Chapter 15

Rational Degree & Algebra Reviews

Rational functions are functions whose *global input-output rule* is of the form

$$x \xrightarrow{RAT} RAT(x) = \frac{POLY_{Num}(x)}{POLY_{Den}(x)}$$

where $POLY_{Num}(x)$ and $POLY_{Den}(x)$ stand for two *positive-exponent* polynomial expressions.

Example 1. The function whose global input-output rule is

$$x \xrightarrow{TAB} TAB(x) = \frac{-3x^2 + 4x - 7}{-5x^4 - 8}$$

is a rational function in which:

- $POLY_{Num}(x)$ is $-3x^2 + 4x - 7$
- $POLY_{Den}(x)$ is $-5x^4 - 8$

15.1 Rational Degree

Because the *upper degree* of polynomial functions is what we used to sort polynomial functions into different *types*, we now try to extend the idea of

rational degree
regular rational function
exceptional rational function

upper degree to the case of rational functions in the hope that this will also help us sort rational functions into different *types*.

Given a rational function whose global input-output rule is

$$x \xrightarrow{RAT} RAT(x) = \frac{POLY_{Num}(x)}{POLY_{Den}(x)}$$

the **rational degree** of this rational function is the upper degree of $POLY_{Num}(x)$ minus the upper degree of $POLY_{Den}(x)$:

$$\text{Rat.Deg. of } \frac{POLY_{Num}(x)}{POLY_{Den}(x)} = \text{UppDeg. of } POLY_{Num}(x) - \text{UppDeg. of } POLY_{Den}(x)$$

Thus, the *rational degree* of a rational function can well be *negative*.

NOTE. The *rational degree* is to rational function very much what the *size* is to arithmetic fractions in "school arithmetic" which distinguishes fractions according to the *size* of the numerator compared to the *size* of the denominator even though, by now, the distinctions are only an inconsequential remnant of history..

What happened is that, historically, the earliest arithmetic fractions were "unit fractions" , that is reciprocals of whole numbers such as one half, one third, one quarter, etc. Later came "Egyptian fractions", that is combinations of (distinct) unit fractions, such as one third and one fifth and one eleventh, etc. A much later development were the "proper fractions", also called "vulgar fractions", such as two thirds, three fifths etc. Later still, came "improper fractions" such as five thirds, seven halves, etc. And finally "mixed numbers", such as three and two sevenths. Today, none of these distinctions matters inasmuch as we treat all fractions in the same manner.

However, while these "school arithmetic" distinctions are based on the *size* of the numerator versus the *size* of the denominator and make no real differences in the way we handle arithmetic fractions, in the case of rational functions, the above distinction based on the *upper degree* of the numerator versus the *upper degree* of the denominator will make a difference—even though no major one—in the way we will handle rational functions of different types.

In fact, by analogy with what we did with *power functions*, we will say that

- Rational functions whose rational degree is either > 1 or < 0, are **regular rational functions**,
- Rational functions whose rational degree is either $= 0$ or $= 1$, are **exceptional rational functions**.

EXAMPLE 2. Find the rational degree of the function $DOUGH$ whose global input-output rule is

$$x \xrightarrow{DOUGH} DOUGH(x) = \frac{+1x^4 - 6x^3 + 8x^2 + 6x - 9}{x^2 - 5x + 6}$$

Since the rational degree is given by

$$\text{Rat.Deg. of } \frac{POLY_{Num}(x)}{POLY_{Den}(x)} = \text{UppDeg. of } POLY_{Num}(x) - \text{UppDeg. of } POLY_{Den}(x)$$

and since, here,

- $POLY_{Num}(x) = +1x^4 - 6x^3 + 8x^2 + 6x - 9$
- $POLY_{Den}(x) = +1x^2 - 5x + 6$

we get from the definition of the upper degree of a polynomial that:

