Chapter 6

Negative-Exponent Power Functions


**Negative-exponent power functions** are functions that divide a given number, called the coefficient, by a given number of copies of the input. More precisely:

- The coefficient can be any finite number.
- The exponent, which can be any negative counting number, says what the function will do to the coefficient:
  - The sign of the exponent says that the coefficient is to be divided by the copies of the input.

**Example 1.**

\[
x \xrightarrow{\text{FLOP}} \text{FLOP}(x) = (+8.273.1)x^{-5} = \frac{+8.273.1}{x \cdot x \cdot x \cdot x \cdot x}
\]

- The size of the exponent gives the number of copies of the input that are to be used.
Example 2.

\[
x \xrightarrow{FLIP} FLIP(x) = (+8.273.1)x^{-5} \\
= +8.273.1 \\
\frac{x \cdot x \cdot x \cdot x \cdot x}{5 \text{ copies of } x}
\]

1. Occasionally, we will be able to make statements that are true both of positive-exponent power functions and of negative-exponent power functions and zero-power functions so, it will be convenient to lump them all together and use power function to mean a function that can be a positive-exponent power function or a negative-exponent power function or a zero-power function. In other words, the exponent in a power function can be any signed counting number.

Example 3. All of the following:
- the regular positive-exponent power function
  \[
x \xrightarrow{FLIP} FLIP(x) = (-13.44)^{+6}
  \]
- the exceptional positive-exponent power function
  \[
x \xrightarrow{FLIP} FLIP(x) = (-13.44)^{+1}
  \]
- the zero-power function
  \[
x \xrightarrow{BLOP} FLOP(x) = (+8.273.1)x^{0}
  \]
- the negative-exponent power function
  \[
x \xrightarrow{FLOP} FLOP(x) = (+8.273.1)x^{-5}
  \]
are power functions.

2. Recall, though, that in Chapter 4, we called regular positive-power functions those positive-power functions whose exponent is different from +1 as distinguished from the exceptional power functions, that is those power functions whose exponent is either +1 or 0. Quite often, we will be able to make statements that are true both of regular positive-exponent power functions and of negative-exponent power functions but not of the exceptional power functions, that is those power functions whose exponent is either +1 or 0. So it will be convenient to call regular power functions those power functions whose exponent is any signed counting number except +1 and 0. Still in other words, regular power functions are either regular positive-exponent power functions or negative-exponent power functions.
**Example 4.** Both the regular positive-exponent power function

\[ x \xrightarrow{FLIP} FLIP(x) = (-13.44)x^6 \]

and the negative-exponent power function

\[ x \xrightarrow{FLOP} FLOP(x) = (+8.273.1)x^{-5} \]

are regular power functions but neither the exceptional positive-exponent power function

\[ x \xrightarrow{BLIP} FLIP(x) = (-13.44)x^{1} \]

nor the zero-power function

\[ x \xrightarrow{BLOP} FLOP(x) = (+8.273.1)x^{0} \]

are regular power functions.

3. The reason we continue our investigation of Algebraic Functions with negative-exponent power functions, and the reason these are also extremely important, is that:
   i. negative-exponent power functions are almost as simple as positive-exponent power functions,
   ii. negative-exponent power functions already exhibit the local qualitative features that we discussed earlier and in terms of which Rational Functions will be investigated in Part Three of this text,
   iii. negative-exponent power functions are the other “building blocks” in terms of which we will “deconstruct” Rational Functions in Part Three of this text.

### 6.1 Input-Output Pairs

As with any function, given the input-output rule, in order to get the output for a given input we must:

- Read and write what the input-output rule says,
- Replace \( x \) in the given input-output rule by the given input
- Identify the resulting specifying phrase.

**Example 5.** Given the input-output rule

\[ x \xrightarrow{FLOP} FLOP(x) = (+5273.1) \cdot x^{-5} \]

and given the input \(-3\), in order to get the input-output pair, we proceed as follows.
a. To get the output:
   i. We read and write what the input-output rule says:
      - The input-output rule reads:
        "The output of FLOP is obtained by dividing +5273.1 by 5 copies of the input."
      - We write
        \[ x \xrightarrow{FLOP} FLOP(x) = \frac{+5,273.1}{\underbrace{x \cdot x \cdot x \cdot x \cdot x}} \]

   ii. We indicate that \( x \) is about to be replaced by the given input \(-3\)

   \[
   \begin{array}{c|c|c|c}
   x & \xrightarrow{FLOP} & FLOP(x) & \xrightarrow{FLOP} x := -3 \\
   \hline
   x & := -3 & \frac{+5,273.1}{\underbrace{x \cdot x \cdot x \cdot x \cdot x}} & := -3 \\
   \end{array}
   \]

   which gives us the following specifying-phrase

   \[
   = \frac{+5,273.1}{(-3) \cdot (-3) \cdot (-3) \cdot (-3)} \]

   iii. We identify the specifying-phrase

   \[
   = \frac{+5,273.1}{-243} \]

   \[
   = -21.7
   \]

   Without the comments, here is how things should look like once all done:

   \[
   \begin{array}{c|c|c|c}
   x & \xrightarrow{FLOP} & FLOP(x) & \xrightarrow{FLOP} \| x := -3 \\
   \hline
   x & := -3 & \frac{+5,273.1}{\underbrace{x \cdot x \cdot x \cdot x \cdot x}} & := -3 \\
   \end{array}
   \]

   \[
   = \frac{+5,273.1}{(-3) \cdot (-3) \cdot (-3) \cdot (-3)} \]

   \[
   = \frac{+5,273.1}{-243} \]

   \[
   = -21.7
   \]

b. Depending on the circumstances, we can then write:

   \[ -3 \xrightarrow{FLOP} -21.47 \]
or

$$FLOP(-3) = -21.47$$

or

$$(-3, -21.47)$$ is an input-output pair for the function $FLOP$.

