Chapter 1

Relations And Functions

To see whether something is changing—or not changing, it is necessary to look at it in relationship with something else.

**EXAMPLE 1.** The amount of income tax changes in relationship with the amount of income, the amount of property tax changes in relationship with the amount of property, the amount of sales tax changes in relationship with the amount of purchases.

More precisely, in order to observe something that is changing—or not changing, we must pair each observed state with some reference state.

**EXAMPLE 2.** We might say that someone’s income tax was $2,753. But that would not be saying much since, for instance, $2,753 was a lot less money in FY 2008 (FY is for Fiscal Year) than it was in, say, FY 1913— the year income tax was first established.

So, in order for $2,753, the observed state, to make sense, we must give it along with the reference state, namely the Fiscal Year.

We will call relation whatever process, device, procedure, agency, converter, exchanger, translator, etc in terms of which the pairing is done and we will call input numbers the numbers that correspond to the reference states and we will call output numbers the numbers that correspond to
the observed states.¹

NOTE. These terms are taken from computer science because they are more suggestive
than the more traditional terms, point for input and value for output.²

An input number together with an output number that it is paired with
by the relation make up what we will call an input-output pair. (Note
that we will be using parentheses to enclose input-output pairs and that this
is yet another use of parentheses.)

EXAMPLE 3. In the previous EXAMPLE 6, (FY2003, §2, 753) is an input-output pair.

NOTE. Eventually, the word “number” will go without saying and we shall just use the
word “input” instead of the phrase “input number” and the word “output” instead of
the phrase “output number”. Occasionally, though, we will have to use the full phrases,
“input number” and “output number”.

1.2 Quantitative Rulers

Very often, we will want to picture the input numbers and the output numbers involved in a relation and we will do that with quantitative rulers
which are pretty much what goes in the real world by the name of “ruler”.

NOTE. In high school, quantitative rulers usually go by the name of number lines,
a term we will not use in this text³.

1. More precisely, the tick-marks on a quantitative ruler must be labeled, in order, and equally spaced.

EXAMPLE 4. The following:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

are all quantitative rulers but the following is not a quantitative ruler even though the
tick-marks are labeled and in order:

¹Educologists will wonder why we do not continue sharply to distinguish the “real
world” from the “paper world” as we did in REASONABLE BASIC ALGEBRA. The answer is
that, by now, we hope being able to take advantage of what Thurston calls “compression”.

²Actually, never ones to shy away from the impenetrable, or perhaps just to show
off, Educologists generally prefer the XIXth century terms, “independent variable” and
“dependent variable”.

³It would be interesting to trace the origin of this remarkably un-enlightening term.
But then, it is probably due to Educologists’ well known craving for the esoteric.
1.2. QUANTITATIVE RULERS

Quantitative rulers are specified by two things:

- The **extent** of a given quantitative ruler consists of both the smallest label and the largest label which we write between **curly brackets** \{ \}.
- The **resolution** of a given quantitative ruler is the space between the labels of two consecutive tick-marks.

**Example 5.** Given the following quantitative ruler

![](quantitative_ruler.png)

- the **extent** of the given ruler is \{-40, +80\}
- the **resolution** of the given ruler is 10

2. Thus, given a quantitative ruler, there are going to be two kinds of number:

- Numbers that fall **within the extent** of the quantitative ruler which will be called **bounded numbers**. It is important to realize that any given number we happen to be interested in can always be viewed as a **bounded number** since we can always draw a quantitative ruler whose extent will **encompass** the given number.

**Example 6.** We can view the number 308,195 as a **bounded number** by using the following quantitative ruler:

![](bounded_number_ruler.png)

In fact, any bunch of numbers can be viewed as a bunch of **bounded numbers** since we need only use a quantitative ruler whose extent encompasses both the smallest number in the bunch and the largest number in the bunch.

