Chapter 10

Affine Functions: Global Analysis

As we will see, affine functions are truly *exceptional* in that we will be able to solve *all* global problems *exactly*. With most other functions, though, we will be able to solve only *some* global problems and mostly only *approximately* so.

10.1 The Essential Question

As always when we set out to investigate any kind of functions, the first thing we must do is to find out if the *offscreen graph* of an *affine function* consists of just the *local graph near* $\infty$ or if it also includes the *local graph near one or more $\infty$-height inputs*.

In other words, we need to ask the **Essential Question**: 

- Do all *bounded inputs* have *bounded outputs*
- or
- Are there *bounded inputs* that are $\infty$-height inputs, that is are there inputs whose nearby inputs have infinite outputs?
Now, given a bounded input $x$, we have that:
- since $a$ is bounded, $ax$ is also bounded
- $b$ is bounded
and so, altogether, we have that $ax + b$ is bounded and that the answer to the Essential Question is:

**THEOREM 1 (Bounded Height).** Under an affine function, all bounded inputs return bounded outputs.

and therefore

**THEOREM 2 (Offscreen Graph).** The offscreen graph of an affine function consists of just the local graph near $\infty$.

### Existence Theorems

The notable inputs are those
- whose existence is forced by the offscreen graph which, by the Bounded Height Theorem for affine functions, consists of only the local graph near $\infty$.
- whose number is limited by the interplay among the three features

Since polynomial functions have no bounded $\infty$-height input, the only way a feature can change sign is near an input where the feature is 0. Thus, with affine functions, the feature-change inputs will also be 0-feature inputs.

None of the theorems, though, will indicate where the notable inputs are. The Location Theorems will be dealt with in the last part of the chapter.

**EXAMPLE 1.** When somebody has been shot dead, we can say that there is a murderer somewhere but locating the murderer is another story.

### 10.2 Slope-sign

Given the affine function $AFFINE_{a,b}$, that is the function specified by the global input-output rule

$$\begin{align*}
x & \xrightarrow{\mathbf{AFFINE}} \mathbf{AFFINE}(x) = ax + b
\end{align*}$$

recall that when $x$ is near $\infty$ the Slope-sign Near $\infty$ Theorem says that:
10.2. **SLOPE-SIGN**

- When $a$ is $+$, $\text{Slope-Sign}|_x \nearrow \infty = (\nearrow, \nearrow)$
- When $a$ is $-$, $\text{Slope-Sign}|_x \nearrow \infty = (\searrow, \searrow)$

1. Since the slope does not change sign as $x$ goes through $\infty$ from the left side of $\infty$ to the right side of $\infty$, the slope need not change sign as $x$ goes across the screen from the left side of $\infty$ to the right side of $\infty$ so there does not have to be a bounded Slope-sign change input:

**Example 2.** Given an affine function whose offscreen graph is

![Output Ruler](image1)

we don’t need a bounded slope-sign change input to join smoothly the local graphs near $\infty$:

![Output Ruler](image2)

2. In fact, not only does there not have to be a bounded slope-sign change input, there cannot be a bounded slope-sign change input since the local linear coefficient is equal to the global linear coefficient $a$ and the slope must therefore be the same everywhere:

**Theorem 3 (Slope-Sign Change Non-Exisence).** An affine function has no bounded Slope-Sign Change input.

3. Another consequence of the fact that the local slope does not depend on $x_0$, and is thus the same everywhere, is that it is a feature of the function $\text{AFFINE}_{a,b}$ itself and so that the function $\text{AFFINE}_{a,b}$ has a global slope specified by the global linear coefficient $a$.

4. Moreover, the slope cannot be equal to 0 somewhere because the slope is equal to $a$ everywhere. So, we also have:

**Theorem 4 (0-Slope Input Non-Existence).** An affine function has no bounded 0-slope input.
10.3 Extremum

From the optimization viewpoint, an affine function has no extremum input, that is no bounded input whose output would be larger (or smaller) than the output of nearby inputs.

THEOREM 5 (Extremum Non-existence). An affine function has no bounded local extremum input.

