Rational functions are functions whose global input-output rule is of the form

\[ x \xrightarrow{RAT} RAT(x) = \frac{POLY_{Num}(x)}{POLY_{Den}(x)} \]

where \( POLY_{Num}(x) \) and \( POLY_{Den}(x) \) stand for two positive-exponent polynomial expressions.

**Example 1.** The function whose global input-output rule is

\[ x \xrightarrow{TAB} TAB(x) = \frac{-3x^2 + 4x - 7}{-5x^4 - 8} \]

is a rational function in which:
- \( POLY_{Num}(x) \) is \(-3x^2 + 4x - 7\)
- \( POLY_{Den}(x) \) is \(-5x^4 - 8\)

1. This definition of a rational function actually encompasses more functions than might at first appear and we will now look at functions whose global input-output rule would not seem to qualify them as rational functions but which are in fact rational functions.

a. Occasionally, either one or both of the two polynomial expressions \( POLY_{Num}(x) \) and \( POLY_{Den}(x) \) may happen to be factored.
EXAMPLE 2. The function whose global input-output rule is

\[ x \xrightarrow{MAB} MAB(x) = \frac{-3x^2 + 4x - 7}{(-5x^4 - 8)^2} \]

is in fact a rational function because

\[ (-5x^4 - 8)^2 = 25x^8 + 80x^4 + 64 \]

so that

\[ MAB(x) = \frac{-3x^2 + 4x - 7}{(-5x^4 - 8)^2} = \frac{-3x^2 + 4x - 7}{25x^8 + 80x^4 + 64} \]

b. A function which outputs a rational expression plus a polynomial expression is in fact a rational function just like a “mixed number” is in fact a fraction.

EXAMPLE 3. The function whose global input-output rule is

\[ x \xrightarrow{MAT} MAT(x) = \frac{7x^2 - 2}{-5x + 3} - 2x^3 + 4 \]

is in fact a rational function because

\[-2x^3 + 4 = (-2x^3 + 4) \cdot 1 = (-2x^3 + 4) \cdot \frac{-5x + 3}{-5x + 3} = (-2x^3 + 4) \cdot (-5x + 3) = -5x + 3 = +10x^4 - 6x^3 - 20x + 12 \]

so that

\[ MAT(x) = \frac{+7x^2 - 2}{-5x + 3} - 2x^3 + 4 = \frac{+7x^2 - 2}{-5x + 3} + \frac{+10x^4 - 6x^3 - 20x + 12}{-5x + 3} = \frac{+10x^4 - 6x^3 + 7x^2 - 20x + 10}{-5x + 3} \]

c. The requirement that \( POLY_{Num}(x) \) and \( POLY_{Den}(x) \) be positive-exponent polynomial expressions is merely intended to normalize things and is not crucial as global input-output rules that involve negative exponents can
easily be changed into input-output rules involving only positive-exponent exponents.

**Example 4.** The function whose global input-output rule is

\[ x \xrightarrow{\text{RAB}} \text{RAB}(x) = \frac{+3x^2 - 2x^{-3}}{-5x + 3} \]

is in fact a rational function because

\[
\begin{align*}
\frac{+3x^2 - 2x^{-3}}{-5x + 3} &= \frac{+3x^2 - 2x^{-3}}{-5x + 3} \cdot 1 \\
&= \frac{+3x^2 - 2x^{-3}}{-5x + 3} \cdot \frac{x^3}{x^3} \\
&= \frac{(+3x^2 - 2x^{-3}) \cdot (x^3)}{(-5x + 3) \cdot (x^3)} \\
&= \frac{+3x^5 - 2}{-5x^4 + 3x^5}
\end{align*}
\]

so that

\[ \text{RAB}(x) = \frac{+3x^5 - 2}{-5x^4 + 3x^5} \]

2. Rational functions are important for a number of reasons but, essentially, these reasons all derive from the fact that positive-exponent polynomial functions are, in fact, too simple because, since positive-exponent polynomial functions have no \( \infty \)-height inputs, positive-exponent polynomial functions are essentially regular positive-exponent power functions with possibly a few bounded fluctuations thrown in.

By contrast, the graphs of rational functions will come in an almost bewildering variety of shapes, essentially because of the possible existence of \( \infty \)-height inputs and thus rational functions will turn out to have much more varied and interesting behaviors.

**Example 5.** Rational functions are extremely important in *imaging* because rational functions can have infinite-height inputs and can therefore have bounded inputs with infinite slope. The reason this is important is that, given a function that outputs a given color, programs recognize the boundary of objects of that color by looking for where the slope is very high, that is where the color changes abruptly.

3. Rational functions are to polynomial functions very much what fractions are to whole numbers. And just like fractions are approximated by decimal numbers, rational functions will be approximated by (Laurent) polynomial functions.
15.1 Rational Degree

Because the upper degree of polynomial functions is what we used to sort polynomial functions into different types, we now try to extend the idea of upper degree to the case of rational functions in the hope that this will also help us sort rational functions into different types.

Given a rational function whose global input-output rule is

\[ x \xrightarrow{\text{RAT}} \text{RAT}(x) = \frac{\text{POLY}_{\text{Num}}(x)}{\text{POLY}_{\text{Den}}(x)} \]

the rational degree of this rational function is the upper degree of \( \text{POLY}_{\text{Num}}(x) \) minus the upper degree of \( \text{POLY}_{\text{Den}}(x) \):

\[
\text{Rat.Deg. of } \frac{\text{POLY}_{\text{Num}}(x)}{\text{POLY}_{\text{Den}}(x)} = \text{UppDeg. of } \text{POLY}_{\text{Num}}(x) - \text{UppDeg. of } \text{POLY}_{\text{Den}}(x)
\]

Thus, the rational degree of a rational function can well be negative.

**Note.** The rational degree is to rational function very much what the size is to arithmetic fractions in “school arithmetic” which distinguishes fractions according to the size of the numerator compared to the size of the denominator even though, by now, the distinctions are only an inconsequential remnant of history.