$$\begin{aligned}
\text{UppDeg. of } +1x^4 - 6x^3 + 8x^2 + 6x - 9 &= \text{Exponent of Highest Term} \\
&= \text{Exponent of } +1x^4 \\
&= 4 \\
\text{UppDeg. of } +1x^2 - 5x + 6 &= \text{Exponent of Highest Term} \\
&= \text{Exponent of } +1x^2 \\
&= 2
\end{aligned}$$

so that the rational degree of the rational function $DOUGH$ is:

$$\begin{aligned}
\text{Rat.Deg. of } \frac{+1x^4 - 6x^3 + 8x^2 + 6x - 9}{+1x^2 - 5x + 6} &= \text{Exponent of } +1x^4 - \text{Exponent of } +1x^2 \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

so that $DOUGH$ is an example of a rational function of degree > 1 and therefore of a *regular* rational function.

Higher Addition Formulas

15.2 Addition Formulas: $(x_0 + h)^n$ when $n > 3$

While we did not investigate polynomial functions of degree higher than 3 we will investigate rational functions that involve power functions of degree higher than 3 and therefore we need to investigate *Addition Formulas* for exponents higher than 3.

1. We already saw two *addition formulas*:

i. The addition formula for exponent 2 as given in **Chapter 10** by **THEOREM 32** (**Addition Formula For Squares**):

$$(x_0 + h)^2 = x_0^2 + 2x_0h + h^2$$

pattern

ii. The addition formula for exponent 3 as given in **Chapter 12** by **THEOREM 40** (**Addition Formula For Cubes**):

$$(x_0 + h)^3 = x_0^3 + 3x_0^2 h + 3x_0 h^2 + h^3$$

which we established both by repeated multiplication and by enlarging the sides of a *square* and of a *cube* of sides x_0 by a small amount h.

2. In the case of exponents higher than 3, though,

- Repeated-multiplication is going to get rapidly more and more painful as the exponent goes up,
- We cannot draw pictures in dimentsions higher than 3.

So, in order to find a *general procedure* for constructing the addition formulas for $(x_0 + h)^3$ when the exponent n is higher than 3, we will first try to look for a **pattern** in each one of the above two addition formulas without letting anything "go without saying" and see what *particular procedure* would give each one of them.

a. When the *exponent* is 3, we have:

$$(x_0 + h)^3 = x_0^3 + 3x_0^2 h + 3x_0 h^2 + h^3$$

that is, writing in all that "goes without saying",

$$= 1x_0^{+3}h^0 + 3x_0^{+2}h^{+1} + 3x_0^{+1}h^{+2} + 1x_0^0 h^{+3}$$

that is, replacing the short hand by the long hand

$$\begin{aligned} = \quad & 1 \bullet x_0 \bullet x_0 \bullet x_0 \\ & + 3 \bullet x_0 \bullet x_0 \bullet h \\ & + 3 \bullet x_0 \bullet h \bullet h \\ & + 1 \bullet h \bullet h \bullet h \end{aligned}$$

So, looking separately at the *powers* and at the *coefficients* in the addition formula for $(x_0 + h)^3$:

- The procedure for finding the *powers* would seem to be:

i. We make a row of 3 copies of x_0 with multiplication signs in-between:	$x_0 \bullet x_0 \bullet x_0$
ii. We tack on 3 copies of the row we wrote in step **i.**	$x_0 \bullet x_0 \bullet x_0$ $x_0 \bullet x_0 \bullet x_0$ $x_0 \bullet x_0 \bullet x_0$ $x_0 \bullet x_0 \bullet x_0$

iii. Starting with the first of the 3 rows that we tacked-on in step **ii.** and going down, we replace in each row, starting from the right, one more copy of x_0 by a copy of h than in the preceding row:

$$\begin{array}{c} x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet h \\ x_0 \bullet h \bullet h \\ h \bullet h \bullet h \end{array}$$

- The procedure for finding the *coefficients*

  1 3 3 1

  is certainly not obvious and we leave it alone for the time being.