10.4 Height-sign

Given the affine function \( AFFINE_{a,b} \), that is the function specified by the global input-output rule

\[
    x \xrightarrow{AFFINE} AFFINE(x) = ax + b
\]

recall that when \( x \) is near \( \infty \) the Height-sign Near \( \infty \) Theorem says that:

- When \( a \) is \(+\), Height-Sign\( \mid _{x \text{ near } \infty} = (+, -) \)
- When \( a \) is \(-\), Height-Sign\( \mid _{x \text{ near } \infty} = (-, +) \)

1. Since the height changes sign as \( x \) goes from the left side to the right side of \( \infty \) across \( \infty \), the height must also change sign as \( x \) goes from the left side to the right side of \( \infty \) across the screen so there has to be at least one bounded Height-sign change input:

EXAMPLE 3. Given the affine function whose offscreen graph is

there has to be a bounded height-sign change input:
2. On the other hand, an affine function can have at most one 0-height input because, if it had more, it would have to have 0-slope inputs in-between the 0-height inputs which an affine function cannot have. So, we have:

**THEOREM 6 (0-Height Existence).** An affine function has exactly one bounded 0-height input and it is a 0-height input: \( x_{\text{Height-sign change}} = x_{0\text{-height}} \)

### 10.5 Bounded Graph

There are two ways to look at the shape of the bounded graph.

1. As a consequence of the Bounded Height Theorem for affine functions, the offscreen graph consists only of the local graph near \( \infty \) and we can obtain the forced bounded graph by extrapolating smoothly the local graph near \( \infty \).

There remains however a question namely whether the extrapolated bounded graph is straight that is has no concavity. However, affine functions have no concavity and that settles the matter: the local graph near \( -\infty \) and the local graph near \( +\infty \) must be lined up and can therefore be joined smoothly with a straight line.

2. In the case of affine functions, it happens that we can also obtain the bounded graph by interpolating local graphs near bounded inputs:

We start from the local graphs near a number of finite points as follows:

We construct local graphs near, say, three different finite inputs, \( x_1, x_2, x_3 \).

They would look something like this:

However, this is not possible because that would mean that inputs such as \( x_4 \) would have two outputs:
As a result, the local graphs near finite inputs must all line up and so the bounded graph must be a straight line:

Of course, the bounded graph must line up with the local graph near $\infty$ as, otherwise, there would have to be a jump in the transition zone.

**REVISITING (IN)EQUATIONS**

In order to deal with location problems, it is necessary to have a tool that works no matter what. The purpose here, then, is:
- To make sure of the algebra that we will need to deal with location problems,
- To have a systematic procedure to refer to,
- To set a language that is consistent

### 10.6 Equation Problems

1. We begin with the terminology that we will use to be extremely clear as to what we are doing. An equation problem will consist of:
   i. A data set from which the number are to be picked/
   
   **Example 4.** If the problem to be dealt with involves the age of human beings, the data set should probably not involve negative numbers and probably not go much farther than about 100.

   ii. An equation that will specify the solution subset of the data set, that is which will select the numbers in the data set that are to be retained as solutions, that is the numbers that turn the equation into a true sentence. Those numbers in the data set that turn the equation into a false sentence will be called non-solutions.

2. The approach that we will follow, which we will call the Reduction Method, will be to reduce the original equation to an equation until we get to an equation that we already know how to solve and we will call that equation the reduced equation. Of course, the reduced equation will
have to be equivalent to the original equation in the sense that the reduced equation will have to have the same solution subset as the original equation. What we will do here will only be to add the same number to both sides or multiply both sides by the same number (other than 0) so that the following will apply:

**THEOREM 7 (Fairness).** Given any equation, as long as, whatever we do onto one side of the verb =, we do exactly the same onto the other side of the verb =, we get an equivalent equation.

**NOTE.** While the Fairness Theorem seems obviously true, making the case that it is true is not that easy because what is not obvious is on what evidence to base the case. We will thus leave this issue for when the reader takes a course in Mathematical Logic.

3. In the case of affine equations, that is of equations of the form

\[ ax + b = c \]

the Reduction Method proceeds as follows:

\[
\begin{align*}
ax + b & = c \\
ax + b - b & = c - b \\
ax & = c - b \\
\frac{ax}{a} & = \frac{c - b}{a} \\
x & = \frac{c - b}{a}
\end{align*}
\]

so that the Fairness Theorem applies and since \( \frac{c - b}{a} \) is the solution of the reduced equation it is also the solution of the original equation.

