Chapter 3

Graphic Local Analysis


When a function is specified by a global input-output rule and we want to get its bounded graph, what we will do instead of plotting a number of input-output pairs and “joining them smoothly”, as taught in high school, will be:

- To get a small number of a certain kind of “partial graphs”—instead of plot points,
- To join “smoothly” these “partial graphs”.

Moreover and as already mentioned, in order to investigate functions specified by a global input-output rule, we will also need to pin down and discuss a number of “features”—and therefore to develop a language to do so—which we will also do in terms of “partial graphs”. In particular, we will have to investigate what we will call notable inputs, that is inputs with some of these “features” and, indeed, one of the most important aspects of our investigation of functions specified by a global input-output rule will be to locate these notable inputs.

3.1 Local Graphs

We begin by introducing the kind of “partial graph” that we will be using.

1. Given the graph of a function, the local graph near ∞ is the part
EXAMPLE 1. Given the function whose quantitative graph is

\[
\begin{array}{cccccc}
\text{Output} & \text{Ruler} \\
\hline
& \text{Input} \text{ Ruler} \\
\end{array}
\]

the local graph near \(+\infty\) is

\[
\begin{array}{cccccc}
\text{Output} & \text{Ruler} \\
\hline
& \text{Input} \text{ Ruler} \\
\end{array}
\]

In fact, though, we will often have to look separately at:

- the local graph near \(+\infty\), i.e. the local graph for inputs left of \(\infty\),
- the local graph near \(-\infty\), i.e. the local graph for inputs right of \(\infty\).

EXAMPLE 2. Given the function whose quantitative graph is

\[
\begin{array}{cccccc}
\text{Output} & \text{Ruler} \\
\hline
& \text{Input} \text{ Ruler} \\
\end{array}
\]

the local graph near \(+\infty\) is
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the local graph near $-\infty$ is

2. Given the graph of a function and given a bounded input $x_0$, a local graph near $x_0$ is the part of the graph for the inputs that are near $x_0$.

**Example 3.** Given the function whose quantitative graph is

the local graph near $+3$ is
3.2 Local Language

Given a function and given an input, which can be \( \infty \) or a bounded input \( x_0 \), the local graph near that input gives us a picture of what we will call the local behavior of the given function near the given input. However, we will also need to describe this local behavior in words and so we will have to develop what we will call a local language.

Since there is no reason for the local behavior to be the same on both sides of the center of the neighborhood, the local language will have to take care separately of the behavior left of the center and of the behavior right of the center.

Given a function and given a local graph near a given input, infinity or bounded, the paren that we will use to describe the local graph that pictures the local behavior will consist of a pair of parentheses to represent the neighborhood with a comma in-between to represent the given bounded input at the center of the neighborhood. Then,

i. The code that we will write into the paren left of the comma will describe the local graph for inputs that are left of the center as seen when facing the center of the neighborhood,

ii. The code that we will write into the paren right of the comma will describe the local graph for inputs that are right of the center as seen when facing the center of the neighborhood.
3.3 PLACE OF A LOCAL GRAPH

We will now introduce the various codes that we can insert in a paren to describe the local behavior of a given function near a given input, whether the given input is infinite, $\infty$, or bounded, $x_0$.

3.3 Place of a Local Graph

The first kind of local feature that we will discuss are the two local features that give, on either side of the given input, the place of the local graph:

1. The height size on a side of a given input depends on the size of the outputs of nearby inputs on that side.

   a. More precisely:
      - The height size is infinite when the outputs for nearby inputs on that side are infinite.

      Graphically, this means that the local graph is in the stretch between the screen and either one of the two $\infty$ output level line,

      ![Graphical representation of the local graph](image)

      We will use the symbol $\infty$ to code into a paren the fact that the height size is infinite.

   **Example 4.** Given the function $XXX$ whose local graph near $+5$ is
bounded

- the height size on the left side of +5 is infinite
- the height size on the right side of +5 is infinite
which we code into a paren as:

\[ \text{Height Size } XXXX \text{ near } +5 = (\infty, \infty) \]

- The height size is bounded when the outputs for nearby inputs on that side are bounded.

Graphically, this means that the side of the local graph is in the stretch across the screen.

We will use the symbol $\flat$ to code into a paren the fact that the height size is bounded.

