Chapter 16

Rational Functions: Local Analysis Near $\infty$

The local analysis of a rational function will go exactly in the same spirit as the local analysis of any of the polynomial functions: Given an input $\text{Imp}$, either $\infty$ or $x_0$, and a wanted local feature,

a. We will localize the global input-output rule near the given input $\text{Imp}$,

b. We will extract from the local input-output rule near the given input $\text{Imp}$ the term that is responsible for the wanted feature near $\text{Imp}$, and then of course we will get the wanted local feature from:

- the local graph of the power function near $\infty$ when $\text{Imp}$ is $\infty$
- the local graph of the power function near 0 when $\text{Imp}$ is $x_0$

The only difference between the case of a rational function and the case of a polynomial function will be in the way we extract the power function from the local input-output rule. In general, the difficulty in the case of rational functions will be in carrying out the extraction with the least amount of work because that requires an awareness of all the possible routes so as to be able to choose the most efficient route.

More precisely, the difference between the local analysis of a polynomial function and the local analysis of a rational function is a technical one:
• In the case of a polynomial function, we were able to extract the term(s) responsible for the wanted feature(s) directly from the local input-output rule. In fact, with a bit of practice, we were even able to “pick” the corresponding term in each one of the power expansions given by the addition formulas and combine them with the corresponding coefficients to get just the term responsible for the wanted feature without having to get the whole local input-output rule.

• In the case of a rational function, the difficulty will be that we will not be able to extract the term responsible for the wanted feature directly from the local input-output rule because the local input-output rule will be in the form of a fraction and it is in the nature of the beast that it is always difficult to deal directly with fractions.

Example 1. Looking at two fractions, say $\frac{6271737}{343022}$ and $\frac{8291737}{464623}$, it is not at all obvious which one is larger and it is not at all easy to compute their sum, difference, product or quotient.

As a result, contrary to the way we dealt with the local analysis of polynomial functions, which was to discuss all aspects of the investigation, from beginning to end, in the case of a rational function we will focus on how to extract from the local input-output rule the term(s) responsible for the wanted feature(s) and we will then let the student take it from there.

Overall Strategy

We begin with a discussion in which we develop a basic strategy for when the wanted features are near $\text{Imp}$, WHERE $\text{IMP}$ CAN BE WITHER $\infty$ OR $x_0$. We will then use this strategy:

i. In this chapter when $\text{Imp}$ is $\infty$, that is when the wanted features are to be found near $\infty$

ii. In the next chapter when $\text{Imp}$ is $x_0$, that is when the wanted features are to be found near $x_0$.

16.1 Local Input-Output Rule Near $\text{Imp}$

The first step will be a localization step which, in the case of a rational function, will consist in localizing both the numerator and the denominator of the rational function:

More precisely, given a rational function $\text{RAT}$ specified by the global input-
output rule

\[ x \xrightarrow{\text{RAT}} \text{RAT}(x) = \frac{\text{POLY}_{\text{Num}}(x)}{\text{POLY}_{\text{Den}}(x)} \]

and given an input \( \text{Imp} \), where \( \text{Imp} \) can be either \( \infty \) or \( x_0 \), the localization near \( \text{Imp} \) will be:

\[ x \big|_{x \text{ near } \text{Imp}} \xrightarrow{\text{RAT}} \text{RAT}(x) \big|_{x \text{ near } \text{Imp}} = \frac{\text{POLY}_{\text{Num}}(x) \big|_{x \text{ near } \text{Imp}}}{\text{POLY}_{\text{Den}}(x) \big|_{x \text{ near } \text{Imp}}} \]

where, in order to get \( \text{POLY}_{\text{Num}}(x) \big|_{x \text{ near } \text{Imp}} \) and \( \text{POLY}_{\text{Den}}(x) \big|_{x \text{ near } \text{Imp}} \) we must:

i. Use neighborhood-centric inputs, that is count the inputs naturally in terms of the neighborhood of \( \text{Imp} \),

ii. Make sure the successive terms in the local input-output rules for the numerator and for the denominator are both in descending order of sizes.

### 16.2 Polynomial Approximations

1. It is at this point that we encounter difficulties.

   a. The first problem will be that the local input output rule of \( \text{RAT} \) near \( \text{Imp} \) is a fraction while we do not know how to extract the term responsible for the wanted feature from anything other than a polynomial. Often, though, a good way, if not the only way, to deal with a fraction is to carry out the division so as to get an approximation of the fraction.

**Example 2.** Given two fractions, say \( \frac{627137}{343022} \) and \( \frac{8291737}{464623} \), once we have obtained decimal approximations for these fractions by carrying out the divisions, for instance

\[ \frac{627137}{343022} = 18.28 + [...] \]
\[ \frac{8291737}{464623} = 17.84 + [...] \]

it becomes quite obvious which fraction is larger and quite easy to compute approximations for their sum, difference, product or quotient.
And, in fact, dividing \( POLY_{\text{Num}}(x)|_{x \text{ near } \text{Imp}} \) by \( POLY_{\text{Den}}(x)|_{x \text{ near } \text{Imp}} \) will give us \( \text{POLY. APPROX. RAT}(x)|_{x \text{ near } \text{Imp}} \), that is a polynomial approximation of the fraction \( \frac{POLY_{\text{Num}}(x)}{POLY_{\text{Den}}(x)}|_{x \text{ near } \text{Imp}} \) from which we will be able to extract the term responsible for the wanted feature. In other words, we could proceed as follows:

\[
x|_{x \text{ near } \text{Imp}} \xrightarrow{\text{RAT}} \text{RAT}(x)|_{x \text{ near } \text{Imp}} = \frac{POLY_{\text{Num}}(x)}{POLY_{\text{Den}}(x)}|_{x \text{ near } \text{Imp}} = \frac{POLY_{\text{Num}}(x)}{POLY_{\text{Den}}(x)}|_{x \text{ near } \text{Imp}}
\]

and, after dividing \( POLY_{\text{Num}}(x)|_{x \text{ near } \text{Imp}} \) by \( POLY_{\text{Den}}(x)|_{x \text{ near } \text{Imp}} \),

\[
= \text{POLY. APPROX. RAT}(x)|_{x \text{ near } \text{Imp}} + [...]\]

b. In practice, though, a second problem is that this could take a lot of work and use a lot of time since:
- \( POLY_{\text{Num}}(x) \) and \( POLY_{\text{Den}}(x) \) could include high degree terms and/or many terms which would complicate the computation of \( POLY_{\text{Num}}(x)|_{x \text{ near } \text{Imp}} \) and \( POLY_{\text{Den}}(x)|_{x \text{ near } \text{Imp}} \).
- Division of polynomials can be a fairly involved and time-consuming procedure.

So, we will
- approximate \( POLY_{\text{Num}}(x)|_{x \text{ near } \text{Imp}} \) to get \( \text{APPROX. POLY}_{\text{Num}}(x)|_{x \text{ near } \text{Imp}} \)
- approximate \( POLY_{\text{Den}}(x)|_{x \text{ near } \text{Imp}} \) to get \( \text{APPROX. POLY}_{\text{Den}}(x)|_{x \text{ near } \text{Imp}} \)
which we will then divide to get a polynomial approximations of \( \text{RAT}(x) \), namely \( \text{POLY. APPROX. RAT}(x)|_{x \text{ near } \text{Imp}} \).

c. In other words,

i. In the first step, which we will refer to as the localization step, we will get

\[
x|_{x \text{ near } \text{Imp}} \xrightarrow{\text{RAT}} \text{RAT}(x)|_{x \text{ near } \text{Imp}} = \frac{\text{APPROX. POLY}_{\text{Num}}(x)}{\text{APPROX. POLY}_{\text{Den}}(x)}|_{x \text{ near } \text{Imp}} + [...]\]

ii. In the second step which we will refer to as extraction step, we will get

\[
x|_{x \text{ near } \text{Imp}} \xrightarrow{\text{RAT}} \text{RAT}(x)|_{x \text{ near } \text{Imp}} = \text{POLY. APPROX. RAT}(x)|_{x \text{ near } \text{Imp}} + [...]\]

which we will call a local approximately polynomial input-output rules—local AP input-output rules for short—that is a local input-output rule that will consist of a polynomial, \( \text{POLY. APPROX. RAT}(x)|_{x \text{ near } \text{Imp}} \) plus “something too small to matter here”. 
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- But then a third problem could arise in that
  \( \text{APPROX. POLY}_{\text{Num}}(x)|_{x \text{ near } \text{Imp}}, \) and \( \text{APPROX. POLY}_{\text{Den}}(x)|_{x \text{ near } \text{Imp}}, \) could be “too short” in the sense that in carrying out the division we would bump into [...] “too soon”, that is before we could extract the term responsible for the feature we want.