**Example 7.** The numbers -176,329, -53.78, +543,830 will be **bounded numbers** for any ruler with an extent that encompasses both -176,329 and +543,830 such as, for instance, \{-200,000, +1,000,000\}

- Numbers that fall **beyond the extent** of the quantitative ruler which we will call **offscreen numbers**\(^4\). Because there is a number of difficulties with **offscreen numbers**, though, we will not deal with them right away and will return to them in **Chapter 2**.

\(^4\)Educologists will surely know why we didn’t use the term “unbounded”.

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\[
\begin{align*}
\text{extent} & \quad \text{curly brackets} & \{ \} \\
\text{resolution} & \quad \text{bounded numbers} & \text{encompass} \\
\text{offscreen numbers} & \end{align*}
\]
1.3 Finite Numbers And Their Neighborhoods

As we just saw, whether a number is bounded or not depends only on the extent of the ruler and thus this is totally independent of the resolution of the ruler. We now investigate matters that do depend on the resolution of the ruler, that is on how the ruler is tick-marked.

1. Given a quantitative ruler and given a bounded number, there are two possible cases

   a. The given bounded number can fall on a tick-mark in which case we will be able to plot the given number by placing a solid dot on the tick-mark and we will say that the bounded number is a finite number for the given ruler.

   EXAMPLE 8. Given the following quantitative ruler

   we can plot the bounded numbers $-500$ and $+1300$ but we cannot plot the bounded number $+800$

   and therefore, for this quantitative ruler, the bounded numbers $-500$ and $+1300$ are finite numbers but the bounded number $+800$ is not a finite number.

   EXAMPLE 9. Given the following quantitative ruler

   we can plot the bounded numbers $-1.4$ and $-0.8$ but we cannot plot the bounded number $-1.1$:

   and therefore, for this quantitative ruler, the bounded numbers $-1.4$ and $-0.8$ are finite numbers but the bounded number $-1.1$ is not a finite number.

   In order to discuss finite numbers in general, that is when the actual finite numbers that we are discussing will remain undisclosed for the duration, we will use the letter $x$ with a subscript, usually $0$ but also $1$, $2$ etc, so that we will use $x_0$ to designate a single given finite number but we will use $x_1$, $x_2$, etc when we have to deal with two or more given finite numbers.

   b. The given bounded number can fall somewhere between two tick-marks and therefore cannot be plotted.

   EXAMPLE 10. Given the following quantitative ruler

   we can plot the bounded numbers $-1.4$ and $-0.8$ but we cannot plot the bounded number $-1.1$:

   and therefore, for this quantitative ruler, the bounded numbers $-1.4$ and $-0.8$ are finite numbers but the bounded number $-1.1$ is not a finite number.
the number $-1.27$

falls between the tick-marks $-1.2$ and $-1.3$ but cannot be plotted.

In fact, this will be rather the more frequent case since there are only so many tick-marks on a quantitative ruler and therefore only so many finite number for that quantitative ruler while there can be many, many numbers.

2. We saw in the previous section that any bunch of numbers we happen to be interested in can be viewed as a bunch of bounded numbers.

Similarly, we can view any single given number we happen to be interested in as a finite number because we can always make up a ruler such that the label of one of the tick-marks is going to be the given number.

However, given a bunch of numbers, it is not always possible to look at all of them as finite numbers because it is not always possible to find a quantitative ruler whose resolution allows us to plot all the given numbers, that is for which all the given numbers are labels of tick-marks.

**Example 11.** It does not seem very feasible to come up with a quantitative ruler on which to plot the numbers $1000, 100, 10, 1, 0.1, 0.01, 0.001$ that is with a quantitative ruler with both the extent necessary to show $0$ and $1000.$ and the resolution necessary to separate $0.001$ from $0.01.$

3. What we will do is to look at each tick-mark on a quantitative ruler as the center of a neighborhood extending between the two halfway-marks surrounding the tick-mark which we will mark with parentheses, yet another use of parentheses.

**Example 12.** Given the quantitative ruler

Then, any given bounded number will fall in the neighborhood of some tick-mark and, naturally, we will say that the bounded number is near the finite number which is the label of the tick-mark.