b. When the exponent is 2, we have

$$(x_0 + h)^2 = x_0^2 + 2x_0h + h^2$$

that is, writing in all that "goes without saying",

$$= 1x_0^2h^0 + 2x_0^1h^1 + 1x_0^0h^2$$

that is, replacing the short hand by the long hand

$$\begin{aligned} = \quad & 1 \bullet x_0 \bullet x_0 \\ & + 2 \bullet x_0 \bullet h \\ & + 1 \bullet h \bullet h \end{aligned}$$

So, looking separately at the *powers* and at the *coefficients* in the addition formula for $(x_0 + h)^2$:

- The procedure for finding the *powers* would seem to be the same as for the previous addition formula:

  i. We make a row of 2 copies of x_0 with multiplication signs in-between:

  $$x_0 \bullet x_0$$

  ii. We tack on 2 copies of the row we wrote in step **i.**

  $$\begin{array}{c} x_0 \bullet x_0 \\ x_0 \bullet x_0 \\ x_0 \bullet x_0 \end{array}$$

  iii. Starting with the first of the 2 rows that we tacked-on in step **ii.** and going down, we replace in each row, starting from the right, one more copy of x_0 by a copy of h than in the preceding row:

  $$\begin{array}{c} x_0 \bullet x_0 \\ x_0 \bullet h \\ h \bullet h \end{array}$$

- The procedure for finding the *coefficients*

  1 2 1

  is certainly not obvious and we leave it alone for the time being.

c. Finally, for the sake of completion, we look at what happens when the exponent is 1. We have

$$(x_0 + h)^1 = x_0 + h$$

that is, writing in all that "goes without saying",

$$= 1x_0^1h^0 + 1x_0^0h^1$$

that is, replacing the short hand by the long hand

$$\begin{aligned} = \quad & 1 \bullet x_0 \\ & + 1 \bullet h \end{aligned}$$

So, looking separately at the *powers* and at the *coefficients* in the "addition formula" for $(x_0 + h)^1$:

- The procedure for finding the *powers* would seem to be the same as for the previous addition formula:

i. We make a row of 1 copy of x_0 with no room for multiplication signs in-between:	x_0
ii. We tack on 1 copy of the row we wrote in step **i.**	x_0 x_0
iii. Starting with the first–and only– row that we wrote in step **ii.**, we replace in that row, the one copy of x_0 by a copy of h:	x_0 h

- The procedure for finding the *coefficients*

  1 1

  is certainly not obvious and we leave it alone for the time being.

3. Putting everything together, the general procedure for constructing the addition formula for $(x_0 + h)^n$ would seem to be,

- The procedure for finding the *powers* seems to be in all cases:

  i. We make a row of n copies of x_0 with multiplication signs in-between:

PASCAL TRIANGLE
parent-entries

ii. We tack on n copies of the row we got in step **i.**

iii. Starting with the first of the n rows that we wrote in step **ii.** and going down, we replace in each row, starting from the right, one more copy of x_0 by a copy of h than in the preceding row:

- So far, the procedures for the *coefficients* were not obvious but when we write them starting with exponent 1 and ending with exponent 3

$$\begin{array}{lccccccc} n \leftarrow 1 & & & 1 & & 1 & & \\ n \leftarrow 2 & & 1 & & 2 & & 1 & \\ n \leftarrow 3 & 1 & & 3 & & 3 & & 1 \end{array}$$

we can see that we can get each entry in what is called the **PASCAL TRIANGLE** by adding its two **parent-entries**, that is the two entries just above it.

EXAMPLE 3. The next line in the **PASCAL TRIANGLE** for $n \leftarrow 4$ would be:

$$\begin{array}{lccccccccc} n \leftarrow 3 & & & 1 & & 3 & & 3 & & 1 \\ & & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \\ n \leftarrow 4 & 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

We thus have two procedures, one for getting the *powers* and the other for getting the *coefficients* and all that remains is to put them together.