2. Depending on the wanted feature(s), the polynomial approximation for \( \text{RAT}(x) \) from which we will extract the term responsible for the wanted feature, namely \( \text{POLY}.\text{APPROX}.\text{RAT}(x)|_{x \text{ near } \text{Imp}}, \) will be either

- \( \text{Princ. TERM RAT}(x)|_{x \text{ near } \text{Imp}} \) when the first term in the result of the division turns out to be the term responsible for the wanted feature,

or,

- \( \text{Princ. PART RAT}(x)|_{x \text{ near } \text{Imp}} \) when the term responsible for the wanted feature does not turn out up front in the result of the division.

On the other hand, since this extraction step involves both division and approximation, there will be two possible routes depending on which of division and approximation we do first and which we do second:

- The short route, that is the route in which we will approximate first and divide second. The short route will lead to Principal TERM of \( \text{RAT}(x)|_{x \text{ near } \text{Imp}} \)

The advantage of the short route will be that since the short route will involve the (short) division of two monomials it will involve very little work but the disadvantage will be that, since the division will stop by itself automatically, we will have no control over the result which may or may not be the term responsible for the wanted feature.

- The long route, that is the route in which we will divide first and approximate second—by stopping the division. The long route will lead to Principal PART of \( \text{RAT}(x)|_{x \text{ near } \text{Imp}} \)
The disadvantage will be that long division involves a significant amount of work but the advantage will be that we will retain full control over what we get since in the long division of two polynomials it will be up to us to stop the division when we get the term responsible for the wanted feature.

3. It will of course be from the familiarity we will gain from experience that we will best be able to navigate this procedure. So, now that we have laid out the general strategy for the local analysis of rational functions, we turn to the implementation of the two steps, localization and extraction, in various cases depending on the wanted feature(s).

**Local Analysis Near $\infty$**

When the wanted features are to be found near $\infty$, the rational degree of the rational function plays a great role inasmuch as it creates a diversity of cases but, on the other hand, the rational degree will tell us up front whether or not the short route will allow us to extract the term responsible for the wanted feature.

**16.3 Local AP Input-Output Rule Near $\infty$**

More precisely, given a rational function $RAT$ specified by the global input-output rule

\[ x \xrightarrow{RAT} RAT(x) = \frac{POLY_{Num}(x)}{POLY_{Den}(x)} \]

when $Imp$ is $\infty$, what we will want will be a polynomial approximation for the output of the local input-output rule near $\infty$

\[ x|_{x \nearrow \infty} \xrightarrow{RAT} RAT(x)|_{x \nearrow \infty} = \frac{POLY_{Num}(x)|_{x \nearrow \infty}}{POLY_{Den}(x)|_{x \nearrow \infty}} \]
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from which to \textit{extract} the term responsible for the wanted feature.

1. Since $\text{Imp}$, the center of the neighborhood, is $\infty$, the \textit{localization step}

\[
\begin{array}{c}
\text{Localize near } \infty \\
\text{POLY}_{\text{Num}}(x) \\
\text{Localize near } \infty \\
\text{POLY}_{\text{Den}}(x) \\
\end{array}
\]

that is the step in which we \textit{localize} both

- $\text{POLY}_{\text{Num}}(x)$ to $\text{POLY}_{\text{Num}}(x) \big|_{x \nearrow \infty}$

and

- $\text{POLY}_{\text{Den}}(x)$ to $\text{POLY}_{\text{Den}}(x) \big|_{x \nearrow \infty}$

is not exactly a strenuous one:

\begin{itemize}
  \item[i.] Inputs cannot be \textit{counted} from $\infty$ and we must do the next best thing that is count inputs from the point \textit{diametrically opposed} to $\infty$, namely from the \textit{origin} 0. Since this is automatically already the case with the \textit{global input-output rule} there is nothing for us to do.
  \item[ii.] The successive terms in the \textit{local input-output rule} must be in \textit{descending order of sizes}. Since $\text{Imp}$ is $\infty$, this means that the terms must be in \textit{descending order of exponents} and this is usually, but not necessarily and not always, the case and we must make sure.
\end{itemize}

2. Depending on the wanted feature(s) and the rational degree, the \textit{extraction step} will take one of the following two routes:

- The \textit{short route} to $\text{Princ. TERM RAT}(x) \big|_{x \nearrow \infty}$, that is:
  \begin{itemize}
    \item[i.] We \textit{approximate} both $\text{POLY}_{\text{Num.RAT}}(x) \big|_{x \nearrow \infty}$ and $\text{POLY}_{\text{Den.RAT}}(x) \big|_{x \nearrow \infty}$ to their \textit{principal term}—that is to just their \textit{highest size term}—which, since $x$ is near $\infty$, is their term with \textit{highest exponent}:
  \[
  \begin{array}{c}
  \text{Localize near } \infty \\
  \text{POLY}_{\text{Num}(x)} \\
  \text{Localize near } \infty \\
  \text{POLY}_{\text{Den}(x)} \\
  \end{array}
  \]
  \[
  \frac{\text{POLY}_{\text{Num}(x)} \big|_{x \nearrow \infty}}{\text{POLY}_{\text{Den}(x)} \big|_{x \nearrow \infty}} \\
  \]
  
  \[
  \frac{\text{Princ. TERM Num}(x) \big|_{x \nearrow \infty} + [...]}{\text{Princ. TERM Den}(x) \big|_{x \nearrow \infty} + [...]}
  \]
  \end{itemize}

  \item[ii.] We \textit{divide} $\text{Princ. TERM Num}(x) \big|_{x \nearrow \infty}$, that is the principal term of the \textit{numerator} of $\text{RAT}$ near $\infty$ by $\text{Princ. TERM Den}(x) \big|_{x \nearrow \infty}$, that is the principal term of the \textit{denominator} of $\text{RAT}$ near $\infty$. The result is $\text{Princ. TERM RAT}(x) \big|_{x \nearrow \infty}$, that is the \textit{principal term} of the rational function $\text{RAT}$ near $\infty$: 

The reason that the short division extracts only $Princ.TERM RAT(x) \mid_{x \nearrow \infty}$ is because the division stops by itself as soon as we hit $[\ldots]$, that is immediately after dividing $Princ.TERM_{Num}(x) \mid_{x \nearrow \infty}$ by $Princ.TERM_{Den}(x) \mid_{x \nearrow \infty}$, that is at

$$Princ.TERM RAT(x) \mid_{x \nearrow \infty} = \frac{Princ.TERM_{Num}(x) \mid_{x \nearrow \infty}}{Princ.TERM_{Den}(x) \mid_{x \nearrow \infty}}$$

$$= \frac{\text{coef. } Princ.TERM_{Num}(x) \mid_{x \nearrow \infty}}{\text{coef. } Princ.TERM_{Den}(x) \mid_{x \nearrow \infty}} \cdot x^{\text{UppDeg.POLY}_{Num}(x) - \text{UppDeg.POLY}_{Den}(x)}$$

$$= \frac{\text{coef. } Princ.TERM_{Num}(x) \mid_{x \nearrow \infty}}{\text{coef. } Princ.TERM_{Den}(x) \mid_{x \nearrow \infty}} \cdot x^{\text{RatDeg.RAT}(x)}$$