**Example 13.** Given the following quantitative ruler
since the number $-1.57$ falls between $-1.6$ and $-1.5$ and $-1.6$ is the closest tick-mark, $-1.57$ is in a neighborhood of $-1.6$.

Or, we can just say that $-1.57$ is near $-1.6$

### 1.4 Quantitative Screens

Given a relation, we will often want to picture the input-output pairs involved in the relation.

1. The simplest way, of course, is to use:
   - a quantitative input ruler, that is a quantitative ruler to plot input numbers
   - a quantitative output ruler, that is a quantitative ruler to plot output numbers
   - links to pair the input and the output into an input-output pair.

   **Example 14.** Given the input-output pair (FY1961, $2,750), we could picture it as follows:

   ![Diagram of quantitative screen and rulers](image)

   Rather obviously, though, this approach is not going to work very well when the relation involves a lot of input-output pairs because the links will very quickly turn into something looking like a pile of overcooked spaghetti.

2. So, for a better way to picture the input-output pairs of a relation, we will use quantitative screens which will consist of:
   - A screen, that is a rectangular area in which we will link the input and the output of an input-output pair,
   - An quantitative input ruler placed under the screen with the extent of the input ruler corresponding to the width of the screen,
1.4. QUANTITATIVE SCREENS

• An quantitative output ruler placed left of the screen with the extent of the output ruler corresponding to the height of the screen.
• Some offscreen-space beyond the screen.

EXAMPLE 15.

3. We now turn to the procedure for linking on a quantitative screen the input and the output of an input-output pair:

i. We plot the input number, that is we mark the input point, that is the tick-mark on the input ruler whose label is the input number,

ii. We draw the input level line, that is the vertical line through the input point,

iii. We plot the output number, that is we mark the output point, that is the tick-mark on the output ruler whose label is the output number,

iv. We draw the output level line, that is the horizontal line through the output point,

v. Then, the plot point that pictures the input-output pair on the screen is at the intersection of the input level line and the output level level line.

EXAMPLE 16. In order to picture the input-output pair $(−2, +5)$,
i. We mark the input point, that is the tick-mark on the input ruler whose label is the input number $-2$.

ii. We draw the input level line, that is the vertical line through the input point.

iii. We mark the output point, that is the tick-mark on the output ruler whose label is the output number $+5$.

iv. We draw the output level line, that is the horizontal level line through the output point.

v. Then, the plot point that pictures the pair $(-2, +5)$ on the screen is at the intersection of the input level line and the output level line.

**NOTE.** The fact that we use “side rulers” instead of “crosshairs” in the middle of the screen does not conform with the usual practice in textbooks—but it does conform with the usual practice just about anywhere else. One reason we will use “side rulers” as opposed to “crosshairs” is that rulers and “0-level lines” are entirely separate entities which we want to keep separate and so we do not want to draw the rulers on top of level lines. Another reason, as we will soon see, will be our heavy use of “qualitative rulers” in which, as we will see, the rulers cannot go across the screen.

4. In this way of picturing input-output pairs, the link consists of part of the input level line and part of the output level line.

**EXAMPLE 17.**
Given the input-output pair $(+3, +4)$ the link consists of the part of the $+3$-input level line from the input point to the plot point and the part of the $+4$-output level line from the plot point to the output point:

However, just the plot-point is a **good picture** of an input-output pair.
1.4. QUANTITATIVE SCREENS

because, once we have drawn the plot-point, we can erase the input point and the output point as well as the two level lines and we are still able to recover the input-output pair of which this plot-point is the picture. All we have to do is to go backwards through the above steps:

i. We draw the input level line (vertical) through the given plot point,

ii. The input point (that represent the input number) is where the input level line intersects the input ruler,

iii. We draw the output level line (horizontal) through the given plot point,

iv. The output point (that represent the output number is where the output level line intersects the output ruler.