**Example 5.** Given the function $XXXX$ whose local graph near +5 is

- the height size on the left side of +5 is bounded
- the height size on the right side of +5 is infinite
which we code into a *paren* as:  
Height Size *XXXX* near $+5 = (0, \infty)$

**b.** We will often need to know more than whether or not the local graph for a given input is *bounded* and, particularly in the case of *Power Functions*, we will need the following:  
The *height size* is *infinitesimal* when the outputs for nearby inputs on that side are *infinitesimal*.

Graphically, this means that the side of the local graph is in a narrow stretch on either side of the 0 output level line.

We will use the symbol 0 to code into a *paren* the fact that the height size is *infinitesimal*.

**EXAMPLE 6.** Given the function *XXXX* whose local graph near $+5$ is

- the height size on the *left side* of $+5$ is *infinitesimal*
- the height size on the *right side* of $+5$ is *infinite*

which we code into a *paren* as:  
Height Size *XXXX* near $+5 = (0, \infty)$

**c.** To summarize:

We will *code* the *height size* into a *paren* (* * *)
- with $\infty$ to say that the height size is *infinite*
- with $b$ to say that the height size is *bounded*
- with 0 to say that the height size is *infinitesimal*.
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Example 7. Given a function $JACK$ and an input $x_0$, a multiple choice question might look like:

- a. Height Size $JACK$ near $x_0 = (\infty, \infty)$
- b. Height Size $JACK$ near $x_0 = (\infty, 0)$
- c. Height Size $JACK$ near $x_0 = (\infty, \infty)$
- d. Height Size $JACK$ near $x_0 = (0, 0)$
- e. None of the preceding.

2. The height sign on a side of a local graph depends on the sign of the outputs for nearby inputs on that side. More precisely,
   - The height sign on a side of a local graph is above 0 when the outputs for inputs on that side are positive.
     Graphically, this means that the side of the local graph is above the 0 output level line.

   We will use the symbol + to code into a paren the fact that the height sign is above 0.

Example 8. Given the function $XXXX$ whose local graph near $+5$ is

   - the height sign on the left side of $+5$ is +
   - the height sign on the right side of $+5$ is +
     which we code into a paren as:
     Height Sign $XXXX$ near $+5 = (+, +)$
   - The height sign of a side of a local graph is below 0 when the outputs
for nearby inputs on that side are negative.

Graphically, this means that the side of the local graph is below the 0 output level line.

We will use the symbol $-$ to code into a paren the fact that the height sign is below 0.

**Example 9.** Given the function $XXX$ whose local graph near $+5$ is

- the height sign on the left side of $+5$ is above 0
- the height sign on the right side of $+5$ is below 0
which we code into a paren as:

Height Sign $XXX$ near $+5 = (+,-)$

To summarize:

> We will code the height sign into a paren $( , )$
- with $+$ to say that the local graph is above 0,
- with $-$ to say that the local graph is below 0.

**Example 10.** Given a function $JACK$ and an input $x_0$, a multiple choice question might look like:

a. Height Sign $JACK$ near $x_0 = (+,+)$  

b. Height Sign $JACK$ near $x_0 = (+,-)$  

c. Height Sign $JACK$ near $x_0 = (-,+)$
3. The **Height Sign-Size** of a local graph is the *height sign* together with the *height size*. However, we will have to keep in mind an unfortunate linguistic “peculiarity”, namely that:

- When the code for the *size* is ∞, the code for the *sign* is written before the code for the *size* just as with *signed numbers*: +∞, −∞.
- But when the code for the *size* is 0 or ♭, the code for the *sign* is written after the code for the *size* and just as if it were an *exponent*: 0+, 0−, ♭+, ♭−.

**NOTE.** Tradition has it that the reason for this peculiarity is that a + or a − in front of 0 or ♭ might be construed as being the *sign* of 0 or ♭ rather than the *side* of 0 that the inputs are on.\(^1\)

**EXAMPLE 11.** Given a function *JILL* and an input \(x_0\), a multiple choice question might look like:

- a. Height Sign-Size *JILL* near \(x_0\) = (+∞, +∞)
- b. Height Sign-Size *JILL* near \(x_0\) = (+∞, −∞)
- c. Height Sign-Size *JILL* near \(x_0\) = (−∞, +∞)
- d. Height Sign-Size *JILL* near \(x_0\) = (−∞, −∞)
- e. None of the preceding.

**EXAMPLE 12.** Given a function *JACK* and an input \(x_0\), a multiple choice question might look like:

- a. Height Sign-Size *JACK* near \(x_0\) = (0+, 0+)
- b. Height Sign-Size *JACK* near \(x_0\) = (0+, 0−)
- c. Height Sign-Size *JACK* near \(x_0\) = (0−, 0+)
- d. Height Sign-Size *JACK* near \(x_0\) = (0−, 0−)
- e. None of the preceding.