- The long route to $Princ.PART RAT(x) \mid_{x \nearrow \infty}$ in which
  i. In order to divide $POLY_{Num}(x) \mid_{x \nearrow \infty}$ by $POLY_{Den}(x) \mid_{x \nearrow \infty}$, we set up the division as a long division, that is $POLY_{Den.RAT}(x) \mid_{x \nearrow \infty}$ dividing into $POLY_{Num.RAT}(x) \mid_{x \nearrow \infty}$;

$$POLY_{Num}(x) \mid_{x \nearrow \infty} \div POLY_{Den}(x) \mid_{x \nearrow \infty}$$

  ii. We approximate by stopping the long division as soon as we have the principal part that has the feature(s) we want:
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$\displaystyle \frac{POLY_{Num}(x)}{POLY_{Den}(x)}$ Localize near $\infty$

$\displaystyle \frac{POLY_{Num}(x)}{POLY_{Den}(x)} \bigg|_{x \nearrow \infty}$

$\displaystyle \frac{POLY_{Den}(x)}{POLY_{Num}(x)} \bigg|_{x \nearrow \infty}$

$\displaystyle \frac{POLY_{Num}(x)}{POLY_{Den}(x)}$ Localize near $\infty$

ii. Approximate $Princ.PART\ RAT(x) \bigg|_{x \nearrow \infty} + \ldots$

3. Which route we will take in each particular case will depend both on the wanted feature(s) near $\infty$ but often also on the rational degree of $RAT$ and so we will now look separately at how we get $Height$-sign $\bigg|_{x \nearrow \infty}$, $Slope$-sign $\bigg|_{x \nearrow \infty}$ and $Concavity$-sign $\bigg|_{x \nearrow \infty}$

**EXAMPLE 3.** Given the rational function $DOUGH$ whose global input-output rule is

$$x \xrightarrow{DOUGH} DOUGH(x) = \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6}$$

find its local rule near $\infty$.

a. We have

$$x \bigg|_{x \nearrow \infty} \xrightarrow{DOUGH} DOUGH(x) \bigg|_{x \nearrow \infty} = \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \bigg|_{x \nearrow \infty}$$

$$= \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \bigg|_{x \nearrow \infty}$$

We now proceed with the two steps:

$\displaystyle \frac{POLY_{Num}(x)}{POLY_{Den}(x)}$ Localize near $\infty$

$\displaystyle \frac{POLY_{Num}(x)}{POLY_{Den}(x)} \bigg|_{x \nearrow \infty}$

$\displaystyle \frac{POLY_{Num}(x)}{POLY_{Den}(x)}$ Localize near $\infty$

$\displaystyle \frac{POLY_{Num}(x)}{POLY_{Den}(x)} \bigg|_{x \nearrow \infty}$

b. The more usual presentation is:

$$x \bigg|_{x \nearrow \infty} \xrightarrow{DOUGH} DOUGH(x) \bigg|_{x \nearrow \infty} = \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \bigg|_{x \nearrow \infty}$$

$$= \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \bigg|_{x \nearrow \infty}$$

We approximate $+12x^5 - 6x^3 + 8x^2 + 6x - 9 \bigg|_{x \nearrow \infty}$ and $-3x^2 - 5x + 6 \bigg|_{x \nearrow \infty}$

$$= \frac{+12x^5 + [\ldots]}{-3x^2 + [\ldots]}$$
and then we divide:

\[ \frac{-12}{3}x^{5-2} + [...], \]

\[ = -4x^3 + [...]. \]

c. Since the degree of the power function

\[ x^{\text{POWER}} \rightarrow \text{POWER}(x) = -4x^3 \]

which approximates DOUGH near \( \infty \) is > 1, the power function \( \text{POWER} \) is regular and has all three features, concavity, slope and height. (This was of course to be expected from the fact that the rational degree of DOUGH is > 1.)

### 16.4 Height-sign Near \( \infty \)

No matter what the rational degree of the given rational function \( \text{RAT} \), \( \text{Princ.} \text{TERM} \text{RAT}(x) \big|_{x \searrow \infty} \) will give us \( \text{Height-sign} \big|_{x \searrow \infty} \) because, no matter what its exponent, any power function has \( \text{Height-sign} \big|_{x \searrow \infty} \). So, no matter what the rational degree of \( \text{RAT} \), the extraction step to get the term responsible for \( \text{Height-sign} \big|_{x \searrow \infty} \) can take the short route to \( \text{Princ.} \text{TERM} \text{RAT}(x) \big|_{x \searrow \infty} \):

\[
\begin{align*}
\text{POLY}_{\text{Num}}(x) \big|_{x \searrow \infty} & \quad \text{Localize near } \infty \quad \text{POLY}_{\text{Num}}(x) \big|_{x \searrow \infty} \quad \text{i. Approximate} \quad \text{Princ.} \text{TERM}_{\text{Num}}(x) \big|_{x \searrow \infty} + [...]
\text{POLY}_{\text{Den}}(x) \big|_{x \searrow \infty} & \quad \text{Localize near } \infty \quad \text{POLY}_{\text{Den}}(x) \big|_{x \searrow \infty} \quad \text{i. Approximate} \quad \text{Princ.} \text{TERM}_{\text{Den}}(x) \big|_{x \searrow \infty} + [...]
\end{align*}
\]

In other words,

**a.** As always, the localization step will be to localize near \( \infty \) both

- \( \text{POLY}_{\text{Num}}(x) \) to \( \text{POLY}_{\text{Num}}(x) \big|_{x \searrow \infty} \)
- \( \text{POLY}_{\text{Den}}(x) \) to \( \text{POLY}_{\text{Den}}(x) \big|_{x \searrow \infty} \)

which near \( \infty \) amounts only to making sure that the terms are in descending order of exponents.

**b.** Inasmuch as \( \text{Princ.} \text{TERM} \text{RAT}(x) \big|_{x \searrow \infty} \) has \( \text{Height} \) no matter what the rational degree, the extraction step to get \( \text{Height-sign} \big|_{x \searrow \infty} \) can take the short route to \( \text{Princ.} \text{TERM} \text{RAT}(x) \big|_{x \searrow \infty} \) no matter what the rational degree is:

**i.** Approximate both
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- $POLY_{Num}(x) \mid_{x \to \infty}$ to $Princ.TERM POLY_{Num}(x) \mid_{x \to \infty}$
- $POLY_{Den}(x) \mid_{x \to \infty}$ to $Princ.TERM POLY_{Den}(x) \mid_{x \to \infty}$

ii. Divide $Princ.TERM POLY_{Num}(x) \mid_{x \to \infty}$ by $POLY_{Den}(x) \mid_{x \to \infty}$ to get $Princ.TERM RAT(x) \mid_{x \to \infty} + [...]$.