### 10.7 Inequation Problems

This involves a general procedure that we will call the **Pasch Procedure** (after the name of the mathematician who first noticed that, while it was quite obvious that in order to go from one side of a point on a straight line to the other side you had to “get across” the point, this turned out to be impossible to “prove” because the question was “on the basis of what” so that it had to be accepted as an axiom).

**EXAMPLE 5.** Given the point \( P \) on one side of the point \( B \) and the point \( Q \) on the other side of the point \( B \), to go from \( P \) to \( Q \), we need to get across the point \( B \)
1. Roughly, given an inequation problem, that is a data set and an inequation, in order to determine the solution subset of that inequation problem, we will proceed in two stages:

I. We will locate the boundary of the solution subset of the inequation problem by finding the solution subset of the associated equation problem.

(In the case of an affine inequation, we have already seen in the previous section how to deal with this stage.)

To graph the boundary point we will use:
- a solid dot when the boundary point is a solution of the inequation
- a hollow dot when the boundary point is a non-solution of the inequation.

II. We will locate the interior of the solution subset of the inequation problem, that is the solution subset of the associated strict inequation problem.

In the case of an affine inequation, the boundary consists of only one boundary point which separates the data set in two sections which we will call Section A and Section B. Then, we will locate the interior as follows:

i. We will pick a test number in Section A and check if the test number is a solution or a non-solution of the given inequation.

ii. We will pick a test number in Section B and check if the test number is a solution or a non-solution of the given inequation.

iii. We will then conclude with the help of

THEOREM 8 (Pasch).

- If the test number in a section is a solution, then all numbers in that same section are included in the solution subset.
- If the test number in a section is a non-solution, then all numbers in that same section are non-included in the solution subset.

EXAMPLE 6. Given the inequation problem in which
- the data set consists of all numbers
- the inequation is

\[ x \geq -13.72 \]

we locate separately.

i. The boundary point of the solution subset of the inequation problem is the solution of the associated equation:

\[ x = -13.72 \]

which, of course, is \(-13.72\) and which we graph as follows since the boundary point is a solution of the inequation.
ii. The interior of the solution subset, that is the solution subset of the associated
strict inequation

\[ x > -13.72 \]

i. The boundary point \(-13.72\) separates the data set in two sections, Section A and
Section B:

While, in the case of affine inequations, we can work directly on the
inequation, this is not generally the case while, as we will see, the advantage
of the PASCH PROCEDURE is that it will work in all cases.

**LOCATION THEOREMS**

Previously, we only established the existence of certain notable features of
affine functions and this investigation was based on graphic considerations.
Here we will investigate the location of the inputs where these notable features occur and this investigation will be based on input-output rule considerations.

10.8 0-Slope Location

We saw earlier that affine functions cannot have a 0-slope input. On the other hand, since the slope is the same everywhere, it is a global feature of the function itself and we have:

**THEOREM 9 (Global Slope-sign).** Given the affine function \(AFFINE_{a,b}\),

- When \(a\) is positive, \(\text{Slope-sign} \ AFFINE = \slash\).
- When \(a\) is negative, \(\text{Slope-sign} \ AFFINE = \backslash\).

10.9 Locating Inputs Whose Output = \(y_0\)

The simplest global problem is when the feature that is required from the output is to be some given number \(y_0\). In other words, given a function, we ask for the input(s), if any, whose output is \(y_0\). (A particular case of this problem is to find the 0-height input(s), if any.)

What we do is:

1. We write the global input-output rule

\[ x \xrightarrow{A_{a,b}} A_{a,b}(x) = ax + b \]

2. Since we want the output to be equal to \(y_0\), that is since we want

\[ A_{a,b}(x) = y_0 \]

we set the equation

\[ ax + b = y_0 \]

3. We then solve using the **REDUCTION METHOD**:

\[ ax + b = y_0 \]
\[ ax + b - b = y_0 - b \]
\[ ax = y_0 - b \]
\[ \frac{ax}{a} = \frac{y_0 - b}{a} \]
\[ x = \frac{y_0 - b}{a} \]
So, \( x_{y_0\text{-height}} = \frac{y_0 - b}{a} \)

**Example 7.** Given the affine function whose global input-output rule is:

\[
x \rightarrow_{A_{a,b}} A_{a,b}(x) = -2.45x + 43.72
\]

find \( x_{y_0\text{-height}} \)

**Note.** The resulting fraction, though, is only “code” that indicates what division is to be made and one should keep in mind that most division do not end by themselves so that the result of a division is usually only approximate.