**EXAMPLE 13.** Given a function *MIKE* and an input \(x_0\), a multiple choice question might look like:

- a. Height Sign-Size *MIKE* near \(x_0\) = (−∞, +♭)
- b. Height Sign-Size *MIKE* near \(x_0\) = (0+, 0−)
- c. Height Sign-Size *MIKE* near \(x_0\) = (−♭, +∞)
- d. Height Sign-Size *MIKE* near \(x_0\) = (0−, 0−)
- e. None of the preceding.

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\(^{1}\)Here it must be admitted that the tradition antedates the birth of Educology.
3.4 ∞-Height Inputs and 0-Height Inputs

It will turn out that most bounded inputs have a bounded output and we will have two kinds of notable inputs:

1. For the purpose of getting the graph of a function, the most important kind of notable inputs will be ∞-height bounded inputs, that is bounded inputs whose nearby inputs have infinite outputs no matter how we set the extent of the output ruler. **NOTE.** In practice, the qualifier “bounded” will go without saying.

We will say that

\[ x_0 \] is a **∞-height input** for \( f \)

and to denote an ∞-height input, we will write:

\[ x_{∞-height} \]

with the name of the function “going without saying”.

We will make the following distinction:

- An **even ∞-height input** is an ∞-height input for which the local graph looks like one of the following:

  ![Even ∞-Height Input Diagram]

- An **odd ∞-height input** is an ∞-height input for which the local graph looks like one of the following:

  ![Odd ∞-Height Input Diagram]

2. For the purpose of locating all the inputs whose output has a given feature, the most important kind of notable inputs will be those near which the height is 0. We will say that

\[ x_0 \] is a **0-height input**\(^2\) for \( f \)

and in order to denote a 0-height input we will write

\[ x_{0-height} \]

\[ x_{even \ 0-height} \]

\[ x_{odd \ 0-height} \]

\[ x_{0-height} \]

\[^2\]Educologists will surely ask why not use **zero point** or **critical point of order 0**.
with the name of the function “going without saying”. Graphically, this means that the local graph near $x_0$ is near the 0-output level line and thus looks like one of the following

or like one of the following

### 3.5 Shape of a Local Graph

The second kind of local features of a local graph that we will discuss are the two local features that describe the shape of the local graph. Here, though, since we are only concerned with the qualitative aspect, we will record only the sign and not the size.

1. The slope sign of the local graph near $x_0$ is whether each side of the local graph near $x_0$ is
   - **sloping up**, that is looks more or less like $\backslash$ in which case we will also say that the slope is positive

EXAMPLE 14.
• **sloping down**, that is looks more or less like \ in which case we will also say that the **slope is negative**

**Example 15.**

In other words, we are not taking the **size** of the **slope** into consideration. Even though + and − are the symbols that are used traditionally,

We will code the **slope sign** into a paren ( , )

- with / to say that the local graph is **sloping up** (positive slope),
- with \ to say that the local graph is **sloping down** (negative slope).

**Example 16.** Given a function \( \text{JACK} \) and an input \( x_0 \), a multiple choice question might look like:

a. **Slope Sign** \( \text{JACK} \) near \( x_0 = (/ , /) \)

b. **Slope Sign** \( \text{JACK} \) near \( x_0 = (/ , \backslash) \)

c. **Slope Sign** \( \text{JACK} \) near \( x_0 = (\backslash , /) \)

d. **Slope Sign** \( \text{JACK} \) near \( x_0 = (\backslash , \backslash) \)

e. None of the preceding.

**NOTE.** Textbooks invariably use the following less systematic language:

- \( f \) is **increasing** near \( x_0 \) to mean **Slope Sign** \( f \) near \( x_0 = (/ , /) \),
- \( f \) is **decreasing** near \( x_0 \) to mean **Slope Sign** \( f \) near \( x_0 = (\backslash , \backslash) \),

and, much less frequently,

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3Educologists will object that while in Rome, one ought to do as Romans do and this is indeed a close call. The use of the symbols / and \ is intended to help the reader: (i) describe the behavior of functions by using more transparent symbols, (ii) keep in mind what is being described by not using the same symbols for **height**, **slope** and **concavity**, (iii) and thus generally by not adding to the present burden whereas, **later**, once completely familiar with the ideas themselves and should the student pursue studies in mathematics, the notational changeover should present very little difficulty, if any.
peaking  
bottoming out  
concavity sign  
bending up  
bending down

- $f$ is **peaking** near $x_0$ to mean Slope Sign $f$ near $x_0 = (\cup, \cap)$.
- $f$ is **bottoming out** near $x_0$ to mean Slope Sign $f$ near $x_0 = (\cap, \cup)$.