**Example 4.** Given the rational function $DOUGH$ specified by the global input-output rule

\[ x \xrightarrow{DOUGH} DOUGH(x) = \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \]

find $Height$-sign $DOUGH \mid_{x \to \infty}$.

a. The localization step is to localize both the numerator and the denominator near $\infty$—which amounts only to making sure that the terms are in descending order of exponents.

\[ \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \]

b. Inasmuch as $Princ.TERM DOUGH(x) \mid_{x \to \infty}$ has $Height$ no matter what the rational degree, the extraction step to get the term responsible for $Height$-sign $\mid_{x \to \infty}$ can take the short route to $Princ.TERM DOUGH(x) \mid_{x \to \infty}$:

i. We approximate

\[ \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \xrightarrow{L. Approximate} \frac{+12x^5}{-3x^2} + [...]
\]

that is we approximate

- the numerator $+12x^5 - 6x^3 + 8x^2 + 6x - 9$ to its principal term, $-12x^5$
- the denominator $-3x^2 - 5x + 6$ to its principal term, $-3x^2$

ii. And then we divide:

\[ \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \xrightarrow{L. Approximate} \frac{+12x^5}{-3x^2} + [...]
\]

where

\[ \frac{+12x^5}{-3x^2} = \frac{+12 \cdot x \cdot x \cdot x \cdot x \cdot x}{-3 \cdot x \cdot x} = \frac{12}{3} x^{5-2} \]
The more usual way to write all this is something as follows:

\[ x \mid_{x \to \infty} \xrightarrow{\text{DOUGH}} \text{DOUGH}(x) \mid_{x \to \infty} = \frac{12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \mid_{x \to \infty} = \frac{12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \mid_{x \to \infty} = \frac{12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \mid_{x \to \infty} = \frac{12x^5 + [\ldots]}{-3x^2 + [\ldots]} = \frac{-12}{3}x^3 - [\ldots]

Whatever we write, the \textit{principal term} of \text{DOUGH} near \infty is \(-\frac{12}{3}x^3\) and it gives

\[ \text{Height-sign} \text{DOUGH} \mid_{x \to \infty} = (-, +) \]

**Example 5.** Given the function \(PAC\) specified by the global input-output rule

\[ x \xrightarrow{\text{PAC}} PAC(x) = \frac{-12x^3 + 7x + 4}{+4x^5 - 6x^4 - 17x^2 - 2x + 10} \]

find \text{Height-sign} \(PAC\mid_{x \to \infty}\).

Inasmuch as \(Princ.\text{TERM} PAC(x) \mid_{x \to \infty}\) has \textit{Height} no matter what the rational degree, the \textit{extraction step} to get the term responsible for \text{Height-sign} \(PAC\mid_{x \to \infty}\) can take the short route to \(Princ.\text{TERM} \text{DOUGH}(x) \mid_{x \to \infty}\):

\[ x \mid_{x \to \infty} \xrightarrow{\text{PAC}} PAC(x) \mid_{x \to \infty} = \frac{-12x^3 + 7x + 4}{+4x^5 - 6x^4 - 17x^2 - 2x + 10} \mid_{x \to \infty} = \frac{-12x^3 + 7x + 4}{+4x^5 - 6x^4 - 17x^2 - 2x + 10} \mid_{x \to \infty} = \frac{-12x^3 + [\ldots]}{+4x^5 + [\ldots]} = \frac{-12}{+4}x^3 + [\ldots] = -3x^2 + [\ldots]

and we get that

\[ \text{Height-sign} PAC \mid_{x \to \infty} = (-, -) \]

**Example 6.** Given the rational function \(SOUTH\) specified by the global input-output rule

\[ x \xrightarrow{\text{SOUTH}} SOUTH(x) = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \]

find \text{Height-sign} of \(SOUTH\) near \(\infty\).
i. We get the local rule near $\infty$ of $SOUTH$

a. We have

$$x|_{x \to \infty} \overset{SOUTH}{\rightarrow} \left. SOUTH(x) \right|_{x \to \infty} = \frac{-3x^2 - 5x + 6}{+12x^5 - 6x^3 + 8x^2 + 6x - 9}|_{x \to \infty}$$

$$= \frac{-3x^2 - 5x + 6}{+12x^5 - 6x^3 + 8x^2 + 6x - 9}|_{x \to \infty}$$

We now proceed with the two steps:

1. Approximate $-3x^2 - 5x + 6$ and $+12x^5 - 6x^3 + 8x^2 + 6x - 9$.

b. The more usual presentation is:

$$x|_{x \to \infty} \overset{SOUTH}{\rightarrow} \left. SOUTH(x) \right|_{x \to \infty} = \frac{-3x^2 - 5x + 6}{+12x^5 - 6x^3 + 8x^2 + 6x - 9}|_{x \to \infty}$$

$$= \frac{-3x^2 - 5x + 6}{+12x^5 - 6x^3 + 8x^2 + 6x - 9}|_{x \to \infty}$$

We approximate $-3x^2 - 5x + 6|_{x \to \infty}$ and $+12x^5 - 6x^3 + 8x^2 + 6x - 9|_{x \to \infty}$

$$= \frac{-3x^2 + [...]}{+12x^5 + [...]}$$

and then we divide:

$$= \frac{-3}{+12} x^{2-5} + [...]$$

$$= \frac{-1}{4} x^{-3} + [...]$$

c. Since the degree of the power function

$$x \overset{POWER}{\rightarrow} \left. POWER(x) \right|_{x \to \infty} = \frac{-1}{4} x^{-3}$$

which approximates $SOUTH$ near $\infty$ is $< 0$, the power function $POWER$ has all three features, concavity, slope and height. (This was of course to be expected from the fact that the rational degree of $SOUTH$ is $< 0$.)
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ii. We get

\[
\text{Height-sign of } SOUTH \text{ near } \infty = (-, +)
\]

**EXAMPLE 7.** Given the rational function \( DOUGH \) specified by the global input-output rule

\[
x \xrightarrow{DOUGH} DOUGH(x) = \frac{+12x^4 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6}
\]

find Height-sign of \( DOUGH \) near \( \infty \)

i. We get the local input-output rule near \( \infty \) of \( DOUGH \) as in Example .

ii. We get the local graph near \( \infty \) of \( DOUGH \) as in Example .

iii. We read the Height-sign near \( \infty \) off the local graph near \( \infty \) as in Chapter 2.—keeping in mind that we must use the Magellan viewpoint—and we get:

\[
\text{Height-sign of } DOUGH \text{ near } \infty = (-, +)
\]

**EXAMPLE 8.** Given the rational function \( BATH \) specified by global input-output rule

\[
x \xrightarrow{BATH} BATH(x) = \frac{+x^3 - 4x^2 + x + 6}{+x^2 - 4x + 3}
\]

find the local rule near \( \infty \) that gives the Height.

i. We begin by localizing both the numerator of \( BATH \) and the denominator of \( BATH \) near \( \infty \),

\[
\begin{align*}
+&x^3 - 4x^2 - 5x + 6 | \text{Localize near } \infty \quad +&x^3 - 4x^2 - 5x + 6 | \text{Localize near } \infty \\
+&x^2 - 4x + 3 | \text{Localize near } \infty \\
&x^2 - 4x + 3 | x \text{ near } \infty \\
&x^2 - 4x + 3 | x \text{ near } \infty 
\end{align*}
\]

which we carry out by just making sure that the terms are in descending order of exponents:

\[
\frac{+x^3 - 4x^2 + x + 6 | x \text{ near } \infty}{+x^2 - 4x + 3 | x \text{ near } \infty} = \frac{+x^3 - 4x^2 + x + 6}{+x^2 - 4x + 3}
\]

ii. Since what we need is only the Height, we approximate first and then divide

\[
\begin{align*}
\frac{+x^3 - 4x^2 - 5x + 6}{+x^2 - 4x + 3} & \quad \text{Approximate} \\
\frac{+x^3 - 4x^2 - 5x + 6}{+x^2 - 4x + 3} & \quad \text{Approximate} \\
\frac{+x^2 + [\ldots]}{+x^2 + [\ldots]} & \quad \text{Divide}
\end{align*}
\]

\[
\frac{+x^2 - 4x + 3}{+x^3 - 4x^2 - 5x + 6} \quad \text{Approximate}
\]
which we carry out as follows:

\[
\begin{align*}
    x \big|_{x \to \infty} \xrightarrow{BATH} BATH(x) \big|_{x \to \infty} &= \frac{+x^3 + x^2 - 5x + 6}{+x^2 - 4x + 3} \bigg|_{x \to \infty} \\
    &= \frac{+x^3 + x^2 - 5x + 6}{+x^2 - 4x + 3} \bigg|_{x \to \infty} \\
    &= +x^3 + [..] \\
    &= +1 \\
    &= +x^3 - 4x + 3 \\
    &= +x^1 + [..]
\end{align*}
\]

so that the local input-output rule near \( \infty \) is

\[
\begin{align*}
    x \big|_{x \to \infty} \xrightarrow{BATH} BATH(x) \big|_{x \to \infty} &= +x + [..]
\end{align*}
\]

which, indeed, gives the Height-sign:

\[
\text{Height-sign } BATH \big|_{x \to \infty} = (+, -)
\]

Observe that this local input-output rule near \( \infty \) also gives the Slope

\[
\text{Slope-sign } BATH \big|_{x \to \infty} = (/, /)
\]

16.5 Slope-sign Near \( \infty \)

In the case of Slope-sign \( RAT \big|_{x \to \infty} \), there are two cases depending on the rational degree of the given rational function.