We set the output equal to 0 and solve the resulting equation:

\[
-2.43x + 43.72 = 0
\]

\[
-2.43x + 43.72 - 43.72 = 0 - 43.72
\]

\[
-2.43x = -43.72
\]

\[
\frac{-2.43}{-2.43} = \frac{-43.72}{-2.43}
\]

\[
x = +\frac{43.72}{2.43}
\]

So, \( x_{y_0\text{-height}} = +\frac{43.72}{2.43} \)

If we want \( x_{y_0\text{-height}} \) with two decimals, the division gives 17.99 and we can write either:

\[
\cdot x_{y_0\text{-height}} = 17.99 + [...]
\]

where [...] stands for something positive too small to matter here,

or

\[
\cdot x_{y_0\text{-height}} = 18.00 + [...]
\]

where [...] stands for something negative too small to matter here.

If we want \( x_{y_0\text{-height}} \) with five decimals, the division gives 17.99176 and we can write either:

\[
\cdot x_{y_0\text{-height}} = 17.99176 + [...]
\]

where [...] stands for something positive too small to matter here,

or

\[
\cdot x_{y_0\text{-height}} = 17.99177 + [...]
\]

where [...] stands for something negative too small to matter here.

### 10.10 Locating Inputs Whose Output > \( y_0 \) Or < \( y_0 \)

Given the affine function \( AFFINE_{a,b} \), we are now ready to deal with the global problem of finding all inputs whose output is smaller (or larger) than some given number \( y_0 \).

The input-output rule of \( AFFINE_{a,b} \) is:

\[
x \rightarrow_{AFFINE_{a,b}} AFFINE_{a,b}(x) = ax + b
\]
If we want, for instance, the output to be larger than $y_0$, we write

\[ AFFINE_{a,b}(x) > y_0 \]

which means that we have to find the solution subset of the inequation

\[ ax + b > y_0 \]

So, according to the Pasch Procedure,

I. We find the boundary point of the inequation problem by finding the solution of the associated equation:

\[ ax + b = y_0 \]

So, we solve using the Reduction Method:

\[
\begin{align*}
ax + b &= y_0 \\
ax + b - b &= y_0 - b \\
ax &= y_0 - b \\
\frac{ax}{a} &= \frac{y_0 - b}{a} \\
x &= \frac{y_0 - b}{a}
\end{align*}
\]

which we graph with a hollow dot since, here, the inequation is strict (i.e. $<$) so that the boundary point $\frac{y_0 - b}{a}$ is a non-solution of the inequation:

II. We determine the interior of the solution subset of the inequation problem by testing.

Since the boundary point $\frac{y_0 - b}{a}$ separates two sections in the data set (i.e. all inputs), Section A and Section B,
we must test each section but since, as we shall see, all that matters will be the sign of \(a\), it will be easier to deal separately with the case when \(a\) is positive and the case when \(a\) is negative:

- **When \(a\) is positive,**
  - We pick \(-\infty\) to test Section A because we know \(-\infty\) is in Section A and because it is easy to check \(-\infty\) in the inequation:
    \[
    a x + b|_{x=-\infty} > y_0 \\
    a(-\infty) + [...] > y_0
    \]
    and since \(a\) is positive,
    \[-\infty + [...] > y_0\]

    Since this is false, \(-\infty\) is a non-solution and, by Pasch Theorem, all inputs in Section A are also non-solutions:

- **When \(a\) is negative,** proceeding in the same manner as above we find that the solution subset of the inequation problem is
10.11 Initial Value Problem

An Initial Value Problem asks the question:

- The function $F$ is affine
- The slope of the function $F$ is to be a given number $a$
- The output returned by the function $F$ for a given input $x_0$ is to be a given number $y_0$.

**Example 8.** Find the global input-output rule of the function $KATE$ given that it is affine, that its slope is $-3$ and that the output for the input $+2$ is $+5$.