**EXAMPLE 18.** Given, the following plot-point

![Plot point diagram](image)

we can recover the input-output pair as follows:

i. We draw the input level line (vertical) through the given plot point,

ii. The input point (that represents the input number) is where the input level line intersects the input ruler,

iii. We draw the output level line (horizontal) through the given plot point,

iv. The output point (that represents the output number) is where the output level line intersects the output ruler.

In other words, a plot point does not involve any loss of information compared to the input-output pair that it pictures.
CHAPTER 1. RELATIONS AND FUNCTIONS

1.5 Functions

Relations can get to be surprisingly complicated and so, from now on, we will only investigate functions, that is relations that meet the requirement that:

No input shall be paired with more than one output.

In other words, given any input, a function may either return one output or no output at all but never more than one output.

However, there is nothing to prevent a function from pairing many inputs with a same, single output. In other words, a function may return the same output for different inputs.

Example 19. A parking meter

is a function because, whatever the amount of money we input, we can be sure that anyone who inputs the same amount of money will get the same parking times as we did. For instance, given an input, say 1 Quarter, the parking meter outputs a definite amount of parking time, say 20 Minutes.

However, since there usually is a maximum parking time, any amounts of money we input above the maximum will return the same amount of parking time namely the maximum parking time.

Educologists will rightfully object that functions should not be allowed to return no output and therefore that we should introduce the notion of domain. But while, of course, pontificating about domains can be gratifying to the instructor, it also complicates the students’ mathematical life quite unnecessarily since, at this point, the loophole thereby opened is quite unlikely to occur to beginners and thus to puzzle them.
1.6. FUNCTIONS SPECIFIED BY A GLOBAL I-O RULE

Example 20. A slot machine is not a function because, given an input, say 1 Quarter, a slot machine can output just about any number of Quarters.

Example 21. We definitely want the relation between taxable income and income tax to be a function because, presumably, we don’t want two persons with the same taxable income to pay different amounts of income tax!

1.6 Functions Specified By A Global I-O Rule

Functions can be specified in several ways:

1. In some sciences, such as Psychology, Sociology, Business, Accounting, etc functions are usually specified by input-output table(s). We will call them tabular functions.

Example 22. A business may be described by its profits/losses over the years, that is by a tabular function specified by the following input-output table:

<table>
<thead>
<tr>
<th>Fiscal Year</th>
<th>Profit/Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>2001</td>
<td>+5,924</td>
</tr>
<tr>
<td>2002</td>
<td>-2,351</td>
</tr>
<tr>
<td>2003</td>
<td>+6,753</td>
</tr>
<tr>
<td>2004</td>
<td>+5,636</td>
</tr>
<tr>
<td>2005</td>
<td>-3,753</td>
</tr>
<tr>
<td>2006</td>
<td>+8,482</td>
</tr>
</tbody>
</table>

2. We will investigate only functions specified directly by a global input-output rule constructed as follows:

i. We must already have or, if not, create a name for the function. (By default, that is in the absence of any other name, the letter f is often used.)

ii. We use a letter, usually x, which we will call unspecified input, as place holder standing for specific inputs, that is we will be allowed at any time to replace the unspecified input x by any specific input we want.

iii. Then, f(x), to be read as “f of x”, yet another use of parentheses, will stand for the output returned by the function f for the unspecified input x.

iv. Finally, we must give a specifying phrase to specify f(x) in terms of x.

Example 23. Let JILL be the function that doubles the input and adds 5 to the result to give the output. Then the function JILL is specified by the global input-output rule

\[ x \xrightarrow{JILL} JILL(x) = 2x + 5 \]

where 2x + 5 is the formula that specifies JILL(x) in terms of x.
3. Then, given a function specified by a global input-output rule, the procedure to identify the output for a given input is as follows:

i. We indicate by which given input we want to replace the unspecified input $x$. To do that:

   a. We draw, to the right of the unspecified input a vertical bar extending a bit below the line, which we read as “where”

   b. We write to the bottom right of the vertical bar:
      - the unspecified input $x$
      - the symbol $\leftarrow$, to be read as “is to be replaced by”,
      - the specific input that we want to replace $x$ with.

ii. We carry out the replacement of the unspecified input by the given specific input.

iii. We identify the resulting specifying phrase.