2. The **concavity sign** of the local graph near $x_0$ is whether each side of the local graph near $x_0$ is

- bending up, that is looks like a *part* of a cup like $\cup$

  ![Example 17](image)

- bending down, that is looks like a *part* of a cap like $\cap$

  ![Example 18](image)

In other words, we are not taking the *size* of the bending into consideration. Even though $+$ and $-$ are the symbols that are used traditionally,$^{5}$

We will **code** the concavity sign into a paren $(\ , \ )$

- with $\cup$ to say that the local graph is *bending up*,
- with $\cap$ to say that the local graph is *bending down*.

**Example 19.** Given a function $JACK$ and an input $x_0$, a multiple choice question might look like:

- a. Concavity Sign $JACK$ near $x_0 = (\cup, \cup)$
- b. Concavity Sign $JACK$ near $x_0 = (\cup, \cap)$
- c. Concavity Sign $JACK$ near $x_0 = (\cap, \cup)$
- d. Concavity Sign $JACK$ near $x_0 = (\cap, \cap)$
- e. None of the preceding.

$^{4}$Educologists use $\max$ (and $\min$). This, though, confuses the slope viewpoint with *optimization*, an entirely different viewpoint which we will introduce presently and which indeed contrast with the slope viewpoint.

$^{5}$Ditto.
3.6 Local Behavior

In order to describe the local behavior of a given function near a given input, we will thus first have to get a local graph near that given input. However, because, when the function is specified by an global input-output rule, the kind of computations necessary to get a local graph will depend on the kind of function being investigated, here we will use functions specified by a graph and get the local graph from the given graph. This will allow us, later, to focus just on the computations necessary to get the local graph from the global input-output rule.

So, in order to describe the local behavior of a given function near a given input:

i. We highlight the local graph near the given input—later we will have to compute the local graph from the global input-output rule,

ii. We read the qualitative features from the local graph,

iii. We write into the paren the code for the qualitative features.

We will consider separately the following three cases even though there isn’t much difference between them:

- Bounded inputs with bounded outputs
- Bounded inputs with infinite outputs (∞-height inputs)
- Infinite inputs

1. When the given input is bounded input with a bounded output, things are completely straightforward.

Example 20. Given the function ABEL specified by the following graph,

in order to describe the local behavior of ABEL near +2:

i. We highlight the local graph near +2:

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6This kind of “approac” is never considered by Educologists, let alone taken advantage of, since, for them, isolated “topics” reign supreme.
ii. We read the qualitative features from the local graph (blown up for convenience)

iii. We write into the paren the code for the qualitative features:

- Height Sign \( ABEL \) near \(+2\) = \((+,+)
- Height Size \( ABEL \) near \(+2\) = \((b,b)
- Height Sign-Size \( ABEL \) near \(+2\) = \((b^+,b^+)
- Slope Sign \( ABEL \) near \(+2\) = \((/ , /)
- Concavity Sign \( ABEL \) near \(+2\) = \((\cup,\cup)

EXAMPLE 21. Given the function \( BETH \) specified by the following graph,
in order to describe the local behavior of \( BETH \) near \(-1\).

i. We highlight the local graph near \(-1\):

ii. We read the qualitative features off the local graph (blown up for convenience)

iii. We write the qualitative features in code:

- Height Sign \( BETH \) near \(-1\) = \((+, -)\)
- Height Size \( BETH \) near \(-1\) = \((0, 0)\)
- Height Sign-Size \( BETH \) near \(-1\) = \((0^+, 0^-)\)
• Slope Sign $BETH$ near $-1 = (\setminus, \setminus)$
• Concavity Sign $BETH$ near $-1 = (\cup, \cap)$

2. When the given input is an $\infty$-height input, the local graph is outside the screen so that near an $\infty$-height input not just the output but also the slope is infinite. However, it is not obvious what the concavity near an $\infty$-height input should be inasmuch as, on the one hand, it is very tempting to say that the concavity near an $\infty$-height input should be infinite too but, on the other hand, the local graph near an $\infty$-height input is essentially straight.

This hints at the fact that there is more to concavity than just bending. But, until such times as we are equipped for discussing these more complicated aspects of concavity, we will leave alone the issue of the concavity near $\infty$-height inputs.

