + \), Slope-sign \( \text{CUBIC} \mid_{\text{near } x_0} = (/, /) \)
- **When** 1st Derivative \( \text{CUBIC}(x_0) = - \), Slope-sign \( \text{CUBIC} \mid_{\text{near } x_0} = (\\backslash, \\backslash) \)
- **When** 1st Derivative \( \text{CUBIC}(x_0) = 0 \), Slope-sign \( \text{CUBIC} \mid_{\text{near } x_0} \) is given by the sign of 2nd Derivative \( \text{CUBIC}(x_0) \)

Using the standard notation, that is thinking of

| \( / \) and \( \backslash \) as being positive, that is as 0
| \( + \)
| \( \\backslash \) and \( \cap \) as being negative, that is as -

allows for a much nicer statement of the theorem, one that brings out the similarity of the Slope-sign Near \( x_0 \) Theorem with the Height-sign Near \( x_0 \) Theorem:

**THEOREM 6 (Slope-sign Near \( x_0 \))** For any cubic function \( \text{CUBIC}_{a,b,c,d} \):

- **When** 1st Derivative \( \text{CUBIC}(x_0) = + \), Slope-sign \( \text{CUBIC} \mid_{\text{near } x_0} = (+, +) \)
- **When** 1st Derivative \( \text{CUBIC}(x_0) = - \), Slope-sign \( \text{CUBIC} \mid_{\text{near } x_0} = (-, -) \)
- **When** 1st Derivative \( \text{CUBIC}(x_0) = 0 \), Slope-sign \( \text{CUBIC} \mid_{\text{near } x_0} \) is given by the sign of 2nd Derivative \( \text{CUBIC}(x_0) \)

3. The local squaring term

\[
\left[ 3ax_0 + b \right] h^2
\]

is what controls the local concavity. This is because both the constant term and the linear term have no concavity and the cubic term contributes very little to the total concavity—as long as \( h \) remains small—and certainly not enough to change the sign of the total concavity.
The local squaring coefficient is therefore what determines the concavity-sign near any finite input \( x_0 \) which will be determined by the sign of 2\(^{\text{nd}}\) Derivative \( \text{CUBIC}(x_0) \):

**THEOREM 7 (Concavity-sign Near } x_0).** For any cubic function \( \text{CUBIC}_{a,b,c,d} \):

- **When** 2\(^{\text{nd}}\) Derivative \( \text{CUBIC}(x_0) = + \), Concavity-sign \( \text{CUBIC}_{\mid_{\text{near } x_0}} = (\cup, \cup) \)
- **When** 2\(^{\text{nd}}\) Derivative \( \text{CUBIC}(x_0) = - \), Concavity-sign \( \text{CUBIC}_{\mid_{\text{near } x_0}} = (\cap, \cap) \)
- **When** 2\(^{\text{nd}}\) Derivative \( \text{CUBIC}(x_0) = 0 \), Concavity-sign \( \text{CUBIC}_{\mid_{\text{near } x_0}} \) is given by the sign of 3\(^{\text{rd}}\) Derivative \( \text{CUBIC}(x_0) \)

Using the standard notation, that is thinking of

\[
\begin{array}{c|c}
\text{\textbackslash} & \cup \text{ and } \cup \text{ as being positive, that is as} \\
+ & + \\
\text{\textbackslash} & \cap \text{ and } \cap \text{ as being negative, that is as} \\
- & - \\
\end{array}
\]

allows for a very striking statement of the Concavity-sign Near } x_0 Theorem which brings out the similarity with the Height-sign Near } x_0 Theorem and the Slope-sign Near } x_0 Theorem:

**THEOREM 6 (Concavity-sign Near } x_0).** For any cubic function \( \text{CUBIC}_{a,b,c,d} \):

- **When** 2\(^{\text{nd}}\) Derivative \( \text{CUBIC}(x_0) = + \), Concavity-sign \( \text{CUBIC}_{\mid_{\text{near } x_0}} = (+, +) \)
- **When** 2\(^{\text{nd}}\) Derivative \( \text{CUBIC}(x_0) = - \), Concavity-sign \( \text{CUBIC}_{\mid_{\text{near } x_0}} = (-, -) \)
- **When** 2\(^{\text{nd}}\) Derivative \( \text{CUBIC}(x_0) = 0 \), Concavity-sign \( \text{CUBIC}_{\mid_{\text{near } x_0}} \) is given by the sign of 3\(^{\text{rd}}\) Derivative \( \text{CUBIC}(x_0) \).