**Example 6.** Given the input-output rule

$$x \xrightarrow{FLOP} FLOP(x) = (+112)x^{-4}$$

and given the input $-5$, in order to get the input-output pair, we proceed as follows:

**a.** To get the output:

i. We read and write what the input-output rule says:

• The input-output rule reads:
  • "The output of $FLOP$ is obtained by dividing $+112$ by 4 copies of the input."
  • We write

$$x \xrightarrow{FLOP} FLOP(x) = (+112)x^{-4}$$

$$= \frac{+112}{x \cdot x \cdot x \cdot x}$$

4 copies of $x$

ii. We indicate that $x$ is to be replaced by the given input $+2$

$$x \xrightarrow{x = +2} FLOP(x) \xrightarrow{x = +2} +112$$

$$= +112$$

4 copies of $x$

which gives us the following specifying-phrase

$$= \frac{+112}{(+2) \cdot (+2) \cdot (+2) \cdot (+2)}$$

4 copies of $+2$

iii. We identify the specifying-phrase

$$= +7$$

Without all the comments, here is how getting the output should look like once it is all
features, of input-output rule
sign, of the exponent
parity, of the exponent
sign, of the coefficient
type
NEP
NEN
NOP
NON
done:

\[
x|_{x=+2} \xrightarrow{FLOP} FLOP(x)|_{x=+2} = (+112)x^{-4}|_{x=+2}
\]
\[
= +112 \quad \frac{x \cdot x \cdot x \cdot x}{x_{x=+2}}
\]
\[
= +112 \quad \frac{(+2) \cdot (+2) \cdot (+2) \cdot (+2)}{(+2)}
\]
\[
= +7
\]

b. Depending on the circumstances, we can then write:

\[
+2 \xrightarrow{FLOP} +7
\]

or

\[
FLOP(+2) = +7
\]

or

\((+2, +7)\) is an input-output pair for the function \(FLOP\)

6.2 Normalized Input-Output Rule

The input-output rule of a negative-exponent power function thus has a number of features but since we will be mostly dealing with qualitative investigations, these features will not be equally important to us.

1. As with positive-exponent power functions, the three features that will be important to us are:
   - The sign of the exponent which, for negative-exponent power functions, is \(-\),
   - The parity of the exponent which can be even or odd depending on whether the number of copies is even or odd,
   - The sign of the coefficient which can be \(+\) or \(-\).

From our point of view,
   - The size of the coefficient will not be an important feature to us because of the requirement in the definition of a negative-exponent power function at the start of this chapter that the coefficient be a finite number.
   - The size of the exponent will not be an important feature to us because the plain number of copies will not matter from our qualitative viewpoint.

2. From the qualitative viewpoint that we will be taking, there will therefore be four types of negative-exponent power functions:
6.2. NORMALIZED INPUT-OUTPUT RULE

<table>
<thead>
<tr>
<th>Sign exponent</th>
<th>Parity exponent</th>
<th>Sign coefficient</th>
<th>TYPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>−</td>
<td>Even</td>
<td>+</td>
<td>NEP</td>
</tr>
<tr>
<td></td>
<td>Odd</td>
<td>+</td>
<td>NOP</td>
</tr>
</tbody>
</table>

**Example 7.** The function $PLIP$ whose input-output rule is

$$x \xrightarrow{PLIP} PLIP(x) = \left(\frac{6836}{-7}\right)x^{-7} = \frac{+6836}{x \cdot \ldots \cdot x}$$

is a power function whose *input-output rule* has the following **features**
- The exponent is **Negative**,
- The exponent is **Odd**.
- The coefficient is **Positive**.

So, the power function $PLIP$ is of type **NOP**.

**Example 8.** The function $MILK$ whose input-output rule is

$$x \xrightarrow{MILK} MILK(x) = \left(\frac{+4500}{-6}\right)x^{-6} = \frac{+4500}{x \cdot \ldots \cdot x}$$

is a power function whose *input-output rule* has the following **features**
- The exponent is **Negative**,
- The exponent is **Even**.
- The coefficient is **Positive**.

So, the power function $MILK$ is of type **NEP**.

3. In fact, as with positive-exponent power functions, a negative-exponent power function being given by an *input-output rule*, what we will do is to **normalize** the *input-output rule*, that is we will strip the exponent and the coefficient of all the information that is irrelevant from our **qualitative** point of view:
- We will normalize the **exponent** as follows:

<table>
<thead>
<tr>
<th>If the exponent is:</th>
<th>we will <strong>normalize</strong> it to:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negative Even</td>
<td>$- even$</td>
</tr>
<tr>
<td>Negative Odd</td>
<td>$- odd$</td>
</tr>
</tbody>
</table>

- We will normalize the **coefficient** as follows:
If the coefficient is: we will normalize it to:

<table>
<thead>
<tr>
<th>Positive</th>
<th>+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negative</td>
<td>−1</td>
</tr>
</tbody>
</table>

**Example 9.** The function $CRIC$ whose input-output rule is

$$x \xrightarrow{CRIC} CRIC(x) = (-\frac{345}{15})x^{-\frac{15}{15}} = \frac{-345}{x \cdot \ldots \cdot x} \text{ 15 copies of } x$$

will be normalized to

$$x \xrightarrow{CRIC} CRIC(x) = (-1)x^{-odd} = \frac{-1}{x \cdot \ldots \cdot x} \text{  odd number of copies of } x$$

**Example 10.** The function $CRIP$ whose input-output rule is

$$x \xrightarrow{CRIC} CRIP(x) = (+\frac{562}{12})x^{-\frac{12}{12}} = \frac{+562}{x \cdot \ldots \cdot x} \text{ 12 copies of } x$$

will be normalized to

$$x \xrightarrow{CRIP} CRIP(x) = (-1)x^{-even} = \frac{-1}{x \cdot \ldots \cdot x} \text{  even number of copies of } x$$

4. So, the four types of negative-exponent power functions have the following normalized input-output rules:

<table>
<thead>
<tr>
<th>TYPE</th>
<th>Normalized input-output rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$NEP$</td>
<td>$x \xrightarrow{NEP} NEP(x) = (+1)x^{-\text{even}}$</td>
</tr>
<tr>
<td>$NEN$</td>
<td>$x \xrightarrow{NEN} NEN(x) = (-1)x^{-\text{even}}$</td>
</tr>
<tr>
<td>$NOP$</td>
<td>$x \xrightarrow{NOP} NOP(x) = (+1)x^{-\text{odd}}$</td>
</tr>
<tr>
<td>$NON$</td>
<td>$x \xrightarrow{NON} NON(x) = (-1)x^{-\text{odd}}$</td>
</tr>
</tbody>
</table>
6.3. LOCAL GRAPH NEAR $\infty$

5. As with regular positive-exponent power functions, because of our use of even and odd as exponent, the above are not global input-output rules of functions but only of types of functions and we often will use gauge power functions.