EXAMPLE 4. In order to construct the addition formula for $(x_0+h)^5$, we proceed as follows:

i. We make a row of 5 copies of x_0 with multiplication signs in-between:

$$x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0$$

ii. We tack on 5 copies of the row we wrote in step **i.**

$$\begin{array}{c} x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \end{array}$$

iii. Starting with the first of the 5 rows that we tacked-on in step **ii.** and going down, we replace in each row, starting from the right, one more copy of x_0 by a copy of h than in the preceding row:

$$\begin{array}{c} x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet x_0 \\ x_0 \bullet x_0 \bullet x_0 \bullet x_0 \bullet h \\ x_0 \bullet x_0 \bullet x_0 \bullet h \bullet h \\ x_0 \bullet h \bullet h \bullet h \bullet h \\ x_0 \bullet h \bullet h \bullet h \bullet h \\ h \bullet h \bullet h \bullet h \bullet h \end{array}$$

iv. We get the coefficients from the **PASCAL TRIANGLE**:

$$\begin{array}{lccccccccccc} n \leftarrow 4 & & & 1 & & 4 & & 6 & & 4 & & 1 \\ & & \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & \\ n \leftarrow 5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

4. *Proving* that all this is indeed the case would involve more work than we are willing to do here and so we will take the following for granted:

THEOREM 1 (BINOMIAL THEOREM). *The addition formula for a binomial of degree n is:*

$$
\begin{aligned}
(x_0+h)^n = \frac{1}{1}x_0^n h^0 \\
+ \frac{n}{1}x_0^{n-1}h^{+1} \\
+ \frac{n(n-1)}{1\cdot 2}x_0^{n-2}h^{+2} \\
+ \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x_0^{n-3}h^{+3} \\
+ \ \dots \\
+ \frac{n(n-1)(n-2)\cdots(1)}{1\cdot 2\cdot 3\cdots n}x_0^0 h^n
\end{aligned}
$$

where it is interesting to note that, no matter what n,

$$\frac{n(n-1)(n-2)\cdots(1)}{1\cdot 2\cdot 3\cdots n} = 1$$

because it corresponds to the fact that the rows in the **PASCAL TRIANGLE** are symmetrical.

5. According to the **BINOMIAL THEOREM**, when $n = 0$ we should have

$$(x_0+h)^0 = x_0^0 h^0$$

and since the coefficient 1 goes without saying:

$$= 1x_0^0 h^0 = 1$$

This is of course as it should be but what this says is that the "zeroth" line in the **PASCAL TRIANGLE** is 1 so that the "complete" **PASCAL TRIANGLE** is:

$n \leftarrow 0$							1							slanted row
$n \leftarrow 1$						1		1						
$n \leftarrow 2$					1		2		1					
$n \leftarrow 3$				1		3		3		1				
$n \leftarrow 4$			1		4		6		4		1			
$n \leftarrow 5$		1		5		10		10		5		1		
$n \leftarrow 6$	1		6		15		20		15		6		1	
......	...	...	...	...	...	...	...	...						

6. Usually, though, we do not need the *complete addition formulas* but only the first three or, at worst, the first four terms and so using the **PASCAL TRIANGLE** as we did above would turn out to be a huge waste of energy.

The key to a more economical way of using the **PASCAL TRIANGLE** is to look at the **slanted rows** and notice that:

- The numbers in the *first* slanted row are all 1 and they are the coefficients of the h^0 powers in the addition formulas for $(x_0+h)^1$, $(x_0+h)^2$, $(x_0+h)^3$, $(x_0+h)^4$, $(x_0+h)^5$, $(x_0+h)^6$, etc, which corresponds to the fact that the coefficient of the h^0 power in x_0^n is 1 no matter what n.
- The numbers in the *second* slanted row are $1, 2, 3, 4, 5, 6, \ldots$ and they are the coefficients of the h^{+1} powers in the addition formulas for $(x_0+h)^1$, $(x_0+h)^2$, $(x_0+h)^3$, $(x_0+h)^4$, $(x_0+h)^5$, $(x_0+h)^6$, etc, which shows that the coefficient of the h^{+1} power in x_0^n is n no matter what n.
- We check that the *third* slanted row, $1, 3, 6, 10, 15, \ldots$ are the coefficients of the h^{+2} powers in the addition formulas for $(x_0+h)^1$, $(x_0+h)^2$, $(x_0+h)^3$, $(x_0+h)^4$, $(x_0+h)^5$, $(x_0+h)^6$, etc, and we observe that the coefficient of the h^{+2} power in x_0^n is the half-product of the two coefficients to its left and therefore equal to $\dfrac{n(n-1)}{2}$ no matter what n.