Example 24. Given the function $JACK$ specified by the global input-output rule

$$x \rightarrow_{JACK} JACK(x) = -4x + 2$$

and given the input $-3$, we get the output as follows.

a. We indicate that $x$ is to be replaced by $-3$

$$x|_{x=-3} \rightarrow_{JACK} JACK(x)|_{x=-3} = -4x + 2|_{x=-3}$$

b. We carry out the replacement:

$$= (-4) \cdot (-3) + 2$$

c. We compute the output:

$$= +12 + 2$$

$$= +14$$

d. We can then write

$$-3 \rightarrow_{JACK} +14$$

or

$$JACK(-3) = +14$$

or

$$(-3, +14)$$

is an input-output pair for the function $JACK$.

1.7 Functions Specified By A Curve

Any curve we draw on a quantitative screen specifies a relation because each point on the curve can be looked upon as being a plot point from which we can recover the input-output pair as we did above.
1.7. FUNCTIONS SPECIFIED BY A CURVE

1. If it happens that no input-level line intersects the curve in more than one point, then the relation meets the requirement that

\[
\text{No input shall be paired with more than one output.}
\]

and the curve thus specifies a function. The curve is then called the \textbf{quantitative global graph} of the function.

The reason we need to use the long phrase “quantitative global graph” as opposed to just the single word “graph” is because we will use many different kinds of graphs which it will be good to distinguish.

To begin with, since quantitative rulers have an extent, we can only get the \textbf{quantitative bounded graph}, that is the part of the quantitative global graph that is on the \textit{screen}.

**EXAMPLE 25.** Given the following curve

![Quantitative Global Graph Example](image)

none of input-level lines intersects the curve in more than one point so that the curve is the \textit{bounded graph} of a function.

2. When a \textit{function} is given by a quantitative global graph, and given an input, we get the output for that input with the following procedure:

i. We mark the \textit{input point}, that is the tick-mark on the \textit{input ruler} whose label is the \textit{input number},

ii. We draw the \textit{input level line}, that is the \textit{vertical} line through the \textit{input point},

iii. We mark the \textit{plot point}, that the point at the intersection of the input level line with the curve,

iv. We draw the \textit{output level line}, that is the \textit{horizontal} line through the \textit{plot point},

v. Then, the \textit{output point} that pictures the \textit{output number} is at the intersection of the output level line and the output ruler.

**EXAMPLE 26.** Given the function specified by the quantitative global graph
and given the input $-2$, find the output.

i. We mark the input point, that is the tick-mark on the input ruler whose label is the input number $-2$

ii. We draw the input level line, that is the vertical line through the input point

iii. We mark the plot point, that is the point where the input level line intersects the curve

iv. We draw the output level line, that is the horizontal level line through the plot point,

v. Then, the output point is where the output level line intersects the output ruler.

1.8 The Fundamental Graphic Problem

We now come to the fact that, given a function specified by a global input-output rule, we will often want to find its quantitative global graph.

One reason is that it is usually impossible to see just from the global input-output rule which inputs have some required “feature”. But with the quantitative global graph, it will often be obvious which input(s), if any, have such a required “feature”.

**Example 27.**

Given the function $MILT$ specified by the quantitative global graph
1.8. THE FUNDAMENTAL GRAPHIC PROBLEM

find the inputs whose output is less than 2. From the quantitative bounded graph, we see that the answer is “All inputs between −4 and +3”.

However, given a function specified by a global input-output rule, getting its quantitative global graph is usually not simple.