What happened is that, historically, the earliest arithmetic fractions were “unit fractions”, that is reciprocals of whole numbers such as one half, one third, one quarter, etc. Later came “Egyptian fractions”, that is combinations of (distinct) unit fractions, such as one third and one fifth and one eleventh, etc. A much later development were the “proper fractions”, also called "vulgar fractions", such as two thirds, three fifths etc. Later still, came "improper fractions" such as five thirds, seven halves, etc. And finally "mixed numbers", such as three and two sevenths. Today, none of these distinctions matters inasmuch as we treat all fractions in the same manner.

However, while these “school arithmetic” distinctions are based on the size of the numerator versus the size of the denominator and make no real differences in the way we handle arithmetic fractions, in the case of rational functions, the above distinction based on the upper degree of the numerator versus the upper degree of the denominator will make a difference—even though no major one—in the way we will handle rational functions of different types.

In fact, by analogy with power functions and for a reason that will appear shortly, we will separate rational functions into the following four types:

- **Type “Dº > 1”** : Rational functions whose rational degree is > 1.
- **Type “Dº < 0”** : Rational functions whose rational degree is < 0.
- **Type “Dº = 0”** : Rational functions whose rational degree is = 0.
- **Type “Dº = 1”** : Rational functions whose rational degree is = 1.

Again by analogy with what we did with power functions, we will say that
• Rational functions of type ">1" and rational functions of type "<0", that is rational functions whose rational degree is either > 1 or < 0, are regular rational functions,
• Rational functions of type "= 0" and rational functions of type "= 1", that is rational functions whose rational degree is either = 0 or = 1, are exceptional rational functions.

and we will deal separately with regular rational functions and exceptional rational functions in the last two chapters.

**Example 6.** Find the rational degree of the function DOUGH whose global input-output rule is
\[
x \xrightarrow{DOUGH} DOUGH(x) = \frac{+1x^4 - 6x^3 + 8x^2 + 6x - 9}{x^2 - 5x + 6}
\]

Since the rational degree is given by
\[
\text{Rat.Deg. of } \frac{POLY_{num}(x)}{POLY_{den}(x)} = \text{UppDeg. of } POLY_{num}(x) - \text{UppDeg. of } POLY_{den}(x)
\]

and since, here,
• \(POLY_{num}(x) = +1x^4 - 6x^3 + 8x^2 + 6x - 9\)
• \(POLY_{den}(x) = +1x^2 - 5x + 6\)

we get from the definition of the upper degree of a polynomial that:
\[
\text{UppDeg. of } +1x^4 - 6x^3 + 8x^2 + 6x - 9 = \text{Exponent of Highest Term} \\
= \text{Exponent of } +1x^4 \\
= 4
\]
\[
\text{UppDeg. of } +1x^2 - 5x + 6 = \text{Exponent of Highest Term} \\
= \text{Exponent of } +1x^2 \\
= 2
\]

so that the rational degree of the rational function DOUGH is:
\[
\text{Rat.Deg. of } \frac{+1x^4 - 6x^3 + 8x^2 + 6x - 9}{+1x^2 - 5x + 6} = \text{Exponent of } +1x^4 - \text{Exponent of } +1x^2 \\
= 4 - 2 \\
= 2
\]

so that DOUGH is an example of a rational function of degree > 1 and therefore of a regular rational function.

**Example 7.** Find the rational degree of the function MEGH whose global input-output rule is
\[
x \xrightarrow{MEGH} MEGH(x) = \frac{x^2 - 5x + 6}{+1x^4 - 6x^3 + 8x^2 + 6x - 9}
\]
Since the rational degree is given by
\[
\text{Rat. Deg. of } \frac{\text{POLY}_{\text{Num}}(x)}{\text{POLY}_{\text{Den}}(x)} = \text{UppDeg. of } \text{POLY}_{\text{Num}}(x) - \text{UppDeg. of } \text{POLY}_{\text{Den}}(x)
\]
and since, here,
- \(\text{POLY}_{\text{Num}}(x) = +1x^2 - 5x + 6\)
- \(\text{POLY}_{\text{Den}}(x) = +1x^4 - 6x^3 + 8x^2 + 6x - 9\)
we get from the definition of the upper degree of a polynomial that:
\[
\text{UppDeg. of } +1x^2 - 5x + 6 = \text{Exponent of Highest Term} \\
= \text{Exponent of } +1x^2 \\
= 2
\]
\[
\text{UppDeg. of } +1x^4 - 6x^3 + 8x^2 + 6x - 9 = \text{Exponent of Highest Term} \\
= \text{Exponent of } +1x^4 \\
= 4
\]
so that the rational degree of the rational function \(\text{MEGH}\) is:
\[
\text{Rat. Deg. of } \frac{+1x^2 - 5x + 6}{+1x^4 - 6x^3 + 8x^2 + 6x - 9} = \text{Exponent of } +1x^2 - \text{Exponent of } +1x^4 \\
= 2 - 4 \\
= -2
\]
so that \(\text{MEGH}\) is an example of a rational function of degree \(<0\) and therefore of a regular rational function.

**Example 8.** Find the rational degree of the function \(\text{LEIGH}\) whose global input-output rule is
\[
x \xrightarrow{\text{LEIGH}} \text{LEIGH}(x) = \frac{-17x^5 - x^4 - 5x^3 - 23x + 4}{8x^5 - x^3 + 4x^2 + 13x - 2}
\]
Since the rational degree is given by
\[
\text{Rat. Deg. of } \frac{\text{POLY}_{\text{Num}}(x)}{\text{POLY}_{\text{Den}}(x)} = \text{UppDeg. of } \text{POLY}_{\text{Num}}(x) - \text{UppDeg. of } \text{POLY}_{\text{Den}}(x)
\]
and since, here,
- \(\text{POLY}_{\text{Num}}(x) = -17x^5 - x^4 - 5x^3 - 23x + 4\)
- \(\text{POLY}_{\text{Den}}(x) = +8x^5 - x^3 + 4x^2 + 13x - 2\)
15.1. RATIONAL DEGREE

we get from the definition of the upper degree of a polynomial that:

\[
\text{UppDeg. of } -17x^5 - x^4 - 5x^3 - 23x + 4 = \text{Exponent of Highest Term}
\]
\[
= \text{Exponent of } -17x^5
\]
\[
= 5
\]

\[
\text{UppDeg. of } + 8x^5 - x^3 + 4x^2 + 13x - 2 = \text{Exponent of Highest Term}
\]
\[
= \text{Exponent of } +8x^5
\]
\[
= 5
\]

so that the rational degree of the rational function \textit{LEIGH} is:

\[
\text{Rat.Deg. of } -17x^5 - x^4 - 5x^3 - 23x + 4 + 8x^5 - x^3 + 4x^2 + 13x - 2 = \text{Exponent of } + 1x^2 - \text{Exponent of } +1x^4
\]
\[
= 5 - 5
\]
\[
= 0
\]

so that \textit{LEIGH} is an example of a rational function of rational degree \(= 0\) and therefore of an exceptional rational function.