In particular, we will often use, for many purposes, the gauge power function \textit{RECIPROCAL} specified by the global input-output rule

$$x \xrightarrow{\text{RECIPROCAL}} \text{RECIPROCAL}(x) = x^{-1}$$

and we will often speak of the \textbf{reciprocal function}

It is of course a prototype for power functions of type \textit{NOP} but rarely used as such, perhaps because there is no standard name for a prototype for functions of type \textit{NEP}.

\section*{LOCAL ANALYSIS}

\subsection*{6.3 Local Graph Near $\infty$}

The way we will get the local graph near $\infty$ from the input-output rule in the case of a negative-exponent power function will be almost exactly the same as what we did in the case of a positive-exponent power function. Again, there are several possible approaches:

1. We could use a bare hands approach and proceed “from scratch”, that is in a manner based uniquely on:
   - The definition of “large-in-size”, namely on the fact that \textit{copies of a number that is large-in-size multiply to a result that is large-in-size}
   - The fact that a \textit{finite} coefficient divided by a result that is \textit{large-in-size} gives an output that is \textit{small-in-size}
   - The “rule of signs” for multiplication

\begin{tabular}{c c c}
+ & - \\
+ & + & - \\
- & - & +
\end{tabular}

namely on the fact that:

- multiplying \textit{copies of a positive} input will result in a \textit{positive} result no matter what the number of copies is,
- multiplying \textit{copies of a negative} input will result:
  - in a \textit{positive} result if the number of copies is \textit{even}
  - in a \textit{negative} result if the number of copies is \textit{odd}
CHAPTER 6. NEGATIVE-EXponent POWER FUNCTIONS

EXAMPLE 11. Given the function \( NADE \) whose input-output rule is

\[
x \xrightarrow{NADE} NADE(x) = (-8)x^{-5}
\]

we want to find the local graph near \( \infty \).

i. We normalize the input-output rule to

\[
x \xrightarrow{NADE} NADE(x) = (-1)x^{-odd}
\]

ii. We look for the place of the local graph near \( \infty \):

- To get the place of the local graph near \( +\infty \), we compute the output for inputs that are \(+large\):

\[
\begin{align*}
x | x := +large \quad \xrightarrow{NADE} \quad NADE(x) | x := +large &= (-1)x^{-odd} | x := +large \\
&= (-1)(+large)^{-odd} \\
&= \frac{-1}{(+large) \cdot \ldots \cdot (+large)} \\
&= \frac{1}{odd \text{ number of copies of } +large}
\end{align*}
\]

and since,

- by the Definition of large, copies of large multiply to large
- by the Rule of Signs, any number of copies of + multiply to +

\[
= -\frac{1}{+large}
\]

and, by the fact that a finite coefficient divided by a result that is large-in-size gives an output that is small-in-size,

\[
= -small
\]

and so we have that

\[NADE(+large) = -small\]

So, we have that the place of the local graph near \( +\infty \) is
6.3. LOCAL GRAPH NEAR $\infty$

- To get the place of the local graph near $-\infty$, we compute the output for inputs that are $-\text{large}$:

\[
x \bigg| \begin{array} {c|c} x := -\text{large} & \text{NADE} \rightarrow \text{NADE}(x) \bigg| x := -\text{large} \\
\end{array}
\]

\[
= (-1)^{x^{\text{odd}}} \bigg| x := -\text{large}
\]

\[
= \frac{-1}{(-\text{large}) \cdot \ldots \cdot (-\text{large})}
\]

and since,
- by the Definition of large, copies of large multiply to large
- by the Rule of Signs, an odd number of copies of $-$ multiply to $-$

\[
= \frac{-1}{-\text{large}}
\]

and, by the fact that a finite coefficient divided by a result that is large-in-size gives an output that is small-in-size,

\[
= +\text{small}
\]

and so we have that

\[
\text{NADE}(-\text{large}) = +\text{small}
\]

So, we have that the place of the local graph near $-\infty$ is

Altogether, we found that the place of the local graph near $\infty$ is:

iii. We look for the shape of the local graph near $\infty$ which involves having to compute with specific inputs.
We look for the slope near $+\infty$, for instance from $+10$ to $+20$.

We compute the output for $+10$:

\[
x \overset{\text{NADE}}{\longrightarrow} \text{NADE}(x) \bigg|_{x:=+10} = (-1)x^{-5} \bigg|_{x:=+10} = (-1)(+10)^{-5} = \frac{-1}{(10) \cdot \ldots \cdot (10)} \text{ (5 copies of 10)}
\]

\[
= \frac{-1}{(+1) \cdot \ldots \cdot (+1)} \text{ followed by 5 zeros (5 copies of 1)}
\]

\[
= -1 \text{ followed by 5 zeros}
\]

\[
= \frac{-1}{+100000} = -\frac{1}{100000}
\]

so we have that

\[
\text{NADE}(+10) = -0.00001
\]

We compute the output for $+20$:

\[
x \overset{\text{NADE}}{\longrightarrow} \text{NADE}(x) \bigg|_{x:=+20} = (-1)x^{-5} \bigg|_{x:=+20} = (-1)(+20)^{-5} = \frac{-1}{(20) \cdot \ldots \cdot (20)} \text{ (5 copies of 20)}
\]

\[
= \frac{-1}{(+2) \cdot \ldots \cdot (+2)} \text{ followed by 5 zeros (5 copies of 2)}
\]

\[
= -1 \text{ followed by 5 zeros}
\]

\[
= \frac{-1}{+3200000} = -\frac{1}{3 \times 3}
\]

\[
= -\frac{3200000 \times 3}{3} = -\frac{9600000}{3}
\]

\[
= -\frac{10000000 + (...)}{3}
\]
so we have that

\[ NADE(+20) = -0.000003 + (\ldots) \]

Now we compute the slope from +10 to +20:

\[
NAD\hat{E}(+20) - NAD\hat{E}(+10) \overset{(+20) - (+10)}{=} \frac{[-0.0000003 + (\ldots)] - [-0.000001]}{+20 - 10} = \frac{+0.000001 + (\ldots)}{+10} = +0.000001 + (\ldots)
\]

The slope near \(+\infty\) is thus small-in-size and positive:

![Diagram showing slope calculation](image)

- We look for the slope near \(-\infty\) and compute, say, the slope from \(-20\) to \(-10\).