1. If the rational function \( RAT \) is either:
   - A regular rational function, that is of rational degree \( > 1 \) (Chapter 4)
   - or \( < 0 \) (Chapter 5)

or

   - An exceptional rational function of rational degree \( = 1 \) (In Chapter 6),

In other words:

If \( RAT \) is anything but an exceptional rational function of rational degree \( = 0 \),

Then \( \text{Princ. TERM RAT(x)} \big|_{x \to \infty} \) will be a power function that will have Slope near \( \infty \) and so the extraction step to get the term responsible for Slope-sign \( \big|_{x \to \infty} \) can take the short route to \( \text{Princ. TERM RAT(x)} \big|_{x \to \infty} \):
In other words,

a. As always, the localization step is to localize near $\infty$ both

- $POLY_{Num}(x)$ to $POLY_{Num}(x) \big|_{x \nearrow \infty}$
- $POLY_{Den}(x)$ to $POLY_{Den}(x) \big|_{x \nearrow \infty}$

which near $\infty$ amounts only to making sure that the terms are in descending order of exponents.

b. Inasmuch as the rational degree of $RAT$ is not $= 0$, the extraction step to get the term responsible for $Slope$-sign $\big|_{x \nearrow \infty}$ can take the short route to $Princ.TERM RAT(x) \big|_{x \nearrow \infty}$:

i. Approximate both

- $POLY_{Num}(x) \big|_{x \nearrow \infty}$ to $Princ.TERM POLY_{Num}(x) \big|_{x \nearrow \infty}$
- $POLY_{Den}(x) \big|_{x \nearrow \infty}$ to $Princ.TERM POLY_{Den}(x) \big|_{x \nearrow \infty}$

ii. Divide $Princ.TERM POLY_{Num}(x) \big|_{x \nearrow \infty}$ by $POLY_{Den}(x) \big|_{x \nearrow \infty}$

to get $Princ.TERM RAT(x) \big|_{x \nearrow \infty}$ $+$ $[...]$.

2. If the rational function $RAT$ is an exceptional rational function whose rational degree $= 0$, then $Princ.TERM RAT(x) \big|_{x \nearrow \infty}$ will be an exceptional power function with exponent $= 0$ (In Chapter 6) and $Princ.TERM RAT(x) \big|_{x \nearrow \infty}$ will not have $Slope$ and so the extraction step to get the term responsible for $Height$-sign $\big|_{x \nearrow \infty}$ will have to take the long route to a $Princ.PART RAT(x) \big|_{x \nearrow \infty}$ that has $Slope$:

In other words

a. As always, the localization step is to localize near $\infty$ both
16.5. SLOPE-SIGN NEAR $\infty$

- $POLY_{\text{Num}}(x)$ to $POLY_{\text{Num}}(x) \big|_{x \to \infty}$
- $POLY_{\text{Den}}(x)$ to $POLY_{\text{Den}}(x) \big|_{x \to \infty}$

which near $\infty$ amounts only to making sure that the terms are in descending order of exponents.

b. Inasmuch as $RAT$ is an exceptional rational function of rational degree $= 0$, $\text{Princ. TERM RAT}(x) \big|_{x \to \infty}$ has no Slope and the extraction step to get the term responsible for $\text{Slope-sign} \big|_{x \to \infty}$ will have to take the long route to $\text{Princ. PART RAT}(x) \big|_{x \to \infty}$:

i. Setup the long division of $POLY_{\text{Den}}(x) \big|_{x \to \infty}$ into $POLY_{\text{Num}}(x) \big|_{x \to \infty}$

ii. Stop the long division when as soon as we get get a $\text{Princ. TERM RAT}(x) \big|_{x \to \infty} + [...]$ that has Slope:

**EXAMPLE 9.** Given the rational function $SOUTH$ specified by the global input-output rule

\[
x \mapsto SOUTH(x) = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9}
\]

find Slope-sign of $SOUTH$ near $\infty$

a. We have

\[
x \big|_{x \to \infty} \mapsto SOUTH(x) \big|_{x \to \infty} = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \bigg|_{x \to \infty}
\]

We now proceed with the two steps:

\[
\begin{align*}
-3x^2 &- 5x + 6 \\
+12x^4 &- 6x^3 + 8x^2 + 6x - 9
\end{align*}
\]

- Approximate

\[
\begin{align*}
-3x^2 &+ [...] \\
+12x^4 &+ [...] \\
\end{align*}
\]

- Divide

\[
- \frac{1}{4} x^3 + [...] \]

b. The more usual presentation is:

\[
x \big|_{x \to \infty} \mapsto SOUTH(x) \big|_{x \to \infty} = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \bigg|_{x \to \infty}
\]

\[
= \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \bigg|_{x \to \infty}
\]

\[
= \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \bigg|_{x \to \infty}
\]
We approximate $\frac{-3x^2 - 5x + 6}{x}$ near $\infty$ and $\frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{x}$ near $\infty$

\[
\begin{align*}
-3x^2 + [...] & \quad +12x^5 + [...]
\end{align*}
\]

and then we divide:

\[
\begin{align*}
= -3x^2 - 5 + [...] & \quad +12x^5 + [...] \\
= \frac{1}{4}x^{-3} + [...] 
\end{align*}
\]

c. Since the degree of the power function

\[
x \xrightarrow{\text{POWER}} \text{POWER}(x) = \frac{1}{4}x^{-3}
\]

which approximates $SOUTH$ near $\infty$ is $< 0$, the power function $\text{POWER}$ has all three features, concavity, slope and height. (This was of course to be expected from the fact that the rational degree of $SOUTH$ is $< 0$.)

ii. We get

Slope-sign of $SOUTH$ near $\infty = (\searrow, \swarrow)$

**Example 10.** Given the rational function $DOUGH$ specified by the global input-output rule

\[
x \xrightarrow{\text{DOUGH}} \text{DOUGH}(x) = \frac{+12x^4 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6}
\]

find Slope-sign of $DOUGH$ near $\infty$

i. We get the local input-output rule near $\infty$ of $DOUGH$ as in Example .

ii. We get the local graph near $\infty$ of $DOUGH$ as in Example .

iii. We read the Slope-sign near $\infty$ off the local graph near $\infty$ as in Chapter 2.—keeping in mind that we must use the Magellan viewpoint—and we get:

Slope-sign of $DOUGH$ near $\infty = (\searrow, \nearrow)$

16.6 Concavity-sign Near $\infty$

In the case of $\text{Concavity-sign RAT}_{|x_\infty}$, there are two cases depending on the rational degree of the given rational function.