\textbf{Example 9.} Find the rational degree of the function \textit{ROUGH} whose global input-output rule is

\[
x \xrightarrow{ROUGH} ROUGH(x) = \frac{-17x^6 - x^5 - 5x^4 - 2x^2 + 3x + 4}{+8x^5 - x^3 + 4x^2 + 13x - 2}
\]

Since the rational degree is given by

\[
\text{UppDeg. of } \frac{POLY_{\text{Num}}(x)}{POLY_{\text{Den}}(x)} = \text{UppDeg. of } POLY_{\text{Num}}(x) - \text{UppDeg. of } POLY_{\text{Den}}(x)
\]

and since, here,

\[
\begin{align*}
POLY_{\text{Num}}(x) &= -17x^6 - x^5 - 5x^4 - 2x^2 + 3x + 4 \\
POLY_{\text{Den}}(x) &= +8x^5 - x^3 + 4x^2 + 13x - 2
\end{align*}
\]

we get from the definition of the upper degree of a polynomial that:

\[
\text{UppDeg. of } -17x^6 - x^5 - 5x^4 - 2x^2 + 3x + 4 = \text{Exponent of Highest Term}
\]
\[
= \text{Exponent of } -17x^6
\]
\[
= 6
\]

\[
\text{UppDeg. of } + 8x^5 - x^3 + 4x^2 + 13x - 2 = \text{Exponent of Highest Term}
\]
\[
= \text{Exponent of } +8x^5
\]
\[
= 5
\]

so that the rational degree of the rational function \textit{ROUGH} is:

\[
\text{Rat.Deg. of } \frac{-17x^6 - x^5 - 5x^4 - 2x^2 + 3x + 4}{+8x^5 - x^3 + 4x^2 + 13x - 2} = \text{Exponent of } + 1x^2 - \text{Exponent of } +1x^4
\]
\[
= 6 - 5
\]
\[
= 1
\]
so that \( \text{ROUGH} \) is an example of a rational function of rational degree \( = 1 \) and therefore of an \textit{exceptional rational function}.

\section*{15.2 Addition Formulas: \((x_0 + h)^n \) when \( n > 3 \)}

While we did not investigate polynomial functions of degree higher than 3 we will investigate rational functions that involve power functions of degree higher than 3 and therefore we need to investigate \textit{Addition Formulas} for exponents higher than 3.

\textbf{1.} We already saw two \textit{addition formulas}:

\begin{enumerate}
\item[i.] The addition formula for exponent 2 as given in \textbf{Chapter 10} by \textbf{THEOREM 32} (\textit{Addition Formula For Squares}):
\[
(x_0 + h)^2 = x_0^2 + 2x_0h + h^2
\]

\item[ii.] The addition formula for exponent 3 as given in \textbf{Chapter 12} by \textbf{THEOREM 40} (\textit{Addition Formula For Cubes}):
\[
(x_0 + h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3
\]

\end{enumerate}

which we established both by repeated multiplication and by enlarging the sides of a \textit{square} and of a \textit{cube} of sides \( x_0 \) by a small amount \( h \).

\textbf{2.} In the case of exponents higher than 3, though,

\begin{itemize}
\item Repeated-multiplication is going to get rapidly more and more painful as the exponent goes up,
\item We cannot draw pictures in dimensions higher than 3.
\end{itemize}

So, in order to find a \textit{general procedure} for constructing the addition formulas for \((x_0 + h)^n \) when the exponent \( n \) is higher than 3, we will first try to look for a \textbf{pattern} in each one of the above two addition formulas without letting anything “go without saying” and see what \textit{particular procedure} would give each one of them.

\textbf{a.} When the \textit{exponent} is 3, we have:

\[
(x_0 + h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3
\]

that is, writing in all that “goes without saying”,

\[
= \sum_{i=0}^{3} x_0^{3-i} h^i = \sum_{i=0}^{3} \binom{3}{i} x_0^{3-i} h^i
\]

that is, replacing the short hand by the long hand

\[= \prod_{i=0}^{3} x_0 \]
So, looking separately at the powers and at the coefficients in the addition formula for \((x_0 + h)^3\):

- The procedure for finding the powers would seem to be:
  
  i. We make a row of 3 copies of \(x_0\) with multiplication signs in-between:
      \[
      x_0 \cdot x_0 \cdot x_0
      \]
  
  ii. We tack on 3 copies of the row we wrote in step i.
      \[
      x_0 \cdot x_0 \cdot x_0
      \]
      \[
      x_0 \cdot x_0 \cdot x_0
      \]
      \[
      x_0 \cdot x_0 \cdot x_0
      \]
  
  iii. Starting with the first of the 3 rows that we tacked-on in step ii. and going down, we replace in each row, starting from the right, one more copy of \(x_0\) by a copy of \(h\) than in the preceding row:
      \[
      x_0 \cdot x_0 \cdot x_0
      \]
      \[
      x_0 \cdot x_0 \cdot h
      \]
      \[
      x_0 \cdot h \cdot h
      \]
      \[
      h \cdot h \cdot h
      \]

- The procedure for finding the coefficients

  \[
  \begin{array}{cccc}
  1 & 3 & 3 & 1 \\
  \end{array}
  \]

  is certainly not obvious and we leave it alone for the time being.

b. When the exponent is 2, we have

\[(x_0 + h)^2 = x_0^2 + 2x_0h + h^2\]

that is, writing in all that “goes without saying”,

\[
= 1x_0^2h^0 + 2x_0^1h^1 + 1x_0^0h^2
\]

that is, replacing the short hand by the long hand

\[
= 1 \cdot x_0 \cdot x_0 \\
+ 2 \cdot x_0 \cdot h \\
+ 1 \cdot h \cdot h
\]