We compute the output for \(-20\):
\[ \frac{-1}{-32 \text{ followed by 5 zeros}} = \frac{1}{3200000} \]
\[ = \frac{1}{3200000 \times 3} = \frac{3}{9600000} = \frac{3}{10000000} + (\ldots) \]

so we have that

\[ \text{NADE}(-20) = +0.000003 + (\ldots) \]

We compute the output for \(-10\):

\[
x \mid x = -10 \quad \text{NADE} \rightarrow \text{NADE}(x) \mid x = -10 = (-1)x^{-5} \mid x = -10 = (-1)(-10)^{-5}
\]
\[= \frac{-1}{(10) \cdot \ldots \cdot (10)} \]
\[= \frac{-1}{(-1) \cdot \ldots \cdot (-1)} \text{ followed by 5 zeros} \]
\[= \frac{-1}{-1} \text{ followed by 5 zeros} \]
\[= \frac{-1}{-100000} \]
\[= +\frac{1}{100000} \]

so we have that

\[ \text{NADE}(-10) = +0.00001 \]

Now we compute the slope from \(-20\) to \(-10\):

\[
\frac{\text{NADE}(-10) - \text{NADE}(-20)}{(-10) - (-20)} = \frac{[+0.00001] - [+0.000003 + (\ldots)]}{(-10) - (-20)}
\]
6.3. LOCAL GRAPH NEAR $\infty$

\[
\begin{align*}
6.3. \text{ LOCAL GRAPH NEAR } \infty &= +0.00001 - 0.0000003 + (\ldots) \\
&= -10 + 20 \\
&= +0.00001 + (\ldots) \\
&= +10 \\
&= +0.00000 1 + (\ldots) \\
&= + \frac{1}{1} + (\ldots)
\end{align*}
\]

and we get again that the slope is small-in-size and positive:

\[
\text{slope} = + \frac{0.000 001}{1} + (\ldots)
\]

- Altogether, the slope near $\infty$ is:

\[
\text{slope} = + \frac{0.000 001}{1} + (\ldots)
\]

- We look for the concavity near $+\infty$ by computing, say, the slope from $+20$ to $+30$ and comparing it with the slope from $+10$ to $+20$ (which we already computed above).

We already computed the output for $+20$

\[
\text{NADE}(+20) = -0.00 000 03 + (\ldots)
\]

We compute the output for $+30$:

\[
x \bigg|_{x=+30} \text{ NADE} \rightarrow \text{NADE}(x) \bigg|_{x=+30} = (-1)x^{-5} \bigg|_{x=+30} = (-1)(+30)^{-5} = \frac{-1}{(+30) \cdot \ldots \cdot (+30)}
\]

\[
5 \text{ copies of } +30
\]
so we have that

\[
NADE(+30) = -0.0000004 + (...) 
\]

Now we compute the slope from +20 to +30:

\[
\frac{NADE(+30) - NADE(+20)}{(+30) - (+20)} = \frac{-[0.0000004 + (...) - [0.0000003 + (...)]]}{10} 
\]

\[
= \frac{-0.0000004 + (...) + 0.0000003 + (...) + 10}{10} 
\]

\[
= \frac{+0.0000003 + (...) + 1}{1} 
\]

\[
= \frac{+0.0000003 + (...) + 1}{1} 
\]

We compare the slope from +20 to +30 which we just computed to be

\[
\frac{0.0000003}{1} + (...) 
\]

with the slope from +10 to +20 which we computed earlier to be

\[
\frac{0.0000001}{1} + (...) 
\]

The slope from +20 to +30 is smaller-in-size than the slope from +10 to +20:
so that the concavity-sign near $+\infty$ has to be $\cap$.

- We look for the concavity near $-\infty$ by computing, say, the slope from $-30$ to $-20$ and comparing it with the slope from $-20$ to $-10$ which we already computed above.

We compute the output for $-30$:

\[
\begin{align*}
  x & \bigg|_{x=-30}^{} \text{NADE} \rightarrow \text{NADE}(x) \bigg|_{x=-30} = (-1)x^{-5} \bigg|_{x=-30} \\
  &= (-1)(-30)^{-5} \\
  &= \frac{(-30) \cdots (-30)}{5 \text{ copies of } -30} \\
  &= \frac{-1}{(-3) \cdots (-3) \text{ followed by } 5 \text{ zeros}} \\
  &= -243 \text{ followed by } 5 \text{ zeros} \\
  &= +24300000 \times 4 \\
  &= +97200000 \times 4 \\
  &= +1000000000 + (...) \\
  &= +1000000000 + (...) \\
\end{align*}
\]

so we have that

\[
\text{NADE}(-30) = +0.000004 + (...) 
\]

We already computed the output for $-20$:

\[
\text{NADE}(-20) = +0.000003 + (...) 
\]
Now we compute the slope from $-30$ to $-20$:

\[
\frac{NADE(-20) - NADE(-30)}{(-20) - (-30)} = \frac{[+0.0000003 + (...) - [+0.0000004 + (...)]}{-20 + 30} \\
= \frac{+0.0000003 + (...) - 0.0000004 + (...) + 10}{-20 + 30} \\
= \frac{+0.0000003 + (...) + 1}{+1} \\
= + \frac{0.0000003 + (...) + 1}{1} + (...) \\
= + \frac{0.0000003 + (...) + 1}{1} + (...) \\
\]

We compare the slope from $-30$ to $-20$ which we just computed to be

\[+ \frac{0.0000003}{1} + (...)\]

with the slope from $-20$ to $-10$ which we computed earlier to be

\[+ \frac{0.0000001}{1} + (...)\]

The slope from $-30$ to $-20$ is smaller-in-size than the slope from $-20$ to $-10$:

so that the concavity-sign near $-\infty$ has to be $\cup$.