1. If the rational function $RAT$ is a *regular* rational function, that is if the rational degree of $RAT$ is either $> 1$ or $< 0$, then $\text{Princ. TERM RAT}(x) |_{x \to \infty}$ will be a *regular power function*, that is a power function whose exponent
16.6. CONCAVITY-SIGN NEAR $\infty$

is either $> 1$ (Chapter 4) or $< 0$ (Chapter 5) and then, in either case, \( \text{Princ. TERM RAT}(x) \big|_{x \to \infty} \) will have Concavity and so the extraction step to get the term responsible for Concavity-sign|$x \to \infty$ can take the short route to \( \text{Princ. TERM Den}(x) \big|_{x \to \infty} \):

\[
\begin{align*}
\text{POLY}_\text{Num}(x) \bigg|_{x \to \infty} & \quad \Rightarrow \quad \text{POLY}_\text{Num}(x) \big|_{x \to \infty} \quad \Rightarrow \quad \text{i. Approximate} \quad \Rightarrow \quad \text{Princ. TERM Num}(x) \big|_{x \to \infty} + [\ldots] \\
\text{POLY}_\text{Den}(x) \bigg|_{x \to \infty} & \quad \Rightarrow \quad \text{POLY}_\text{Den}(x) \big|_{x \to \infty} \\
\end{align*}
\]

In other words,

a. As always, the localization step is to localize near $\infty$ both

- \( \text{POLY}_\text{Num}(x) \) to \( \text{POLY}_\text{Num}(x) \big|_{x \to \infty} \)
- \( \text{POLY}_\text{Den}(x) \) to \( \text{POLY}_\text{Den}(x) \big|_{x \to \infty} \)

which near $\infty$ amounts only to making sure that the terms are in descending order of exponents.

b. Inasmuch as \( \text{RAT} \) is a regular rational function, the extraction step to get the term responsible for Concavity-sign|$x \to \infty$ can take the short route to \( \text{Princ. TERM Den}(x) \big|_{x \to \infty} \):

i. Approximate both

- \( \text{POLY}_\text{Num}(x) \big|_{x \to \infty} \) to \( \text{Princ. TERM POLY}_\text{Num}(x) \big|_{x \to \infty} \)
- \( \text{POLY}_\text{Den}(x) \big|_{x \to \infty} \) to \( \text{Princ. TERM POLY}_\text{Den}(x) \big|_{x \to \infty} \)

ii. Divide \( \text{Princ. TERM POLY}_\text{Num}(x) \big|_{x \to \infty} \) by \( \text{POLY}_\text{Den}(x) \big|_{x \to \infty} \) to get \( \text{Princ. TERM RAT}(x) \big|_{x \to \infty} \) + [\ldots].

2. If the rational function \( \text{RAT} \) is an exceptional rational function that is if the rational degree of \( \text{RAT} \) is either $1$ or $0$ then \( \text{Princ. TERM RAT}(x) \big|_{x \to \infty} \) will be an exceptional power function with exponent either $1$ or $0$ (Chapter 7) and in both cases \( \text{Princ. TERM RAT}(x) \big|_{x \to \infty} \) will not have Concavity and the extraction step to get the term responsible for Concavity-sign|$x \to \infty$ will have to take the long route to a \( \text{Princ. PART RAT}(x) \big|_{x \to \infty} \) that does have Concavity.
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\[ \frac{\text{POLY}_{\text{Num}}(x)}{\text{POLY}_{\text{Den}}(x)} \bigg|_{x \to \infty} \quad \text{Localize near } \infty \]

\[ \frac{\text{POLY}_{\text{Num}}(x)}{\text{POLY}_{\text{Den}}(x)} \bigg|_{x \to \infty} \]

\[ \text{Localize near } \infty \]

\[ \frac{\text{POLY}_{\text{Den}}(x)}{\text{POLY}_{\text{Num}}(x)} \bigg|_{x \to \infty} \]

\[ \frac{\text{Princ. PART RAT}(x)}{x \to \infty} + […] \]

In other words:

a. As always, the localization step is to localize near ∞ both
   - \( \text{POLY}_{\text{Num}}(x) \) to \( \text{POLY}_{\text{Num}}(x) \big|_{x \to \infty} \)
   - \( \text{POLY}_{\text{Den}}(x) \) to \( \text{POLY}_{\text{Den}}(x) \big|_{x \to \infty} \)

b. Inasmuch as RAT is an exceptional rational function, \( \text{Princ. TERM RAT}(x) \big|_{x \to \infty} \) will have no concavity and the extraction step to get the term responsible for \( \text{Concavity-sign} \big|_{x \to \infty} \) will have to take the long route to \( \text{Princ. PART RAT}(x) \big|_{x \to \infty} \):
   i. Set up the long division of \( \text{POLY}_{\text{Den}}(x) \big|_{x \to \infty} \) into \( \text{POLY}_{\text{Num}}(x) \big|_{x \to \infty} \)
   ii. Stop the long division when as soon as we get a \( \text{Princ. PART RAT}(x) \big|_{x \to \infty} \) that has \( \text{Concavity} \):

Example 11. Given the rational function \( \text{BATH} \) specified by the global input-output rule

\[ x \overset{\text{BATH}}{\to} \text{BATH}(x) = \frac{+x^3 - 5x^2 + x + 6}{+x^2 - 4x + 3} \]

find \( \text{Concavity-sign} \ \text{BATH} \big|_{x \to \infty} \).

a. The localization step is to localize both the numerator and the denominator near ∞—which amounts only to making sure that the terms are in descending order of exponents.

\[ \frac{+x^3 - 5x^2 + x + 9}{+x^2 - 4x + 3} \]

\[ \frac{+x^3 - 5x^2 + x + 9}{x \to \infty} \]

\[ +x^2 - 4x + 3 \]

\[ +x^2 - 4x + 3 \bigg|_{x \to \infty} \]

b. Since \( \text{Princ. TERM} \ \text{BATH}(x) \big|_{x \to \infty} \) has no concavity, the extraction step to get \( \text{Concavity-sign} \ \text{BATH} \big|_{x \to \infty} \) must take the long route to a \( \text{Princ. PART} \ \text{BATH}(x) \big|_{x \to \infty} \) that has concavity:
   i. We set up the division as a long division:

\[ +x^2 - 4x + 3 \] dividing into \( +x^3 - 5x^2 + x + 9 \).
ii. We approximate by stopping the long division as soon as we have the principal part of the quotient that has Concavity:

\[
\begin{align*}
\frac{x^3 - 5x^2 + x + 9}{x^2 - 4x + 3} & \approx \frac{x^3 - 5x^2 + x + 9}{x^2 - 4x + 3} \\
& \approx \frac{x^3}{x^2} + \frac{0x - 6}{0x - 6} + \cdots \\
& = x - 6x^{-1} + \cdots
\end{align*}
\]

that is we stop with \(-6x^{-1}\) since it is the term responsible for Concavity.