So, looking separately at the powers and at the coefficients in the addition formula for \((x_0 + h)^2\):
• The procedure for finding the powers would seem to be the same as for the previous addition formula:

i. We make a row of 2 copies of \( x_0 \) with multiplication signs in-between:

\[
\begin{align*}
  &x_0 \cdot x_0 \\
\end{align*}
\]

ii. We tack on 2 copies of the row we wrote in step i.

\[
\begin{align*}
  &x_0 \cdot x_0 \\
  &x_0 \cdot x_0 \\
\end{align*}
\]

iii. Starting with the first of the 2 rows that we tacked-on in step ii. and going down, we replace in each row, starting from the right, one more copy of \( x_0 \) by a copy of \( h \) than in the preceding row:

\[
\begin{align*}
  &x_0 \cdot x_0 \\
  &x_0 \cdot h \\
  &h \cdot h \\
\end{align*}
\]

• The procedure for finding the coefficients is certainly not obvious and we leave it alone for the time being.

c. Finally, for the sake of completion, we look at what happens when the exponent is 1. We have

\[
(x_0 + h)^1 = x_0 + h
\]

that is, writing in all that “goes without saying”,

\[
= 1x_0^1h^0 + 1x_0^0h^1
\]

that is, replacing the short hand by the long hand

\[
= 1 \cdot x_0 \\
+ 1 \cdot h
\]

So, looking separately at the powers and at the coefficients in the “addition formula” for \((x_0 + h)^1\):

• The procedure for finding the powers would seem to be the same as for the previous addition formulas:

i. We make a row of 1 copy of \( x_0 \) with no room for multiplication signs in-between:

\[
x_0
\]

ii. We tack on 1 copy of the row we wrote in step i.
iii. Starting with the first—and only—row that we wrote in step ii., we replace in that row, the one copy of $x_0$ by a copy of $h$:

- The procedure for finding the coefficients

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

is certainly not obvious and we leave it alone for the time being.

3. Putting everything together, the general procedure for constructing the addition formula for $(x_0 + h)^n$ would seem to be,

- The procedure for finding the powers seems to be in all cases:
  i. We make a row of $n$ copies of $x_0$ with multiplication signs in-between:
  ii. We tack on $n$ copies of the row we got in step i.
  iii. Starting with the first of the $n$ rows that we wrote in step ii. and going down, we replace in each row, starting from the right, one more copy of $x_0$ by a copy of $h$ than in the preceding row:
- So far, the procedures for the coefficients were not obvious but when we write them starting with exponent 1 and ending with exponent 3

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$n$</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

we can see that we can get each entry in what is called the PASCAL TRIANGLE by adding its two parent-entries, that is the two entries just above it.

**Example 10.** The next line in the PASCAL TRIANGLE for $n \leftarrow 4$ would be:

$\begin{array}{cccccc}
1 & 4 & 6 & 4 & 1
\end{array}$

We thus have two procedures, one for getting the powers and the other for getting the coefficients and all that remains is to put them together.

**Example 11.** In order to construct the addition formula for $(x_0 + h)^5$, we proceed as follows:

i. We make a row of 5 copies of $x_0$ with multiplication signs in-between:

\[ x_0 \cdot x_0 \cdot x_0 \cdot x_0 \cdot x_0 \]

ii. We tack on 5 copies of the row we wrote in step i.

\[ x_0 \cdot x_0 \cdot x_0 \cdot x_0 \cdot x_0 \]
\[ x_0 \cdot x_0 \cdot x_0 \cdot x_0 \cdot x_0 \]
\[ x_0 \cdot x_0 \cdot x_0 \cdot x_0 \cdot x_0 \]
\[ x_0 \cdot x_0 \cdot x_0 \cdot x_0 \cdot x_0 \]
\[ x_0 \cdot x_0 \cdot x_0 \cdot x_0 \cdot x_0 \]
iii. Starting with the first of the 5 rows that we tacked-on in step ii. and going
down, we replace in each row, starting from the right, one more copy of $x_0$ by a copy
of $h$ than in the preceding row:

\[
\begin{align*}
  x_0 \cdot x_0 \cdot x_0 \cdot x_0 \cdot x_0 \\
  x_0 \cdot x_0 \cdot x_0 \cdot x_0 \cdot h \\
  x_0 \cdot x_0 \cdot x_0 \cdot h \cdot h \\
  x_0 \cdot h \cdot h \cdot h \cdot h \\
  h \cdot h \cdot h \cdot h \cdot h
\end{align*}
\]

iv. We get the coefficients from the PASCAL TRIANGLE:

4. Proving that all this is indeed the case would involve more work than
we are willing to do here and so we will take the following for granted:

**THEOREM 1 (BINOMIAL THEOREM).** The addition formula for
a binomial of degree $n$ is:

\[
(x_0 + h)^n = \frac{1}{1} x_0^n h^0 + \frac{n}{1} x_0^{n-1} h^1 + \frac{n(n-1)}{1 \cdot 2} x_0^{n-2} h^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x_0^{n-3} h^3 + \ldots + \frac{n(n-1)(n-2) \cdots (1)}{1 \cdot 2 \cdot 3 \cdots n} x_0^0 h^n
\]

where it is interesting to note that, no matter what $n$,

\[
\frac{n(n-1)(n-2) \cdots (1)}{1 \cdot 2 \cdot 3 \cdots n} = 1
\]

because it corresponds to the fact that the rows in the PASCAL TRIANGLE are symmetrical.

5. According to the BINOMIAL THEOREM, when $n = 0$ we should have

\[
(x_0 + h)^0 = x_0^0 h^0
\]
15.2. \((x_0 + h)^n\) WHEN \(n > 3\)

and since the coefficient 1 goes without saying:

\[ = 1 \cdot x_0^0 h^0 = 1 \]

This is of course as it should be but what this says is that the “zeroth” line in the PASCAL TRIANGLE is 1 so that the “complete” PASCAL TRIANGLE is:

\[
\begin{array}{cccccccc}
n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & & & & & & & \\
n & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
n & & 1 & 2 & 3 & 4 & 5 & 6 \\
n & & & 1 & 3 & 3 & 1 & 1 \\
n & & & & 1 & 4 & 6 & 4 \\
n & & & & & 1 & 5 & 10 \\
n & & & & & & 1 & 15 \\
n & & & & & & & 1 \\
\end{array}
\]

6. Usually, though, we do not need the complete addition formulas but only the first three or, at worst, the first four terms and so using the PASCAL TRIANGLE as we did above would turn out to be a huge waste of energy.