**iv.** So, finally, we have that the local graph of $NADE$ near $\infty$ is

This bare hands approach, even though it would be very safe because it does not require any memorization, would be much too slow, in particular when the local graph near $\infty$ is not an end in itself but only a means towards...
other ends such as, for instance, local features near \( \infty \), the essential bounded graph or the essential global graph. It thus worthwhile to invest some time in making the case for some theorems and then invoke these theorems.

2. At one extreme, we could make the case for the following

**THEOREM 1 (Local Graph Near \( \infty \)).** The local graphs near \( \infty \) for negative-exponent power functions are:

![Graph representation]

and then memorize the theorem and just invoke it in each particular case.

While this would of course be extremely fast, it would also be extremely dangerous in that we would be totally dependent on our remembering the theorem perfectly with no chance of becoming aware of an error we might have made by misremembering the theorem and, even less, of recovering from that error.

3. We will take a more reasonable, somewhat *in-between* approach which will use the following two theorems:

i. A theorem that says how the local graph near \( -\infty \) can be flipped from the local graph near \( +\infty \):

**THEOREM 2 (Local Place Near \( -\infty \)).** For a negative-exponent power function, the local place near \( -\infty \) is obtained from the local place near \( +\infty \) according to the parity of the exponent:

- When the exponent is even, the local place near \( -\infty \) is flipped horizontally from the local graph near \( +\infty \)
When the exponent is odd, the local place near \(-\infty\) is flipped diagonally from the local graph near \(+\infty\).

**Theorem 3 (Local Shape Near \(\infty\)).** For negative-exponent power functions, as inputs go towards \(\infty\), the local graph near \(\infty\) tends to become horizontal and so its shape is forced by the place as follows:

The case for the **Local Shape Near \(\infty\) Theorem** is not much harder to make since it follows essentially what we did in the bare hands approach but
is therefore somewhat lengthy and we will leave it to the Supplement.

4. The procedure that we will be using will be the same as the one we used for regular positive-exponent power functions:

---

**Example 12.** Given the positive-exponent power function $NATE$ whose input-output rule is

$$x \xrightarrow{\text{NATE}} NATE(x) = (-33.14159)x^{-12}$$

we want to find the local graph near $\infty$.

i. We normalize the input-output rule to

$$x \xrightarrow{\text{NATE}} NATE(x) = (-1)x^{-\text{even}}$$

$$= \frac{-1}{x \cdot \ldots \cdot x}$$

even number of copies of $x$

ii. We compute the output for inputs that are $+\text{large}$ to get the local place near $+\infty$:

$$+\text{large} \xrightarrow{\text{NATE}} NATE(+\text{large}) = (-1)(+\text{large})^{-\text{even}}$$

$$= \frac{-1}{(+\text{large}) \cdot \ldots \cdot (+\text{large})}$$

even number of copies of $+\text{large}$

and since,

- by the Definition of large, copies of large multiply to large
- by the Rule of Signs, any number of copies of $+$ multiply to $+$

$$= \frac{-1}{+\text{large}}$$

$$= -\text{small}$$
So, we have that the place of the local graph of \( NATE \) near \(+\infty\) is

\[
\begin{array}{c}
\text{Output Ruler} \\
\downarrow \\
0^- \\
\downarrow \\
\text{Input Ruler} \\
\rightarrow +\infty \\
\end{array}
\]

iii. Since the exponent is even, we get from the Local Place Near \(-\infty\) Theorem that the place of the local graph near \(-\infty\) is flipped horizontally from the local graph near \(+\infty\) to

\[
\begin{array}{c}
\text{Output Ruler} \\
\downarrow \\
0^+ \\
\downarrow \\
\text{Input Ruler} \\
\cdots \rightarrow -\infty \\
\end{array}
\]

iv. And then the Local Shape Near \( \infty \) Theorem says that the local graph of \( NATE \) near \( \infty \) must be

\[
\begin{array}{c}
\text{Output Ruler} \\
\downarrow \\
0^- \\
\downarrow \\
\text{Input Ruler} \\
\cdots \rightarrow -\infty \\
\end{array}
\]

**Example 13.** Given the positive-exponent power function \( NAVE \) whose input-output rule is

\[
x \xrightarrow{NAVE} NAVE(x) = (-33.14159)x^{-7} = \frac{-33.14159}{x \cdot \ldots \cdot x} \quad \text{7 copies of } x
\]

we want to find the local graph near \( \infty \).

i. We normalize the input-output rule to

\[
x \xrightarrow{NAVE} NAVE(x) = (-1)x^{-\text{odd}} = \frac{-1}{x \cdot \ldots \cdot x} \quad \text{odd number of copies of } x
\]
ii. We compute the output for inputs that are $+\text{large}$ to get the local place near $+\infty$:

\[
+\text{large} \xrightarrow{N A V E} N A V E(+\text{large}) = (-1)(+\text{large})^{-\text{odd}} = -1 \cdot (+\text{large}) \cdot \ldots \cdot (+\text{large})
\]

and since,
- by the Definition of large, copies of large multiply to large
- by the Rule of Signs, any number of copies of $+$ multiply to $+$

\[
= \frac{-1}{+\text{large}} = -\text{small}
\]

So, we have that the place of the local graph of $N A V E$ near $+\infty$ is

iii. Since the exponent is odd, we get from the Local Place Near $-\infty$ Theorem that the place of the local graph near $-\infty$ is flipped diagonally from the local graph near $+\infty$ to

iv. And then the Local Shape Near $\infty$ Theorem says that the local graph of $N A V E$ near $\infty$ must be
5. While the arithmetic involved with the bare hands procedure is quite a bit harder with negative-exponent power functions than it was with positive-exponent power functions, other than that, we saw that the familiarity with the procedures we acquired with positive-exponent power functions transferred completely to negative-exponent power functions.