Example 22. Given the function $DAVE$ specified by the following graph,

in order to describe the local behavior of $DAVE$ near $+2$.

i. We highlight the local graph near $+2$:

ii. We read the qualitative features off the local graph (blown up for convenience)
iii. We write into the paren the code for the qualitative features:

- Height Sign $DAVE$ near $+1 = (+, -)$
- Height Size $DAVE$ near $+1 = (\infty, \infty)$
- Height Sign-Size $DAVE$ near $+1 = (+\infty, -\infty)$
- Slope Sign $DAVE$ near $+1 = (/, /)$
- Concavity Sign $DAVE$ near $+1 = (\cup, \cap)$

3. When the given input is \textit{infinity}, things are still the same but, when writing into the paren the code for the qualitative features, we have to make sure that we are facing $\infty$ so that $+\infty$ is to our left and $-\infty$ is to our right as opposed to what happens when we are facing what we will call a \textbf{Magellan screen}.

\textbf{Example 23.} Given the function $CATH$ specified by the following graph,
in order to describe the local behavior of \( CATH \) near \( \infty \).

i. We highlight the local graph near \( \infty \):

ii. We read the qualitative features off the local graph (blown up for convenience) keeping in mind that \( +\infty \) is to the left of \( \infty \) and \( -\infty \) is to the right of \( \infty \):

iii. We write into the paren the code for the qualitative features while making sure that we are facing \( \infty \):
3.7 Feature-Sign Change Inputs

The second kind of notable inputs are sign-change inputs that is inputs where at least one of the local qualitative features changes, that is, more precisely, the inputs for which the sign of at least one of the features near that input is different on either side of that input.\(^7\)

1. Given a function \(f\) and an input \(x_0\), when height sign is different from one side of \(x_0\) to the other side of \(x_0\), that is when Height Sign \(f\) near \(x_0\) = \((+, -)\) or \((-,-)\), we will say that \(x_0\) is a height sign change input\(^8\) for \(f\)

and to denote a height sign change input we will write

\[x_{\text{height sign change}}\]

with the name of the function “going without saying”.

**Example 24.** Given the function \(JANE\) whose local graph near \(-4\) is

we will say that

\(-4\) is a height sign change input for \(JANE\)

and we will write

\[x_{\text{height sign change}} = -4\]

**Example 25.** Given the function \(KANE\) whose local graph near \(-4\) is

\(^7\)Educologists will surely cringe at this terminology even though, if nothing else, it has the double merit of being systematic and self-explanatory.

\(^8\)Here, Educologists do not seem to feel the need for a term.
slope sign change input

we will say that

\[ +1 \text{ is a height sign change input for } KANE \]

and we will write

\[ x_{\text{height sign change}} = +1 \]

2. Given a function \( f \) and an input \( x_0 \), when \( \text{slope sign} \) is different from one side of \( x_0 \) to the other side of \( x_0 \), that is when Height Slope \( f \) near \( x_0 = (\nabla, \downarrow) \) or \( (\downarrow, \nabla) \), we will say that

\[ x_0 \text{ is a slope sign change input}^9 \text{ for } f \]

and to denote a slope sign change input we will write

\[ x_{\text{slope sign change}} \]

with the name of the function “going without saying”.

**Example 26.** Given the function \( MARY \) whose local graph near \( +5 \) is

we will say that

\[ +5 \text{ is a slope sign change input for } MARY \]

and we will write

\[ x_{\text{slope sign change}} = +5 \]

\(^9\text{Ditto.}\)
**Example 27.** Given the function $LARS$ whose local graph near $-4$ is

we will say that

$+1$ is a slope sign change input for $LARS$

and we will write

$x_{\text{slope sign change}} = +1$

**3.** Given a function $f$ and an input $x_0$, when concavity sign is different from one side of $x_0$ to the other side of $x_0$, that is when Height Slope $f$ near $x_0 = (\cup, \cap)$ or $\cap, \cup)$, we will say that

$x_0$ is a concavity sign change input$^{10}$ for $f$

and to denote a slope sign change input we will write

$x_{\text{concavity sign change}}$

with the name of the function, $f$, “going without saying”.