EXAMPLE 5. Instead of getting the coefficient of h^{+2} in the addition formula for $(x_0+h)^5$ as the sum of its two "parents", 4 and 6:

$$
\begin{array}{ccccccc}
n \leftarrow 4 & & 1 & 4 & & 6 & 4 & 1 \\
 & & & & \searrow \quad \swarrow & & & \\
n \leftarrow 5 & 1 & 5 & & \boxed{10} & & 10 & 5 & 1
\end{array}
$$

cycles
step

we can get the coefficient of h^{+2} in the addition formula for $(x_0 + h)^5$ as the half-product of the two numbers to its left, 5 and 4:

$$\begin{array}{lccccccccccc}
n \leftarrow 4 & & & 1 & & 4 & & & 6 & & 4 & & 1 \\
 & & & & & & \searrow & & & & & & \\
n \leftarrow 5 & & 1 & & 5 & \longrightarrow & & 10 & & 10 & & 5 & & 1
\end{array}$$

As a result, we only need, and can construct only, the first two slanted rows of the **PASCAL TRIANGLE**.

EXAMPLE 6. In the addition formula for exponent 5, we get the first *three* coefficients just by constructing only the first *two* slanted rows:

$$\begin{array}{l|ccccccccccccc}
n \leftarrow 0 & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & \swarrow & & \searrow & \\
n \leftarrow 1 & & & & & & & & & 1 & & & & \mathbf{1} \\
 & & & & & & & & \swarrow & & \searrow & & \swarrow & \\
n \leftarrow 2 & & & & & & & 1 & & & & \mathbf{2} & & \\
 & & & & & & \swarrow & & \searrow & & \swarrow & & & \\
n \leftarrow 3 & & & & & 1 & & & & \mathbf{3} & & & & \\
 & & & & \swarrow & & \searrow & & \swarrow & & & & & \\
n \leftarrow 4 & & & 1 & & & & \mathbf{4} & & & & & & \\
 & & \swarrow & & \searrow & & \swarrow & & \searrow & & & & & \\
n \leftarrow \mathbf{5} & 1 & & & & \mathbf{5} & & \longrightarrow & & \mathbf{10} & & & &
\end{array}$$

where 10 is the half-product of the second entries in the last two horizontal rows.

POLYNOMIAL DIVISIONS

15.3 Division in Descending Exponents

Since *decimal numbers* are combinations of powers of TEN, it should not be surprising that the procedure for dividing decimal numbers should also work for *polynomials* which are combinations of powers of x.

The *procedure* consists of successive **cycles**, one for each monomial in the quotient. During each of these *cycles*, we go through four **step**s:

Step I. We find each *monomial* of the *quotient* by dividing the *first monomial* in the divisor into the *first monomial* of the previous partial remainder or, if there is not yet a partial remainder, from the *dividend*.

Step II. We find the *partial product* by multiplying the *full divisor* by the *monomial* of the quotient we found in Step I.

Step III. We find the *partial remainder* by subtracting the *partial product* we found in Step II from the previous partial remainder or, if there is not yet a partial remainder, from the *dividend*.

Step IV. We decide if we want to:

- *stop* the division,

or

- *continue* the division and go through another *cycle*.