6.4 Types of Local Graphs Near $\infty$

Proceeding in the same manner as above, we get the local graph near $\infty$ for each one of the four types of negative-exponent power function. They are shown in the table below as seen from two viewpoints:

i. As seen from “not too far”, that is we see the screen and only the part of the local graph near $\infty$ that is near the transition, that is for inputs that are large but not “that” large so that we can still see the slope and the concavity.

ii. As seen from “faraway”. Indeed, in order to see really large inputs, we need to be “faraway” but then the parts of the local graph near $\infty$ that we see are essentially straight and vertical.

<table>
<thead>
<tr>
<th>Input-output rule</th>
<th>From “not too far”</th>
<th>From “faraway”</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \xrightarrow{\text{NEP}} NEP(x) = +x^{\text{even}}$</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
</tr>
</tbody>
</table>

Continued on next page
6.5. THE ESSENTIAL QUESTION

We now need to find out if the outlying graph includes just the local graph near $\infty$ or if it also includes local graph(s) near $\infty$-height inputs.

1. In other words, before we can proceed, we need to answer the Essential Question:

- Do all bounded inputs have bounded outputs (for some extent of the output ruler)

or

- Is there one (or more) bounded input that is an $\infty$-height input, namely whose nearby inputs have infinite outputs (no matter what the extent of the output ruler)?

Since, in the case of negative-exponent power functions, the coefficient is to be divided by copies of the input, the input 0 would immediately seem to
create a difficulty since we cannot divide by 0. The **Essential Question**, though, asks whether nearby inputs have infinite outputs so that the answer is:

**THEOREM 4 (Height).** For negative-exponent power functions, 0 has infinite height, that is nearby inputs have infinite height.

Making the case is based on the fact that dividing a finite coefficient by copies of a small input results in a large output.

**EXAMPLE 14.** Given the function $NYK$ whose input-output rule is

$$x \xrightarrow{MYK} NYK(x) = +20x^{-5}$$

and given an input near 0, that is given a small input, we compute the output:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x := -\text{small}$</th>
<th>$NYK(x)$</th>
<th>$x := -\text{small}$</th>
<th>$x := -\text{small}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x := -\text{small}$</td>
<td>$NYK(x)$</td>
<td>$(+20)x^{-5}$</td>
<td>$(+20)x^{-5}$</td>
</tr>
<tr>
<td></td>
<td>$x := -\text{small}$</td>
<td>$NYK(x)$</td>
<td>$+20 \cdot \ldots \cdot x$</td>
<td>$+20 \cdot \ldots \cdot x$</td>
</tr>
<tr>
<td></td>
<td>$x := -\text{small}$</td>
<td>$NYK(x)$</td>
<td>odd number of copies of $x$</td>
<td>odd number of copies of $x$</td>
</tr>
<tr>
<td></td>
<td>$x := -\text{small}$</td>
<td>$NYK(x)$</td>
<td>$(-\text{small}) \cdot \ldots \cdot (-\text{small})$</td>
<td>$(-\text{small}) \cdot \ldots \cdot (-\text{small})$</td>
</tr>
<tr>
<td></td>
<td>$x := -\text{small}$</td>
<td>$NYK(x)$</td>
<td>odd number of copies of $-\text{small}$</td>
<td>odd number of copies of $-\text{small}$</td>
</tr>
<tr>
<td></td>
<td>$x := -\text{small}$</td>
<td>$NYK(x)$</td>
<td>$+20$</td>
<td>$+20$</td>
</tr>
<tr>
<td></td>
<td>$x := -\text{small}$</td>
<td>$NYK(x)$</td>
<td>$-\text{small}$</td>
<td>$-\text{small}$</td>
</tr>
<tr>
<td></td>
<td>$x := -\text{small}$</td>
<td>$NYK(x)$</td>
<td>$large$</td>
<td>$large$</td>
</tr>
</tbody>
</table>

So the outlying graph of a negative-power function does not include just the local graph near $\infty$ but also the local graph near 0. This affects the “story line” for negative-exponent power functions which, until now, was exactly the same as the “story line” for positive-power functions because now there is a major difference:

- In the case of **positive-exponent** power functions, once we had answered the essential question (in the negative), the “story line” proceeded with getting:
  - i. the **essential bounded graph** on the basis of the local graph near $\infty$,
  - ii. the **existence of notable input(s)** on the basis of the essential bounded graph,
  - iii. the **local graph near 0** on the basis of 0 being the most likely possibility for being the notable input,
  - iv. the **different types of local graphs near 0**.
6.6 LOCAL GRAPH NEAR 0

- In the case of negative-exponent power functions, once we have answered the essential question (in the positive), we have no choice and the “story line” must proceed with getting
  i. the local graph near 0 on the basis of 0 being an $\infty$-height input,
  ii. the different types of local graphs near 0,
  iii. the essential bounded graph on the basis of the local graph near $\infty$ and the local graph near 0,
  iv. the existence of notable input(s) on the basis of the essential bounded graph,
After which, just as with positive-power functions, we will conclude with the essential global graphs.

6.6 Local Graph near 0

The way we will get the local graph near 0 will follow very closely the way we got the local graph near $\infty$ as well as the way we got the local graphs near $\infty$ and near 0 in the case of positive-power functions.

1. We could use a bare hands approach, that is proceed “from scratch”, that is in a manner based uniquely on:
   - The definition of “small-in-size”, namely on the fact that copies of a number that is small-in-size multiply to a result that is small-in-size
   - The fact that a finite coefficient divided by a result that is small-in-size gives an output that is large-in-size
   - The “rule of signs” for multiplication

<table>
<thead>
<tr>
<th>+</th>
<th>−</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>−</td>
<td>−</td>
</tr>
</tbody>
</table>

namely on the fact that:
- multiplying copies of a positive input will result in a positive result no matter what the number of copies is,
- multiplying copies of a negative input will result:
  in a positive result if the number of copies is even
  in a negative result if the number of copies is odd
This approach, even though it would be very safe because it does not require any memorization, would be much too slow, in particular when the local graph near 0 is not an end in itself but only a means towards other ends such as, for instance, local features near 0, the essential bounded graph or the essential global graph.