The key to a more economical way of using the PASCAL TRIANGLE is to look at the slanted rows and notice that:

- The numbers in the first slanted row are all 1 and they are the coefficients of the \(h^0\) powers in the addition formulas for \((x_0+h)^1\), \((x_0+h)^2\), \((x_0+h)^3\), \((x_0+h)^4\), \((x_0+h)^5\), \((x_0+h)^6\), etc, which corresponds to the fact that the coefficient of the \(h^0\) power in \(x_0^n\) is 1 no matter what \(n\).
- The numbers in the second slanted row are 1, 2, 3, 4, 5, 6, . . . and they are the coefficients of the \(h^1\) powers in the addition formulas for \((x_0+h)^1\), \((x_0+h)^2\), \((x_0+h)^3\), \((x_0+h)^4\), \((x_0+h)^5\), \((x_0+h)^6\), etc, which shows that the coefficient of the \(h^1\) power in \(x_0^n\) is \(n\) no matter what \(n\).
- We check that the third slanted row, 1, 3, 6, 10, 15, . . . are the coefficients of the \(h^2\) powers in the addition formulas for \((x_0+h)^1\), \((x_0+h)^2\), \((x_0+h)^3\), \((x_0+h)^4\), \((x_0+h)^5\), \((x_0+h)^6\), etc, and we observe that
the coefficient of the $h^{+2}$ power in $x_0^n$ is the half-product of the two coefficients to its left and therefore equal to $\frac{n(n-1)}{2}$ no matter what $n$.

**Example 12.** Instead of getting the coefficient of $h^{+2}$ in the addition formula for $(x_0 + h)^5$ as the sum of its two "parents", 4 and 6:

\[
\begin{array}{cccccc}
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 5 & 1 \\
\end{array}
\]

we can get the coefficient of $h^{+2}$ in the addition formula for $(x_0 + h)^5$ as the half-product of the two numbers to its left, 5 and 4:

\[
\begin{array}{cccccc}
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

As a result, we only need, and can construct only, the first two slanted rows of the PASCAL TRIANGLE.

**Example 13.** In the addition formula for exponent 5, we get the first three coefficients just by constructing only the first two slanted rows:

\[
\begin{array}{cccccc}
1 & \\
1 & 1 \\
1 & 2 \\
1 & 3 \\
1 & 4 \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 5 & 10 \\
1 & 5 & 10 \\
\end{array}
\]

where 10 is the half-product of the second entry in the last two horizontal rows.

### 15.3 Division in Descending Exponents

Since *decimal numbers* are combinations of powers of TEN, it should not be surprising that the procedure for dividing decimal numbers should also work for *polynomials* which are combinations of powers of $x$.

The *procedure* consists of successive *cycles*, one for each monomial in the quotient. During each of these *cycles*, we go through four *steps*:

**Step I.** We find each *monomial* of the *quotient* by dividing the *first monomial* in the divisor into the *first monomial* of the previous partial remainder or, if there is not yet a partial remainder, from the *dividend*. 

---

**Note:** The text is a continuation of the mathematical content as provided in the original document.
Step II. We find the partial product by multiplying the full divisor by the monomial of the quotient we found in Step I.

Step III. We find the partial remainder by subtracting the partial product we found in Step II from the previous partial remainder or, if there is not yet a partial remainder, from the dividend.

Step IV. We decide if we want to:
- stop the division,
- continue the division and go through another cycle.

Example 14. In order to compute \(-12x^3 + 11x^2 - 17x + 1\) \(-3x^2 + 5x - 2\), we set up the division of \(-3x^2 + 5x - 2\) “into” \(-12x^3 + 11x^2 - 17x + 1\) and we proceed as follows:

**Cycle 1.** Step I. We find the first monomial in the quotient by dividing the first monomial in the divisor, \(-3x^2\), into the first monomial of the dividend, \(-12x^3\), which give us \(\frac{-12x^3}{-3x^2} = 4x\) as first term in the quotient:

\[
\begin{array}{c|ccccc}
\text{-3x}^2 & +5x & -2 & \text{-12x}^3 & +11x^2 & -17x & +1 \\
\hline
+4x & & & & & & \\
\end{array}
\]

Step II. We find the first partial product by multiplying the full divisor by the first monomial in the quotient:

\[
\begin{array}{c|ccccc}
\text{-3x}^2 & +5x & -2 & \text{-12x}^3 & +11x^2 & -17x & +1 \\
\hline
+4x & & & & & & \\
\end{array}
\]

First partial product:

\[
\begin{array}{c|ccccc}
\text{-3x}^2 & +5x & -2 & \text{-12x}^3 & +11x^2 & -17x & +1 \\
\hline
\text{-12x}^3 & +20x^2 & -8x & & & & \\
\end{array}
\]

Step III. We find the first partial remainder by subtracting the first partial product from the full dividend:

\[
\begin{array}{c|ccccc}
\text{-3x}^2 & +5x & -2 & \text{-12x}^3 & +11x^2 & -17x & +1 \\
\hline
+4x & & & & & & \\
\end{array}
\]

\[
\begin{array}{c|ccccc}
\text{-3x}^2 & +5x & -2 & \text{-12x}^3 & +11x^2 & -17x & +1 \\
\hline
\text{-12x}^3 & +20x^2 & -8x & & & & \\
\end{array}
\]

But to subtract the first partial product means to add the opposite of the first partial product to the full dividend:

\[
\begin{array}{c|ccccc}
\text{-3x}^2 & +5x & -2 & \text{-12x}^3 & +11x^2 & -17x & +1 \\
\hline
\text{-12x}^3 & -20x^2 & +8x & & & & \\
\hline
\text{+0x}^3 & -9x^2 & -9x & +1 & & & \\
\end{array}
\]

First remainder:

Step IV. We decide if we want to stop or continue the division.
- If we decide to stop the division, the quotient of the division is \(+4x\).
 CHAPTER 15.  PREVIEW & ALGEBRA REVIEWS

– the remainder of the division is $-9x^2 - 8x + 1$

 If we don’t care about the remainder, we write:

$$\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2} = +4x + [...]$$

where we write $+ [...]$ as a reminder that $\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2}$ is not exactly equal to $+4x$ since there was a remainder.