**Example 28.** Given the function $NATE$ whose local graph near $-4$ is

we will say that

$+5$ is a concavity sign change input for $NATE$

and we will write

$^{10}$Here again, Educologists will deplore our lack of respect for tradition and denounce our failure to use the term “inflection point”.

concavity sign change input

$x_{\text{concavity sign change}}$
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0-slope input

\[ x_{\text{concavity sign change}} = +5 \]

**Example 29.** Given the function \( P E T E \) whose local graph near \(-4\) is

\[ \text{Output Ruler} \]
\[ \text{Input Ruler} \]
\[ \text{Screen} \]

we will say that

\(+1\) is a concavity sign change input for \( P E T E \)

and we will write

\[ x_{\text{concavity sign change}} = +1 \]

### 3.8 0-Slope and 0-Concavity Inputs

A third kind of notable inputs is bounded inputs whose nearby inputs have outputs whose slope or whose concavity is near 0. **Note.** In practice, the qualifier “bounded” will often go without saying.

1. When it is the slope near \( x_0 \) which is near 0, we will say that

\[ x_0 \text{ is a 0-slope input}^{11} \]

for \( f \)

and to denote a 0-slope input we will write

\[ x_{0\text{-slope}} \]

with the name of the function “going without saying”.

Graphically, since the height, which is equal to the output for \( x_0 \), may but has no reason to be 0, this means that the local graph near \( x_{0\text{-slope}} \) looks like one of the following:

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11Here educologists will want to know why not **critical point of order 1** or just **critical point** or **stationary point**.
3.9 Extremum Inputs

In many applications of the Theory of Functions to the real world, one needs to compare the output of a given bounded input to the outputs of nearby inputs and we now come to the third kind of notable inputs which are extremum inputs, that is bounded inputs whose output is either absolutely larger or absolutely smaller than the output of all nearby inputs.

- When the output for a bounded input \( x_0 \) is absolutely larger than the output for all nearby inputs, we will say that \( x_0 \) is a maximum input for \( f \) and to denote a maximum input we will write

\[ x_0 \text{ maximum input} \]

2. When it is the concavity near \( x_0 \) which is near 0, we will say that \( x_0 \) is a 0-concavity input\(^\text{12}\) for \( f \) and to denote a 0-concavity input we will write

\[ x_0 \text{ concavity} \]

with the name of the function “going without saying”.

Graphically, since both the height and the slope may but have no reason to be 0, this means that the local graph near \( x_0 \)-concavity looks like one of the following:

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\(^{12}\)And finally, Educologist will wonder why not critical point of order 2?
with the name of the function “going without saying”.
We will say that the output for \( x_{\text{maximum}} \) is a **maximum output**.
From the graphic viewpoint, the local graph near a maximum input is entirely below the output-level line for the maximum input.

**Example 30.**

\[
\begin{array}{c}
\text{Input} \\
\text{Ruler} \\
\text{Output} \\
\text{Ruler} \\
\text{Screen}
\end{array}
\]

\[
\text{Output level line for maximum input}
\]

\[
\text{max output}
\]

\[
\text{max output}
\]

**Example 31.**

\[
\begin{array}{c}
\text{Input} \\
\text{Ruler} \\
\text{Output} \\
\text{Ruler} \\
\text{Screen}
\end{array}
\]

\[
\text{Output level line for maximum input}
\]

\[
\text{max output}
\]

\[
\text{max output}
\]

- When the output for a bounded input \( x_0 \) is absolutely smaller than the output for all nearby inputs, we will say that \( x_0 \) is a **minimum input** for \( f \) and to denote a minimum input we will write

\[
\begin{array}{c}
\text{Input} \\
\text{Ruler} \\
\text{Output} \\
\text{Ruler} \\
\text{Screen}
\end{array}
\]

\[
\text{Output level line for maximum input}
\]

\[
\text{max output}
\]

\[
\text{max output}
\]

with the name of the function “going without saying”.
We will say that the output for \( x_{\text{minimum}} \) is a **minimum output**.
From the graphic viewpoint, the local graph near a minimum input is entirely above the output-level line for the minimum input.

**Example 32.**
3.9. EXTREMUM INPUTS

**Example 33.**

**NOTE.** It is most important to realize that the words “maximum” and “minimum” can be very misleading:

- When the word “maximum” is used with the word input, it does not mean that the input itself is absolutely larger than all other nearby inputs. (However, according to the above definition, when the word “maximum” is used with the word output, it does mean that the output itself is absolutely larger than the outputs for all nearby inputs.)

- When the word “minimum” is used with the word input, it does not mean that the input itself is absolutely smaller than all other nearby inputs. (However, according to the above definition, when the word “minimum” is used with the word output, it does mean that the output itself is absolutely smaller than the outputs for all nearby inputs.)

**NOTE.** In practice, the qualifier “bounded” will go without saying.