1. In high school we were taught to proceed as follows:

i. Pick a few inputs and use the global input-output rule to compute the outputs and then plot the input-output pairs. In fact, to facilitate plotting, the screen often comes in the form of graph-paper, that is already equipped with input level lines and output level lines in the shape of a grid:

ii. Then, we were supposed to “join smoothly” these few plot-points and the resulting curve would be the quantitative bounded graph of the function.

2. The trouble with that supposed “procedure” is that, while there is no problem with the plotting step, we have absolutely no basis whatsoever to believe that the curve we draw in the second step will be anywhere like the quantitative bounded graph of the function. Here are two reasons why:

a. We would be getting something for nothing and thus, on general principles, we should not expect to get the information contained in all the input-output pairs specified by the global input-output rule from just the
knowledge of the input-output pairs we actually computed. At least in mathematics, there is no such thing as a free lunch.\footnote{That Educologists have no problem with free lunches is nothing new as they never let questions about the contents get in their way. In fact, Educologists make it a point never to care about what they teach but only about how to teach “it”. In particular, they never let themselves be embarrassed by logic which they consider as totally beside the point at best and absolutely confusing at worst.}

b. And then, given a number of plot-points, there is never a unique way to “join them smoothly.”\footnote{This is perhaps where the dishonesty of Educologists is most patent as one would expect them at least to acknowledge this rather unfortunate fact. But they never do.}

**Example 28.** Given the function $RAT$ specified by the following input-output table:

<table>
<thead>
<tr>
<th>Inputs</th>
<th>$-4$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$+1$</th>
<th>$+2$</th>
<th>$+4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outputs</td>
<td>$-1$</td>
<td>$+3$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$-2$</td>
<td>$+3$</td>
</tr>
</tbody>
</table>

and therefore the following plot:

But now the question is how we are to “join smoothly” these plot points and the answer is that, on the face of it any one of the following curves, for instance, is just as likely or as unlikely as any of the others to be the quantitative graph:
The advice we were usually given at this point was “to get more plot points” but this ignores two issues:

- One issue is the question of how many plot-points would be needed to guarantee that there is only one way to joint the plot points to get the graph. As we will see, there generally isn’t any such number. In fact, even computer generated plots cannot be trusted.

**Example 29.** Given the function $CAT$ specified by the global input-output rule

$$x \xrightarrow{CAT} CAT(x) = \frac{x^3 - 1}{x - 2}$$

here are six computer generated plots that look like quantitative bounded graphs:

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Here again, Educologists never bother even to raise the issue.
Which one should we decide is the quantitative bounded graph?

- The other issue is that having very many plot points can turn out to make it impossible to “join them smoothly”.

**Example 30.** The function $SINE$, which is of major importance in Physics, Engineering, etc, is specified indirectly, so doesn’t belong to this text, but what matters here is Strang’s Famous Computer Plot of $SINE$:
What is the quantitative bounded graph? What made Strang’s computer plot famous is that the quantitative bounded graph of $SINE$ is a lot simpler than the above computer plot would suggest.

3. Finally, the immense variety of possible global input-output rules would seem to make it quite impossible to find a “universal procedure” for getting the quantitative bounded graph for every function specified by a global input-output rule. Therefore, our goal will have to be somewhat more modest and, in fact, we will take a qualitative viewpoint, that is we will not deal with specific inputs and we will mostly investigate qualitative bounded graphs, that is bounded graphs that will give us an idea of the “shape” of the quantitative bounded graph. As such, qualitative bounded graphs will be a help for “joining smoothly” plot points but, much more importantly, will also turn out to be crucial in finding inputs with required “features”.

4. And so, after we have discussed a few more issues in this chapter and developed some “technology” in the next two chapters, our primary goal in this text will be to investigate the:

**FUNDAMENTAL PROBLEM.** Given a function specified by a global input-output rule, find its qualitative bounded graph.

In fact, we will investigate the FUNDAMENTAL PROBLEM systematically throughout a whole range of kinds of functions. We will start with extremely simple functions, the “power functions”, and then move incrementally all the way up to the crowning glory: “rational functions”.