EXAMPLE 7. In order to compute $\dfrac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5 - 2}$, we set up the division of $-3x^2 + 5 - 2$ "into" $-12x^3 + 11x^2 - 17x + 1$

$$\begin{array}{rrr|rrrr} -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -17x & +1 \end{array}$$

and we proceed as follows:

CYCLE 1. Step I. We find the *first monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into the *first monomial of the dividend*, $-12x^3$, which give us $\dfrac{-12x^3}{-3x^2} = +4x$ as first term in the quotient:

$$\begin{array}{rrr|rrrr} & & & +4x & & & \\ \hline -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -17x & +1 \end{array}$$

Step II. We find the *first partial product* by multiplying the *full divisor* by the *first monomial in the quotient*:

$$\begin{array}{lrrr|rrrr} & & & & +4x & & & \\ \hline & -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -17x & +1 \\ \text{First partial product:} & & & & -12x^3 & +20x^2 & -8x & \end{array}$$

Step III. We find the *first partial remainder* by *subtracting* the first partial product from the full dividend:

$$\begin{array}{rrr|rrrr} & & & +4x & & & \\ \hline -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -17x & +1 \\ & & \ominus & -12x^3 & +20x^2 & -8x & \end{array}$$

But to *subtract* the first partial product means to *add the opposite* of the first partial product to the full dividend:

$$\begin{array}{lrrr|rrrr} & & & & +4x & & & \\ \hline & -3x^2 & +5x & -2 & -12x^3 & +11x^2 & -17x & +1 \\ & & & \oplus & +12x^3 & -20x^2 & +8x & \\ \hline \text{First remainder:} & & & & +0x^3 & -9x^2 & -9x & +1 \end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4x$.
  - the *remainder* of the division is $-9x^2 - 8x + 1$

  If we don't care about the *remainder*, we write:

$$\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2} = +4x + [...]$$

where we write $+\,[...]$ as a reminder that $\dfrac{-12x^3+11x^2-17x+1}{-3x^2+5x-2}$ is not exactly equal to $+4x$ since there was a *remainder*.

- If we decide to *continue* the division, we begin a new cycle

CYCLE 2. Step I. We find the *second monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into *the first monomial in the first partial remainder*, $-9x^2$, which gives us $\dfrac{-9x^2}{-3x^2} = +3$ for the second term of the quotient:

$$\begin{array}{rrrrrrrr}
 & & & & +4x & +3 & & \\
\cline{5-8}
-3x^2 & +5x & -2 & \big) & -12x^3 & +11x^2 & -17x & +1 \\
 & & & & -12x^3 & +20x^2 & -8x & \\
\cline{5-8}
 & & & & & -9x^2 & -9x & +1
\end{array}$$

Step II. We find the *second partial product* by multiplying the *full divisor* by the *second monomial in the quotient*:

$$\begin{array}{rrrrrrrr}
 & & & & +4x & +3 & & \\
\cline{5-8}
-3x^2 & +5x & -2 & \big) & -12x^3 & +11x^2 & -17x & +1 \\
 & & & & -12x^3 & +20x^2 & -8x & \\
\cline{5-8}
 & & & & & -9x^2 & -9x & +1 \\
\multicolumn{5}{l}{\text{Second partial product:}} & -9x^2 & +15x & -6
\end{array}$$

Step III. We find the *second partial remainder* by *subtracting* the second partial product from the first partial remainder:

$$\begin{array}{rrrrrrrr}
 & & & & +4x & +3 & & \\
\cline{5-8}
-3x^2 & +5x & -2 & \big) & -12x^3 & +11x^2 & -17x & +1 \\
 & & & & -12x^3 & +20x^2 & -8x & \\
\cline{5-8}
 & & & & & -9x^2 & -9x & +1 \\
 & & & & \ominus & -9x^2 & +15x & -6 \\
\cline{6-8}
\end{array}$$

But to *subtract* the second partial product means to *add the opposite* of the second partial product to the first partial remainder:

$$\begin{array}{rrrrrrrr}
 & & & & +4x & +3 & & \\
\cline{5-8}
-3x^2 & +5x & -2 & \big) & -12x^3 & +11x^2 & -17x & +1 \\
 & & & & -12x^3 & +20x^2 & -8x & \\
\cline{5-8}
 & & & & & -9x^2 & -9x & +1 \\
 & & & & \oplus & +9x^2 & -15x & +6 \\
\cline{6-8}
\multicolumn{5}{l}{\text{Second remainder:}} & +0x^2 & -24x & +7
\end{array}$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4x+3$.
  - the *remainder* of the division is $-24x+7$

  If we don't care about the *remainder*, we write:

$$\frac{-12x^3+11x^2-17x+1}{-3x^2+5x-2} = +4x+3+[...]$$

where we write $+\,[...]$ as a reminder that $\dfrac{-12x^3+11x^2-17x+1}{-3x^2+5x-2}$ is not exactly equal to $+4x+3$ since there was a *remainder*.