- If we decide to continue the division, we begin a new cycle

**CYCLE 2.**

**Step I.** We find the second monomial in the quotient by dividing the first monomial in the divisor, $-3x^2$, into the first monomial in the first partial remainder, $-9x^2$, which gives us $\frac{-9x^2}{-3x^2} = +3$ for the second term of the quotient:

\[
\begin{array}{c|ccc}
 & -12x^3 & +11x^2 & -17x & +1 \\
-12x^3 & +20x^2 & -8x &  \\
\hline
0 & -9x^2 & -9x & +1 \\
\end{array}
\]

**Step II.** We find the second partial product by multiplying the full divisor by the second monomial in the quotient:

\[
\begin{array}{c|ccc}
 & -12x^3 & +11x^2 & -17x & +1 \\
-12x^3 & +20x^2 & -8x &  \\
\hline
0 & -9x^2 & -9x & +1 \\
\end{array}
\]

Second partial product:

\[
-9x^2 + 15x - 6
\]

**Step III.** We find the second partial remainder by subtracting the second partial product from the first partial remainder:

\[
\begin{array}{c|ccc}
 & -12x^3 & +11x^2 & -17x & +1 \\
-12x^3 & +20x^2 & -8x &  \\
\hline
0 & -9x^2 & -9x & +1 \\
\end{array}
\]

But to subtract the second partial product means to add the opposite of the second partial product to the first partial remainder:

\[
\begin{array}{c|ccc}
 & -12x^3 & +11x^2 & -17x & +1 \\
-12x^3 & +20x^2 & -8x &  \\
\hline
0 & -9x^2 & -9x & +1 \\
\end{array}
\]

Second remainder:

\[
9x^2 + 15x + 6
\]

**Step IV.** We decide if we want to stop or continue the division.

- If we decide to stop the division,
  - the quotient of the division is $+4x + 3$.  


the remainder of the division is \(-24x + 7\).

If we don’t care about the remainder, we write:

\[
\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2} = +4x + 3 + [...]
\]

where we write + [...] as a reminder that \(-12x^3 + 11x^2 - 17x + 1\) is not exactly equal to \(+4x + 3\) since there was a remainder.

- If we decide to continue the division, we begin a new cycle

**CYCLE 3.**

**Step I.** We find the third monomial in the quotient by dividing the first monomial in the divisor, \(-3x^2\), into the first monomial in the second partial remainder, \(-24x\) that is \(-\frac{24x}{-3x^2} = +8x^{-1}\)

\[
\begin{array}{c}
-3x^2 + 5x - 2 \\
\hline
+4x + 3 + 8x^{-1}
\end{array}
\]

\[
\begin{array}{c}
-12x^3 + 11x^2 - 17x + 1 \\
-12x^3 + 20x^2 - 8x
\end{array}
\]

\[
\begin{array}{c}
-9x^2 - 9x + 1 \\
+9x^2 - 15x + 6
\end{array}
\]

\[
-24x + 7
\]

**Step II.** We find the third partial product by multiplying the full divisor by the third monomial in the quotient:

\[
\begin{array}{c}
-3x^2 + 5x - 2 \\
\hline
+4x + 3 + 8x^{-1}
\end{array}
\]

\[
\begin{array}{c}
-12x^3 + 11x^2 - 17x + 1 \\
-12x^3 + 20x^2 - 8x
\end{array}
\]

\[
\begin{array}{c}
-9x^2 - 9x + 1 \\
+9x^2 - 15x + 6
\end{array}
\]

\[
-24x + 7
\]

**Step III.** We find the third partial remainder by subtracting the third partial product from the first partial remainder:

\[
\begin{array}{c}
-3x^2 + 5x - 2 \\
\hline
+4x + 3 + 8x^{-1}
\end{array}
\]

\[
\begin{array}{c}
-12x^3 + 11x^2 - 17x + 1 \\
-12x^3 + 20x^2 - 8x
\end{array}
\]

\[
\begin{array}{c}
-9x^2 - 9x + 1 \\
+9x^2 - 15x + 6
\end{array}
\]

\[
-24x + 7
\]

But to subtract the second partial product means to add the opposite of the second partial product to the first partial remainder:
CHAPTER 15. PREVIEW & ALGEBRA REVIEWS

\[ -3x^2 + 5x - 2 \]

\[
\begin{array}{cccc}
+4x & +3 & +8x^{-1} \\
\hline
-12x^3 & +11x^2 & -17x & +1 \\
-12x^3 & +20x^2 & -8x & \\
\hline
-9x^2 & -9x & +1 & \\
+9x^2 & -15x & +6 & \\
\hline
-24x & +7 & \\
+24x & -40 & +16x^{-1} & \\
\hline
0x & -33 & +16x^{-1} & \\
\end{array}
\]

Third remainder:

**Step IV.** We decide if we want to stop or continue the division.

- If we decide to stop the division,
  - the quotient of the division is \(+4x + 3 + 8x^{-1}\).
  - the remainder of the division is \(-33 + 16x^{-1}\).

If we don’t care about the remainder, we write:

\[\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2} = +4x + 3 + 8x^{-1} + [\ldots]\]

where we write \(+ [\ldots]\) as a reminder that \(\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2}\) is not exactly equal to \(+4x + 3 + 8x^{-1}\) since there was a remainder.

- If we decide to continue the division, we begin a new cycle.

Just as, in arithmetic, we need not stop a division when the quotient reaches a monomial with exponent 0 because we can always divide a monomial into another and we can have negative exponents. In fact, again just as in arithmetic, there are cases where we absolutely need to go beyond the exponent 0 and use negative exponents. (See Epilogue.)