2. At the other extreme, we could just make the case for the following
THEOREM 5 (Local Graph Near 0). The local graphs near 0 for negative-exponent power functions are:

and then memorize the theorem and just invoke it in each particular case. While this would of course be extremely fast, it would also be extremely dangerous in that we would be totally dependent on our remembering the theorem perfectly with no chance of becoming aware of an error we might have made by misremembering the theorem and, even less, of recovering from that error.

3. The somewhat in-between approach which we will take will be to use a compressed procedure much like the compressed procedure we used near $\infty$ and will use the following, very similar, two theorems:

i. A theorem that says how the local graph near $0^-$ can be obtained from the local graph near $0^+$ by a flip:

THEOREM 6 (Local Place Near $0^-$). For a negative-exponent power function, the local place near $0^-$ is obtained from the local place near $0^+$ according to the parity of the exponent:

- When the exponent is even, the local place near $0^-$ is flipped horizontally from the local graph near $0^+$

- When the exponent is odd, the local place near $0^-$ is flipped diagonally from the local graph near $0^+$
The case for the **Local Place Near $0^-$ Theorem** is entirely based on the **Rule of Signs** and so the phrasing of the case is left to the reader.

ii. A theorem that says how the *shape* of the local graph of a *positive-exponent* power function is **forced** by the *place*.

**THEOREM 7 (Shape Near 0).** For negative-exponent power functions, as inputs go towards 0, the local graph near 0 tends to become vertical and so its shape is forced by the place as follows:

The case for the **Shape Near 0 Theorem** is not much harder to make either but, because the proof is somewhat lengthy, we will leave it to the Supplement.

The *compressed procedure* itself is exactly the same as the compressed procedure we used to get the local graph near $\infty$ with just $\infty$ replaced by 0 and of course *large* replaced by *small*:

1. *Normalize* the input-output rule,
2. Get the *place* of the local graph near $0^+$, by *computing* the output for inputs that are $+small$,
3. Get the *place* of the local graph near $0^-$ by using the **Local Place Near $0^-$ Theorem**—instead of *computing* the output for inputs that are $-small$,
4. Get the *local graph* near $\infty$ by using the **Local Shape Near 0 Theorem**—instead of computing all these *slopes*.
EXAMPLE 15. Given the function \( DATE \) whose input-output rule is

\[ x \xrightarrow{DATE} DATE(x) = (-13.14159) \cdot x^{-24} \]

we want to find the local graph near 0.

i. We normalize the input-output rule to

\[ x \xrightarrow{DATE} DATE(x) = (-1)x^{-even} \]

ii. To get the local place near \( 0^+ \), we compute the output for inputs that are \(+small\):

\[
\begin{align*}
x \bigg|_{x=+small} \xrightarrow{DATE} DATE(x) \bigg|_{x=+small} &= (-1)x^{-even} \bigg|_{x=+small} \\
&= (-1) \cdot (+small)^{-even} \\
&= -1 \\
&= \frac{(+small) \cdot \ldots \cdot (+small)}{\text{even number of copies of } +small}
\end{align*}
\]

and since,
- by the Definition of \( small \), copies of \( small \) multiply to \( small \)
- by the Rule of Signs, copies of \(+\) always multiply to \(+\)

\[
\frac{-1}{(+small)}
\]

and by the fact that a finite coefficient divided by a result that is small-in-size gives an output that is large-in-size

\[ = -large \]

So, we have that the place of the local graph of \( DATE \) near \( 0^+ \) is
iii. To get the place of the local graph near $0^-$, we use the **Local Place Near $0^-$ Theorem** which says that, since the exponent is even, the local place near $0^-$ is flipped **horizontally** from the local place near $0^+$:

Even exponent: horizontal-flip

iv. Then the **Local Shape Near $0$ Theorem** says that the local graph of $DATE$ near $0$ must be

**Example 16.** Given the negative-exponent power function $VARE$ whose input-output rule is

$$x \xrightarrow{VARE} VARE(x) = (+83.17)x^{-13}$$

we want to find the local graph of $VARE$ near $0$.

i. We **normalize** the input-output rule to

$$x \xrightarrow{VARE} VARE(x) = (+1)x^{-\text{odd}}$$

ii. To get the local place near $0^+$, we **compute** the output for inputs that are $+\text{small}$:

$$x \bigg|_{x=+\text{small}} \xrightarrow{VARE} VARE(x) \bigg|_{x=+\text{small}} = (+1)x^{-\text{odd}} \bigg|_{x=+\text{small}} = (+1) \cdot (+\text{small})^{+\text{odd}} = +1 \cdot (+\text{small}) \cdot \ldots \cdot (+\text{small})$$

odd number of copies of $+\text{small}$
and since,
- by Definition of small, copies of small multiply to small
- by the Rule of Signs, copies of + always multiply to +

\[
\text{and, by the fact that a finite coefficient divided by a result that is small-in-size gives an output that is large-in-size}
\]

\[
= +1
\]
\[
= +\text{large}
\]

So, we have that the place of the local graph of VARE near 0+ is

iii. To get the place of the local graph near 0−, we use the Local Place Near 0− Theorem which says that, since the exponent is odd, the local place near 0− is flipped diagonally from the local place near 0+

iv. Then the Local Shape Near 0 Theorem says that the local graph of VARE near 0 must be
6.7 Types of Local Graphs Near 0

Proceeding in the same manner as above, we get the local graph near 0 of each one of the four types of negative-exponent power function. They are shown in the table below as seen from two points of view:

i. As seen from “not too far”, that is we see only the part of the screen that is around 0,0 so that we can still see the slope and the concavity. Of course, there is nothing on the screen and we only see what is in the transition.

ii. As seen from “faraway”, we cannot see the concavity for small inputs but only the slope and we see the local graphs near 0 as essentially straight and vertical.