- If we decide to *continue* the division, we begin a new cycle

CYCLE 3. Step I. We find the *third monomial in the quotient* by dividing the *first monomial in the divisor*, $-3x^2$, into *the first monomial in the second partial remainder*, $-24x$ that is $\dfrac{-24x}{-3x^2} = +8x^{-1}$

$$
\begin{array}{r|rrrrr}
 & +4x & +3 & +8x^{-1} & & \\
\hline
-3x^2\ +5x\ -2 & -12x^3 & +11x^2 & -17x & +1 & \\
 & -12x^3 & +20x^2 & -8x & & \\
 & & -9x^2 & -9x & +1 & \\
 & & +9x^2 & -15x & +6 & \\
 & & & -24x & +7 &
\end{array}
$$

Step II. We find the *third partial product* by multiplying the *full divisor* by the *third monomial in the quotient*:

$$
\begin{array}{r|rrrrr}
 & +4x & +3 & +8x^{-1} & & \\
\hline
-3x^2\ +5x\ -2 & -12x^3 & +11x^2 & -17x & +1 & \\
 & -12x^3 & +20x^2 & -8x & & \\
 & & -9x^2 & -9x & +1 & \\
 & & +9x^2 & -15x & +6 & \\
 & & & -24x & +7 & \\
\text{Third partial product:} & & & -24x & +40 & -16x^{-1}
\end{array}
$$

Step III. We find the *third partial remainder* by *subtracting* the third partial product from the first partial remainder:

$$
\begin{array}{r|rrrrr}
 & +4x & +3 & +8x^{-1} & & \\
\hline
-3x^2\ +5x\ -2 & -12x^3 & +11x^2 & -17x & +1 & \\
 & -12x^3 & +20x^2 & -8x & & \\
 & & -9x^2 & -9x & +1 & \\
 & & +9x^2 & -15x & +6 & \\
 & & & -24x & +7 & \\
 & \ominus & & -24x & +40 & -16x^{-1}
\end{array}
$$

But to *subtract* the second partial product means to *add the opposite* of the second partial product to the first partial remainder:

$$
\begin{array}{r|rrrrr}
 & +4x & +3 & +8x^{-1} & & \\
\hline
-3x^2\ +5x\ -2 & -12x^3 & +11x^2 & -17x & +1 & \\
 & -12x^3 & +20x^2 & -8x & & \\
 & & -9x^2 & -9x & +1 & \\
 & & +9x^2 & -15x & +6 & \\
 & & & -24x & +7 & \\
 & \oplus & & +24x & -40 & +16x^{-1} \\
\text{Third remainder:} & & & 0x & -33 & +16x^{-1}
\end{array}
$$

Step IV. We decide if we want to *stop* or *continue* the division.

- If we decide to *stop* the division,
  - the *quotient* of the division is $\boxed{+4x + 3 + 8x^{-1}}$.
  - the *remainder* of the division is $\boxed{-33 + 16x^{-1}}$

  If we don't care about the *remainder*, we write:

$$\boxed{\frac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2} = +4x + 3 + 8x^{-1} + [...]}$$

  where we write $+$ [...] as a reminder that $\dfrac{-12x^3 + 11x^2 - 17x + 1}{-3x^2 + 5x - 2}$ is not exactly equal to $+4x + 3 + 8x^{-1}$ since there was a *remainder*.
- If we decide to *continue* the division, we begin a new cycle

Just as, in arithmetic, we need not stop a division when the quotient reaches a monomial with exponent 0 because we can always divide a monomial into another and we can have *negative* exponents. In fact, again just as in arithmetic, there are cases where we absolutely need to go beyond the exponent 0 and use negative exponents. (See Epilogue.)