**Example 15.** In order to divide \(-12x^3 + 11x^2 - 17x + 4\) by \(-3x^2 + 5x - 2\) down to a \(x^{-2}\) monomial, we write (in the anglo-saxon tradition):

\[
\begin{array}{cccc}
+4x & +3 & +8x^{-1} & +10x^{-2} \\
\hline
-12x^3 & +11x^2 & -17x & +4 \\
+12x^3 & -20x^2 & +8x & \\
\hline
-9x^2 & -9x & +6 & \\
+9x^2 & -15x & +10 & \\
\hline
-24x & -40 & +16x^{-1} & \\
+24x & -30 & -16x^{-1} & +20x^{-2} & \\
-30 & +50x^{-1} & +20x^{-2} & \\
\hline
-66x^{-1} & +20x^{-2} & \\
\end{array}
\]

The quotient is

\(+4x + 3 + 8x^{-1} + 10x^{-2}\)

and the remainder is
15.4 Default Rules for Division

Since mathematicians are lazy,

- mathematicians do not write the + sign in front of the coefficients of leading monomials,
- mathematicians do not write monomials with 0 coefficient,

and, most dangerously,

- mathematicians want to write only one stage in Step III but there are two traditions concerning what then to write, as a result, in Step II:
  - In the latin tradition, in Step II, we write the partial product, that is what we get it from the multiplication, and so in Step III, when it comes to subtracting, we visualize the opposite of the partial product we wrote in Step II and we oplus what we visualize. The advantage is that each line is exactly what we get from the previous operation.

  **Example 16.**

  \[
  -3x^2 + 5x - 2 \quad \frac{+4x}{-12x^3 + 11x^2 - 16x + 1} \]
  \[
  \frac{-12x^3 + 20x^2 - 8x}{-9x^2 - 8x + 1}
  \]

  - In the anglo-saxon tradition, we anticipate the subtraction to be done in Step III and in Step II we write the opposite of the partial product so in Step III we oplus what we wrote in Step II.

  **Example 17.**

  \[
  -3x^2 + 5x - 2 \quad \frac{+4x}{-12x^3 + 11x^2 - 16x + 1} \]
  \[
  \frac{+12x^3 - 20x^2 + 8x}{-9x^2 - 8x + 1}
  \]

  From now on we will of course follow the anglo-saxon tradition.

  **Example 18.** In order to compute \( \frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} \), we divide \( 2x + 1 \) into \( 6x^3 + 13x^2 + 13x + 7 \):

  **Cycle 1. Step I.** We find the first monomial in the quotient by short division:

  \[
  \begin{array}{c}
  3x^2 \\
  \hline
  2x + 1 \end{array}
  \]

  \[
  6x^3 + 13x^2 + 13x + 7
  \]

  **Step II.** We get the first opposite product by writing the opposite of the result of the full multiplication
\[ \frac{3x^2}{2x + 1} \cdot \frac{6x^3 + 13x^2 + 13x + 7}{-6x^3 - 3x^2} \]

**Step III.** We get the *first remainder* by oplussing the first opposite product

\[ \frac{6x^3 + 13x^2 + 13x + 7}{-6x^3 - 3x^2} = 10x^2 + 13x \]

**Step IV.** We decide if we want to stop or continue the division

- If we decide to *stop* the division,
  - the *quotient* of the division is \( +3x^2 \).
  - the *remainder* of the division is \( +10x^2 + 13x \)

If we don’t care about the remainder, we write:

\[ \frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^2 + (\ldots) \]

where we write \( + (\ldots) \) as a reminder that \( \frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} \) is not exactly equal to \( +3x^2 \) since there was a remainder.

- If we decide to *continue* the division, we begin a new cycle

**CYCLE 2. Step I.** We find the *second monomial* in the quotient by short division:

\[ \frac{3x^2}{2x + 1} \cdot \frac{6x^3 + 13x^2 + 13x + 7}{-6x^3 - 3x^2} \]

\[ \frac{6x^3 + 13x^2 + 13x + 7}{-6x^3 - 3x^2} = 10x^2 + 13x \]

**Step II.** We get the *second opposite product* by writing the opposite of the result of the full multiplication

\[ \frac{3x^2 + 5x}{2x + 1} \cdot \frac{6x^3 + 13x^2 + 13x + 7}{-6x^3 - 3x^2} \]

\[ \frac{6x^3 + 13x^2 + 13x + 7}{-6x^3 - 3x^2} = 10x^2 + 13x \]

**Step III.** We get the *second remainder* by oplussing the first opposite product
15.4. DEFAULT RULES FOR DIVISION

\[ \frac{3x^2 + 5x}{2x + 1} \]

\[ \frac{6x^3 + 13x^2 + 13x + 7}{-6x^3 - 3x^2} \]

\[ 10x^2 + 13x \]

\[ -10x^2 - 5x \]

\[ \frac{8x + 7}{8x + 7} \]

**Step IV.** We decide if we want to stop or continue the division
- If we decide to stop the division,
  * the quotient of the division is \( +3x^2 + 5x \).
  * the remainder of the division is \( +8x + 7 \).

If we don’t care about the remainder, we write:

\[ \frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^2 + 5x + (...) \]

where we write \(+ (...)\) as a reminder that \( \frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} \) is not exactly equal to \( +3x^2 + 5x \) since there was a remainder.

- If we decide to continue the division, we begin a new cycle

**CYCLE 3. Step I.** We find the third monomial in the quotient by short division:

\[ \frac{3x^2 + 5x + 4}{2x + 1} \]

\[ \frac{6x^3 + 13x^2 + 13x + 7}{-6x^3 - 3x^2} \]

\[ 10x^2 + 13x \]

\[ -10x^2 - 5x \]

\[ \frac{8x + 7}{8x + 7} \]

**Step II.** We get the third opposite product by writing the opposite of the result of the full multiplication

\[ 3x^2 + 5x + 4 \]

\[ \frac{6x^3 + 13x^2 + 13x + 7}{-6x^3 - 3x^2} \]

\[ 10x^2 + 13x \]

\[ -10x^2 - 5x \]

\[ \frac{8x + 7}{8x + 7} \]

\[ -8x - 4 \]

**Step III.** We get the third remainder by oplussing the third opposite product
Step IV. We decide if we want to stop or continue the division
- If we decide to stop the division,
  * the quotient of the division is \( +3x^2 + 5x + 4 \).
  * the remainder of the division is \( +3 \).
If we don’t care about the remainder, we write:
\[
\frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} = +3x^2 + 5x + 4 + (...) \\
\]
where we write \( + (...) \) as a reminder that \( \frac{6x^3 + 13x^2 + 13x + 7}{2x + 1} \) is not exactly equal to \( +3x^2 + 5x + 4 \) since there was a remainder.
- If we decide to continue the division, we begin a new cycle

- When writing the partial remainders, we do not write the monomials beyond those that result from subtracting the partial product.