The more usual way to write all this is:

\[
x \big|_{x \to \infty} \text{BATH} \quad \text{BATH}(x) \big|_{x \to \infty} = \frac{+x^3 - 5x^2 + x + 9}{+x^2 - 4x + 3} \big|_{x \to \infty}
\]

and then we divide (in the latin manner):
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\[ +x^2 - 4x + 3 \]
\[ +x^3 - 5x^2 + x + 9 \]
\[ +x^3 - 4x^2 + 3x \]
\[ 0x^3 - x^2 - 2x + 9 \]
\[ -x^2 + 4x - 3 \]
\[ 0x^2 - 6x + 12 \]

Whichever way we write it, $Princ.\ PART\ BATH(\frac{f(x)}{g(x)})|_{x\ near\ \infty} = +x - 1 - 6x^{-1}$ and its third term, $-6x^{-1}$, gives $Concavity\-sign\ BATH|_{x\ near\ \infty} = (\cap, \cup)$

**EXAMPLE 12.** Given the rational function $SOUTH$ specified by the global input-output rule

\[ x \rightarrow SOUTH(x) = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \]

find Concavity-sign of $SOUTH$ near $\infty$

i. We get the local graph near $\infty$ of $SOUTH$

a. We have

\[ x|_{x\ near\ \infty} \rightarrow SOUTH(x)|_{x\ near\ \infty} = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \]

We now proceed with the two steps:

\[ \approx \frac{-3x^2 + [...]}{+12x^3 + [...]} \]

b. The more usual presentation is:

\[ x|_{x\ near\ \infty} \rightarrow SOUTH(x)|_{x\ near\ \infty} = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \]

We approximate $-3x^2 - 5x + 6|_{x\ near\ \infty}$ and $+12x^3 + 8x^2 + 6x - 9|_{x\ near\ \infty}$
and then we divide:

\[
\frac{-3}{+12} x^{2-5} + [\ldots] \\
\frac{-1}{4} x^{-3} + [\ldots]
\]

c. Since the degree of the power function

\[
x \xrightarrow{\text{POWER}} \text{POWER}(x) = \frac{1}{4} x^{-3}
\]

which approximates SOUTH near \( \infty \) is < 0, the power function POWER has all three features, concavity, slope and height. (This was of course to be expected from the fact that the rational degree of SOUTH is < 0.)

ii. We get

Concavity-sign of SOUTH near \( \infty \) = (\( \cap \), \( \cap \))

**Example 13.** Given the rational function DOUGH specified by the global input-output rule

\[
x \xrightarrow{\text{DOUGH}} \text{DOUGH}(x) = \frac{+12x^4 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6}
\]

find Concavity-sign of DOUGH near \( \infty \)

i. We get the local input-output rule near \( \infty \) of DOUGH as in Example .

ii. We get the local graph near \( \infty \) of DOUGH as in Example .

iii. We read the Concavity-sign near \( \infty \) off the local graph near \( \infty \) as in Chapter 2.—keeping in mind that we must use the Magellan viewpoint—and we get:

Concavity-sign of DOUGH near \( \infty \) = (\( \cap \), \( \cap \))

### 16.7 Local Graph Near \( \infty \)

1. Finding the local graph near \( \infty \) is a local issue in that the inputs that are involved are all near a given input, namely \( \infty \). That the given input is \( \infty \) makes it a bit different from when the given input is a bounded input \( x_0 \) but only because a Magellan picture is needed to see \( \infty \).

2. The local graph near \( \infty \) can be something that we have to find as such but it can also be something we have to find because we are being asked for a feature-sign near \( \infty \) such as the concavity-sign near \( \infty \), the slope-sign near \( \infty \) or the height-sign near \( \infty \).

Also, finding the local graph near \( \infty \) is the first step in the procedure we use to get the essential global graph which is something we would have to do if we were asked for any global information, such as concavity-sign change input(s), slope-sign change input(s), height-sign change input(s), etc.
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3. In order to get the local graph near $\infty$ of a rational function, we proceed as usual that is we must first get the local rule near $\infty$.

**Example 14.** Given the rational function $SOUTH$ whose global input-output rule is

$$x \xrightarrow{SOUTH} SOUTH(x) = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9}$$

find its local graph near $\infty$.

i. We get the local input-output rule near $\infty$ as in **Example 1**.

We have:

$$x \big|_{x \nearrow \infty} \xrightarrow{SOUTH} SOUTH(x) \big|_{x \nearrow \infty} = \frac{-3x^2 - 5x + 6}{+12x^4 - 6x^3 + 8x^2 + 6x - 9} \big|_{x \nearrow \infty}$$

We approximate separately the numerator and the denominator:

$$= -3x^2 + [...]$$

$$+12x^5 + [...]$$

We divide the approximations:

$$= \frac{-3}{12} x^{-5} + [...]$$

$$= -\frac{1}{4} x^{-3} + [...]$$

ii. Since the degree of the power function

$$x \xrightarrow{POWER} POWER(x) = -\frac{1}{4} x^{-3}$$

is $< 0$, the power function $POWER$ is regular and has both concavity and slope. So, the local graph of the power function $POWER$ near $\infty$ will be approximately the graph near $\infty$ of the rational function $SOUTH$.

The local graph near $\infty$ of the rational function $SOUTH$ is therefore:
**Example 15.** Given the rational function \( \text{DOUGH} \) whose global input-output rule is

\[
x \xrightarrow{\text{DOUGH}} \text{DOUGH}(x) = \frac{+12x^4 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6}
\]

find its local graph near \( \infty \).

i. We get the local input-output rule near \( \infty \).

We have:

\[
\left. x \right|_{x \to \infty} \xrightarrow{\text{DOUGH}} \text{DOUGH}(x) \bigg|_{x \to \infty} = \frac{+12x^5 - 6x^3 + 8x^2 + 6x - 9}{-3x^2 - 5x + 6} \bigg|_{x \to \infty}
\]

We approximate separately the numerator and the denominator:

\[
= +12x^5 + \ldots \quad \frac{-3x^2 - 5x + 6}{x \to \infty}
\]

We divide the approximations:

\[
= -\frac{12}{-3}x^3 - 2 + \ldots
\]

ii. Since the degree of the power function

\[
x \xrightarrow{\text{POWER}} \text{POWER}(x) = -4x^3
\]

is \( > 1 \), the power function \( \text{POWER} \) is **regular** and has both concavity and slope. So, the local graph of the power function \( \text{POWER} \) near \( \infty \) will be approximately the graph near \( \infty \) of the rational function \( \text{DOUGH} \).

The local graph near \( \infty \) of the rational function \( \text{DOUGH} \) is therefore:

**Example 16.** Given the rational function \( \text{BATH} \) specified by the global input-output rule

\[
x \xrightarrow{\text{BATH}} \text{BATH}(x) = \frac{+x^3 + x^2 - 5x + 6}{+x^2 - 4x + +3}
\]

as in **Example 1**, find the local graph near \( \infty \).

i. We get the local input-output rule near \( \infty \) that gives all three features as we did in **Example 1**:

\[
x \bigg|_{x \to \infty} \xrightarrow{\text{BATH}} \text{BATH}(x) \bigg|_{x \to \infty} = +x + 5 + 27x^{-1} + \ldots
\]

ii. So the local graph near \( \infty \) of the function \( \text{BATH} \) is
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**Example 17.** Find the local rule near $\infty$ of the function whose global input-output rule is

$$x \xrightarrow{\text{CAP}} \text{CAP}(x) = \frac{-12x^3 + 7x + 4}{+4x^3 - 17x^2 - 2x + 10}$$

Near $\infty$, we have

$$x \big|_{x \text{ near } \infty} \xrightarrow{\text{CAP}} \left. \text{CAP}(x) \right|_{x \text{ near } \infty} = \frac{-12x^3 + 7x + 4}{+4x^3 - 17x^2 - 2x + 10} \big|_{x \text{ near } \infty}$$

$$= \frac{-12x^3 + 7x + 4 + 4x^3 + 4}{+4x^3 - 17x^2 - 2x + 10} \big|_{x \text{ near } \infty}$$

$$= \frac{-12x^3 + 7x + 4}{+4x^3 - 17x^2 - 2x + 10} \big|_{x \text{ near } \infty}$$

$$= -12x^3 + 7x + 4 + 4x^3 + 4$$

$$= -12x^3 + 7x + 4 + 4$$

$$= -12x^3 + 7x + 4$$

and since the power function

$$x \xrightarrow{\text{PWR}} \text{PWR}(x) = -3x^0$$

is exceptional and lacks both concavity and slope it will give us the height-sign near $\infty$ of $\text{CAP}$ but no other local information near $\infty$.