<table>
<thead>
<tr>
<th>Input-output rule</th>
<th>From “not too far”</th>
<th>From “faraway”</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \xrightarrow{NEP} NEP(x) = +x^{even}$</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>$x \xrightarrow{NEN} NEN(x) = -x^{even}$</td>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>$x \xrightarrow{NOP} NOP(x) = +x^{odd}$</td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
</tr>
</tbody>
</table>

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CHAPTER 6. NEGATIVE-EXPONENT POWER FUNCTIONS

<table>
<thead>
<tr>
<th>Input-output rule</th>
<th>From “not too far”</th>
<th>From “faraway”</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \rightarrow NON)</td>
<td>( NON(x) = -x^{+\text{odd}} )</td>
<td></td>
</tr>
</tbody>
</table>

6.8 Essential Bounded Graph

As a consequence of the **Height Theorem** for negative-exponent power functions, we will obtain the essential bounded graph by interpolating smoothly:
- the local graph near \(-\infty\) with the local graph near \(0^-\)
- the local graph near \(0^+\) with the local graph near \(+\infty\)

**EXAMPLE 17.** Given the function \( KYK \) whose input-output rule is

\[
x \rightarrow NOP \rightarrow NOP(x) = (+52.92) \cdot x^{-13}
\]

we want to find the essential global graph.

i. We get the local graph near \(\infty\) as usual:

ii. We ask the **Essential Question**:

- Do all bounded inputs have bounded outputs
  or
- Is there one (or more) bounded input whose output is infinite?

Since the exponent \(-13\) is negative, by the **Height Theorem** for negative-exponent power functions, \(0\) is going to have \(\infty\)-height.

So, we must get the local graph near \(0\) as usual:
6.9. EXISTENCE OF NOTABLE INPUTS

Since, in the case of negative-exponent power functions, the outlying graph includes not only the local graph near \( \infty \) but also the local graph near 0, we can discuss the existence—but emphatically not the location—of certain notable inputs on the sole basis of the local graph near \( \infty \) and the local graph near 0. As it happens, though, in the case of negative-exponent power functions, there are no notable inputs:
1. Since the features at the transitions are always the same, there is no feature-sign change input.

\textbf{Example 18.} Given the negative-exponent power function whose outlying graph is

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example18_graph.png}
\end{figure}

there does not have to be any feature-sign change input:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example18_no_change.png}
\end{figure}

2. Since 0 is an $\infty$-height input, there is no 0-feature input.

\textbf{Example 19.} Given the negative-exponent power function whose outlying graph is

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example19_graph.png}
\end{figure}

there does not have to be any feature-sign change input as shown on the right side and 0-feature inputs would require a fluctuation as shown on the left side:
3. Since 0 is an \( \infty \)-height input, there is no extremum input.

**Example 20.** Given the negative-exponent power function whose outlying graph is

there does not have to be any extreme input as the only possible one would be 0 being a minimum input being clipped by too small an extent of the output ruler:

However, this cannot be the case because, as we saw earlier, nearby inputs get larger and larger outputs.

### 6.10 Essential Global Graph

We get the essential global graph of a regular positive-exponent power function by assembling all the information we found above, that is:

- We draw the local graph near \( \infty \),
- We draw the local graph near 0,
• We join smoothly across the screen the local graphs.

As with positive-power functions, there are two ways in which we can draw
the essential global graph. We can make:

i. A Mercator drawing in which we draw the global graph on a flat
surface such as a blackboard, a computer screen, or a piece of paper. This
lets the essential bounded graph play a central role and relegates ∞ where
we cannot see it.

ii. A Magellan drawing in which we draw the global graph on a Magellan
screen, which shows ∞ but introduces distortions and is a lot harder to draw
as we usually cannot really draw it on a sphere but can only make perspective
drawings as in this text.

A Magellan drawing provides a better insight in what is happening near 0.
In particular, we can see better the difference between:

• An ∞ height input for an odd negative-power function

  \begin{example}
  \textbf{Example 21.} Given the function } JUDY \text{ whose input-output rule is }
  \begin{align*}
  x \xrightarrow{JUDY} JUDY(x) &= (-12.84) \cdot x^{-3} \\
  &= \frac{-12.84}{x \cdot x \cdot x} \\
  \end{align*}
  \end{example}

  here is the Magellan drawing of the global graph of } JUDY: 

  ![Magellan drawing of JUDY](image)

• An ∞ height input for an even negative-exponent power functions.

  \begin{example}
  \textbf{Example 22.} Given the function } KRIS \text{ whose input-output rule is }
  \begin{align*}
  x \xrightarrow{KRIS} KRIS(x) &= (+44.08) \cdot x^{-4} \\
  &= \frac{+44.08}{x \cdot x \cdot x \cdot x} \\
  \end{align*}
  \end{example}

  here is the Magellan drawing of the global graph of } KRIS: 

  ![Magellan drawing of KRIS](image)
6.11 Types of Global Graphs

Each type of power function has a different qualitative global graph. Here again, the qualitative global graph is shown both from “nearby” and from “faraway”.

<table>
<thead>
<tr>
<th>Input-output rule</th>
<th>Essential Graph</th>
<th>From “faraway”</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \overset{NEP}{\longrightarrow} NEC(x) = x^{-even} )</td>
<td><img src="image" alt="NEP Graph" /></td>
<td><img src="image" alt="NEP Faraway" /></td>
</tr>
<tr>
<td>( x \overset{NEN}{\longrightarrow} NEN(x) = -x^{-even} )</td>
<td><img src="image" alt="NEN Graph" /></td>
<td><img src="image" alt="NEN Faraway" /></td>
</tr>
<tr>
<td>( x \overset{NOP}{\longrightarrow} NOP(x) = 0^{-odd} )</td>
<td><img src="image" alt="NOP Graph" /></td>
<td><img src="image" alt="NOP Faraway" /></td>
</tr>
</tbody>
</table>

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Continued on next page
## CHAPTER 6. NEGATIVE-EXPONENT POWER FUNCTIONS

<table>
<thead>
<tr>
<th>Input-output rule</th>
<th>Essential Graph</th>
<th>From “faraway”</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \overset{NON}{\longrightarrow} NON(x) = -x^{-\text{odd}} )</td>
<td><img src="image.png" alt="Graph" /></td>
<td><img src="image.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>