**Example 19.**

\[
\begin{array}{cccc}
-3x^2 + 5x - 2 & +4x & +3 \\
-12x^3 & +11x^4 & -16x & +1 \\
-12x^3 & -20x^2 & +8x & \\
-9x^2 & -8x & & \\
+9x^2 & -15x & +6 & \\
-23x & +7 & & \\
\end{array}
\]

The danger here is that, when we do the next subtraction, we may subtract from 0 rather than from the monomial that was left unwritten in the partial remainder.

### 15.5 Comparison With Arithmetic

The procedure to divide polynomials is in fact a lot simpler than the procedure for dividing in arithmetic:
- There is never any “carryover”
- The first term of each partial remainder always has coefficient 0
- There are no Trials in **Step I** because, when we divide the first monomial in the divisor into the first monomial of a partial remainder, we always
get a coefficient for the corresponding monomial in the quotient and the worst that can happen is that this coefficient is a fraction.

15.6 Division in Ascending Order Of Exponents

Fortunately, the procedure is exactly the same as in the case of division in descending order of exponents and so we will just look at an example.

**Example 20.** In order to compute \( \frac{-12 + 23h - h^2 - 2h^3}{-3 + 2h} \), we divide \(-3 + 2h\) into \(-12 + 23h - h^2 - 2h^3\):

\[
\begin{array}{cccc}
-3 + 2h & -12 & +23h & -h^2 - 2h^3 \\
\text{First opposite partial product:} & +4 & -5h & -3h^2 \\
\text{First remainder:} & +12 & -8h \\
\text{Second opposite partial product:} & +15h & -h^2 & -2h^3 \\
\text{Second remainder:} & -15h & +10h^2 \\
\text{Third opposite partial product:} & +9h^2 & -2h^3 \\
\text{Third remainder:} & -9h^2 & +6h^3 \\
\end{array}
\]

- If we decide to stop the division,
  - the quotient of the division is \( +4 - 5h - 3h^2 \).
  - the remainder of the division is \( +4h^3 \). Observe that if we replace the unspecified numerator \( h \) by, say, 0.2, then the remainder is equal to \( 4 \cdot 0.2^3 = 4 \cdot 0.008 = 0.032 \) which is indeed small.

If we don’t care about the remainder, we write:

\[
\frac{-12 + 23h - h^2 - 2h^3}{-3 + 2h} = +4 - 5h - 3h^2 + [..]
\]

where we write \( +[..] \) as a reminder that \( \frac{-12 + 23h - h^2 - 2h^3}{-3 + 2h} \) is not exactly equal to \( +4 - 5h - 3h^2 \) since there was a remainder.

- If we were to decide to continue the division, we would begin a new cycle.

**Example 21.** In order to divide \( 2x^3 + 5x^2 - 6 \) by \( 3x - 1 \) we write (in the anglo-saxon tradition):
\[ 3x - 1 \]
\[
\begin{array}{c}
\frac{2}{3} x^2 + \frac{17}{9} x + \frac{17}{27} \\
- \frac{2}{3} x^3 + 5x^2 - 6 \\
\frac{17}{3} x^2 \\
- \frac{17}{3} x^2 + \frac{17}{9} x \\
\frac{17}{9} x - 6 \\
- \frac{17}{9} x + \frac{17}{27} \\
\frac{145}{27}
\end{array}
\]

The quotient is
\[ + \frac{2}{3} x^2 + \frac{17}{9} x + \frac{17}{27} \]

The remainder is
\[ - \frac{145}{27} \]

### 15.7 Graphic Difficulties

Finally, when there is one or more $\infty$-height bounded input(s), beginners often encounter difficulties when trying to interpolate smoothly the outlying graph of a rational function.

The difficulties are caused by the fact that, when we draw the local graph near $\infty$ and the local graphs near the $\infty$-height inputs from the local input-output rules, we are only concerned with drawing the local graphs themselves from the local input-output rules. In particular, when we draw the local graph near $\infty$ and the local graphs near the $\infty$-height inputs, we want to bend them enough to show the concavity but we often end up bending them too much to interpolate them.

But then, what often happens as a result is that, when we want to interpolate, the local graphs may not line up well enough for us to interpolate them (smoothly).

**EXAMPLE 22.** Given the rational function whose outlying graph was drawn as
so as to show the concavity. Here is what can happen when we attempt to interpolate

![Graphic](image1)

Of course, this is absolutely impossible since, according to this global graph, there would be inputs, such as $x_0$, with more than one output, $y_1, y_2, \ldots$:

![Graphic](image2)

But if we unbend the local graphs just a bit as in

![Graphic](image3)

we have no trouble interpolating:

![Graphic](image4)
The way to avoid this difficulty is not to wait until we have to interpolate but to catch any problem as we draw the local graphs by mentally extending the local graphs slightly into the transitions.

**EXAMPLE 23.** Given the rational function whose outlying graph was drawn as

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{Input} & \text{Ruler} & \text{Output} & \text{Ruler} \\
\hline
+\infty & +\infty & -\infty & -\infty \\
\end{array}
\]

so as to show the concavity, we can already see by extending the local graphs just a little bit into the transitions that this will cause a lot of trouble when we try to interpolate the local graph:

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{Input} & \text{Ruler} & \text{Output} & \text{Ruler} \\
\hline
+\infty & +\infty & -\infty & -\infty \\
\end{array}
\]

So, here, we bend the local graph near \(\infty\) a little bit more and we unbend the local graphs near the \(\infty\)-height inputs a little bit:

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{Input} & \text{Ruler} & \text{Output} & \text{Ruler} \\
\hline
+\infty & +\infty & -\infty & -\infty \\
\end{array}
\]

We check again by extending the local graphs just a little bit into the transitions:
and indeed now we have no trouble interpolating.