**Example 18.** Find the local rule near $\infty$ of the function $\text{DAC}$ whose global input-output rule is

$$x \xrightarrow{\text{DAC}} \text{DAC}(x) = \frac{-12x^3 + 7x + 4}{+4x^3 - 17x^2 - 2x + 10}$$
Near $\infty$, we have

$$x|_{x \nearrow \infty}^{\text{DAC}} \rightarrow CAP(x)|_{x \nearrow \infty} = \frac{-12x^3 + 7x + 4}{+4x^3 - 17x^2 - 2x + 10}|_{x \nearrow \infty}$$

$$= \frac{-12x^3 + 7x + 4}{+4x^3 - 17x^2 - 2x + 10}|_{x \nearrow \infty} = -12x + 7 + [\ldots]$$

$$= +4x^3 + [\ldots]$$

$$= -12 + 3 + [\ldots]$$

and since the power function

$$x \rightarrow PWR \rightarrow PWR(x) = -3x^0$$

is exceptional and lacks concavity it will give us the height-sign near $\infty$ and the slope-sign near $\infty$ of $DAC$ but no other local information near $\infty$.

**Example 19.** Given the rational function $HEATH$ specified by global input-output rule

$$x \rightarrow HEATH \rightarrow HEATH(x) = \frac{+x^2 - 5x + 6}{+x^2 - 4x + 3}$$

find the local rule near $\infty$ that gives the Height.

i. We begin by localizing both the numerator of $HEATH$ and the denominator of $HEATH$ near $\infty$,

$$\begin{align*}
\frac{x^2 - 5x + 6}{+x^2 - 4x + 3} & \quad \text{Localize near } \infty \\
\frac{x^2 - 5x + 6}{+x^2 - 4x + 3} & \quad \text{Localize near } \infty
\end{align*}$$

which we carry out by just making sure that the terms are in descending order of exponents:

$$\begin{align*}
+\frac{x^2 - 5x + 6}{x^2 - 4x + 3} & \quad \text{x near } \infty \\
+\frac{x^2 - 5x + 6}{x^2 - 4x + 3} & \quad \text{x near } \infty
\end{align*}$$

ii. Since what we need is only the Height, we approximate first and then divide
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\[
\frac{x^2 - 5x + 6}{x^2 - 4x + 3} \div \frac{+x^2 + [\ldots]}{+x^2 + [\ldots]}
\]

which we carry out as follows:

\[
x \mid_{x \text{ near } \infty} \xrightarrow{\text{HEATH}} \text{HEATH}(x) \mid_{x \text{ near } \infty} = \frac{+x^2 - 5x + 6}{+x^2 - 4x + 3} \mid_{x \text{ near } \infty} = \frac{+x^2 - 5x + 6}{+x^2 - 4x + 3} \mid_{x \text{ near } \infty}
\]

so that the local input-output rule near $\infty$ is

\[
x \mid_{x \text{ near } \infty} \xrightarrow{\text{HEATH}} \text{HEATH}(x) \mid_{x \text{ near } \infty} = +1 + [\ldots]
\]

which, indeed, gives only the Height-sign

**Example 20.** Given the rational function $\text{HEATH}$ specified by global input-output rule

\[
x \xrightarrow{\text{HEATH}} \text{HEATH}(x) = \frac{+x^2 - 5x + 6}{+x^2 - 4x + 3}
\]

find the local rule near $\infty$ that gives the Slope.

i. We begin by localizing both the numerator of $\text{HEATH}$ and the denominator of $\text{HEATH}$ near $\infty$,

\[
\frac{x^2 - 5x + 6}{x^2 - 4x + 3} \mid_{x \text{ near } \infty}
\]

which we carry out by just making sure that the terms are in descending order of exponents:

\[
\frac{+x^2 - 5x + 6}{+x^2 - 4x + 3} \mid_{x \text{ near } \infty} = \frac{+x^2 - 5x + 6}{+x^2 - 4x + 3}
\]
ii. Since what we need is the *slope*, we divide and will stop the division as soon as we get the term in the quotient that gives the slope.

\[
\frac{x^2 - 5x + 6}{x^2 - 4x + 3} \quad \text{Approximate}\]

\[
+ x^2 + [...] \quad \text{Approximate}\]

which we carry out as follows (in the *latin* manner)

\[
\begin{array}{ccc}
+ & x^2 - 4x + 3 & )
\end{array}
\begin{array}{cccc}
+1 & -x^{-1} & +[...] \\
+ & x^2 & -5x & + 6 \\
+ & x^2 & -4x & + 3 \\
- & 0 & x^2 & -x + 3
\end{array}
\]

Note that after we divided \( +x^2 \) into \( -x \) which gave us \(-x^{-1}\) in the quotient, we stopped the division because we now have a term that gives both *slope* and *concavity*. So, there was no reason to multiply and subtract.

So, the local input-output rule near \( \infty \) is:

\[
x \big|_{x \near \infty} = +1 - x^{-1} + [...]\]

which, along with *height-sign*, also gives *slope-sign* and *concavity-sign*

**Example 21.** Given the rational function \( HEATH \) specified by the global input-output rule

\[
x \xrightarrow{HEATH} HEATH(x) = \frac{+x^2 - 5x + 6}{+x^2 - 4x + 3}
\]

as in Example 1, find the local graph near \( \infty \).

i. We get the *local input-output rule* near \( \infty \) that gives all three features as we did in Example 1:

\[
x \big|_{x \near \infty} HEATH(x) \big|_{x \near \infty} = +1 - x^{-1} + [...]\]

ii. So the local graph near \( \infty \) of the function \( HEATH \) is
EXAMPLE 22. Given the rational function $HEATH$ specified by the global input-output rule

$$x \overset{HEATH}{\rightarrow} HEATH(x) = \frac{+x^2 - 5x + 6}{+x^2 - 4x + 3}$$

as in Example 1, find Height-sign of $HEATH$ near $\infty$

i. We get the local input-output rule near $\infty$ of $HEATH$ by approximating first as in Example 1.

$$x|_{x \rightarrow \infty} \overset{HEATH}{\rightarrow} HEATH(x)|_{x \rightarrow \infty} = +1 + [...]

ii. So, keeping in mind that we must use the Magellan viewpoint, the Height-sign of $HEATH$ near $\infty$, is:

Height-sign $HEATH|_{x \rightarrow \infty} = (+, +)$

EXAMPLE 23. Given the rational function $HEATH$ specified by the global input-output rule

$$x \overset{HEATH}{\rightarrow} HEATH(x) = \frac{+x^2 - 5x + 6}{+x^2 - 4x + 3}$$

as in Example 2, find Slope-sign of $HEATH$ near $\infty$

i. We get the local input-output rule near $\infty$ of $HEATH$ by approximating first as in Example 1.

$$x|_{x \rightarrow \infty} \overset{HEATH}{\rightarrow} HEATH(x)|_{x \rightarrow \infty} = +1 - x^{-1} + [...]

ii. Here it is usually safer to construct the local graph near $\infty$:

iii. Keeping in mind that we must use the Magellan viewpoint, we see from the local graph near $\infty$ that:

Slope-sign $HEATH|_{x \rightarrow \infty} = (\searrow, /)$

EXAMPLE 24. Given the rational function $HEATH$ specified by the global input-output rule

$$x \overset{HEATH}{\rightarrow} HEATH(x) = \frac{+x^2 - 5x + 6}{+x^2 - 4x + 3}$$

as in Example 2, find Concavity-sign of $HEATH$ near $\infty$
i. We get the local input-output rule near $\infty$ of $HEATH$ by approximating first as in Example 1.

\[
x|_{x \to \infty}^\text{HEATH} \quad HEATH(x)|_{x \to \infty} = +1 - x^{-1} + [...]\]

ii. Here it is usually safer to construct the local graph near $\infty$:

iii. Keeping in mind that we must use the Magellan viewpoint, we see from the local graph near $\infty$ that:

Concavity-sign $HEATH|_{x \to \infty} = (\cap, \cup)$