We use all three given pieces of information:

i. Since we are given that $KATE$ is an affine function, we give temporary names for the dilation coefficient, say $a$, and for the constant term, say $b$, and we write the global input-output rule of $KATE$ in terms of these names:

$$x \xrightarrow{KATE_{a,b}} KATE_{a,b}(x) = ax + b$$

ii. By the Local Slope Theorem, the slope is equal to the dilation coefficient:

$$-3 = a$$

which give the equation $a = -3$

iii. Since the output for the input $+2$ is $+5$, we write

$$KATE_{a,b}(x)|_{x=+2} = +5$$

$$ax + b|_{x=+2} = +5$$

$$a(+2) + b = +5$$

which give the equation $2a + b = +5$

iv. So we must solve the system of two equations for two unknowns $a$ and $b$:

$$\begin{aligned}
2a + b &= +5 \\
\end{aligned}$$

This kind of system is very simple to solve since we need only replace $a$ by $-3$ in the second equation to get the equation:

$$2(-3) + b = +5$$
which we solve using the Reduction Method:

\[-6 + b = +5\]
\[-6 + b + 6 = +5 + 6\]
\[b = +11\]

v. So, the global input-output rule for KATE is

\[x \xrightarrow{\text{KATE}_{-3,+11}} \text{KATE}_{-3,+11}(x) = -3x + 11\]

10.12 Boundary Value Problem

A Boundary Value Problem asks the question:

What is the input-output rule of a function \( F \), given that:

- The function \( F \) is affine
- The output returned by the function \( F \) for a given input \( x_1 \) is to be a given number \( y_1 \).
- The output returned by the function \( F \) for a given input \( x_2 \) is to be a given number \( y_2 \).

**Example 9.** Find the global input-output rule of the function DAVE given that it is affine, that the output for the input +2 is \(-1\) and that the output for the input \(-4\) is \(-19\).

We use all three pieces of information that we are given:

i. Since we are given that DAVE is an affine function, we give temporary names for the dilation coefficient, say \( a \), and for the constant term, say \( b \), and we write the global input-output rule of DAVE in terms of these names:

\[x \xrightarrow{\text{DAVE}_{a,b}} \text{DAVE}_{a,b}(x) = ax + b\]

ii. Since the output for the input +2 is \(-1\) we write:

\[\text{DAVE}_{a,b}(x)|_{x=+2} = -1\]
\[ax + b|_{x=+2} = -1\]
\[a(+2) + b = -1\]

which give the equation \( +2a + b = -1 \)

iii. Since the output for the input \(-4\) is \(-19\) we write:

\[\text{DAVE}_{a,b}(x)|_{x=-4} = -19\]
\[ax + b|_{x=-4} = -19\]
\[a(-4) + b = -19\]

which give the equation \(-4a + b = -19\)
iv. So we must solve the system of two equations for two unknowns \( a \) and \( b \):

\[
\begin{align*}
+2a + b &= -1 \\
-4a + b &= -19
\end{align*}
\]

This kind of system is a bit more complicated to solve but since \( b \) appears in both equations, we replace one of the two equations, say the second one, by “the first one minus the second one”:

\[
\begin{align*}
+2a + b &= -1
\end{align*}
\]

This gives us:

\[
\begin{align*}
+2a + b &= -1 \\
+2a + b + 4a - b &= -1 + 19
\end{align*}
\]

that is

\[
\begin{align*}
+2a + b &= -1 \\
+6a &= +18
\end{align*}
\]

that is

\[
\begin{align*}
+2a + b &= -1 \\
+6a &= +18 \\
+6 &= +6
\end{align*}
\]

that is

\[
\begin{align*}
+2a + b &= -1 \\
a &= +3
\end{align*}
\]

and now we replace in the first equation \( a \) by +3:

\[
\begin{align*}
+2a + b &= -1 |_{a:=+3} \\
a &= +3
\end{align*}
\]

that is

\[
\begin{align*}
+2(+3) + b &= -1 \\
a &= +3
\end{align*}
\]

that is

\[
\begin{align*}
+6 + b &= -1 \\
a &= +3
\end{align*}
\]

and we reduce the first equation

\[
\begin{align*}
+6 + b - 6 &= -1 - 6 \\
a &= +3
\end{align*}
\]

which gives us, finally

\[
\begin{align*}
b &= -7 \\
a &= +3
\end{align*}
\]

v. So the global input-output rule of \( DAVE \) is

\[
x \xrightarrow{DAVE_{+3,-7}} DAVE_{+3,-7}(x) = +3x - 7
\]