Example 8. In order to divide $-12x^3 + 11x^2 - 17x + 4$ by $-3x^2 + 5x - 2$ down to a x^{-2} monomial, we write (in the *anglo-saxon* tradition):

$$\begin{array}{r|llllll}
 & +4x \quad +3 & +8x^{-1} & +10x^{-2} & & & \\
\hline
-3x^2 \ +5x \ -2 & -12x^3 & +11x^2 & -17x & +4 & & \\
 & +12x^3 & -20x^2 & +8x & & & \\
\cline{2-4}
 & & -9x^2 & -9x & & & \\
 & & +9x^2 & -15x & +6 & & \\
\cline{3-5}
 & & & -24x & +10 & & \\
 & & & +24x & -40 & +16x^{-1} & \\
\cline{4-6}
 & & & & -30 & -16x^{-1} & \\
 & & & & +30 & -50x^{-1} & +20x^{-2} \\
\cline{5-7}
 & & & & & -66x^{-1} & +20x^{-2}
\end{array}$$

The quotient is

$$+4x + 3 + 8x^{-1} + 10x^{-2}$$

and the remainder is

$$-66x^{-1} + 20x^{-1}$$

15.4 Default Rules for Division

Since mathematicians are lazy,

- mathematicians do not write the + sign in front of the coefficients of leading monomials,
- mathematicians do not write monomials with 0 coefficient,

and, most dangerously,

- mathematicians want to write only one stage in Step III but there are two traditions concerning what then to write, as a result, in Step II:
  - In the *latin* tradition, in Step II, we write the *partial product*, that is what we get it from the *multiplication*, and so in Step III, when it comes to subtracting, we visualize the *opposite of the partial product* we wrote in Step II and we oplus what we *visualize*. The advantage is that each line is exactly what we get from the previous operation.

    EXAMPLE 9.

$$\begin{array}{r|rrrr} & +4x & +3 & & \\ \hline -3x^2 \ +5x \ -2 & -12x^3 & +11x^2 & -16x & +1 \\ & -12x^3 & +20x^2 & -8x & \\ \hline & & -9x^2 & -8x & +1 \end{array}$$

  - In the *anglo-saxon* tradition, we anticipate the subtraction to be done in Step III and in Step II *we write the opposite of the partial product* so in Step III we oplus what we *wrote* in Step II.

    EXAMPLE 10.

$$\begin{array}{r|rrrr} & +4x & +3 & & \\ \hline -3x^2 \ +5x \ -2 & -12x^3 & +11x^2 & -16x & +1 \\ & +12x^3 & -20x^2 & +8x & \\ \hline & & -9x^2 & -8x & +1 \end{array}$$

From now on we will of course follow the *anglo-saxon* tradition.

EXAMPLE 11. In order to compute $\dfrac{6x^3 + 13x^2 + 13x + 7}{2x + 1}$, we divide $2x + 1$ into $6x^3 + 13x^2 + 13x + 7$:

CYCLE 1. Step I. We find the *first monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & 3x^2 \\ \hline 2x+1 & 6x^3 + 13x^2 + 13x + 7 \end{array}$$

Step II. We get the *first opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & 3x^2 \\ \hline 2x+1 & 6x^3 + 13x^2 + 13x + 7 \\ & \underline{-\,6x^3 \ -\,3x^2} \end{array}$$

Step III. We get the *first remainder* by oplussing the first opposite product

$$\begin{array}{r|l} & \quad 3x^2 \\ \cline{2-2} 2x+1 & 6x^3 + 13x^2 + 13x + 7 \\ & \underline{-6x^3 \;\; -3x^2} \\ & \qquad 10x^2 + 13x \end{array}$$

Step IV. We decide if we want to stop or continue the division

– If we decide to *stop* the division,

  * the *quotient* of the division is $+3x^{+2}$.
  * the *remainder* of the division is $+10x^{+2} + 13x$

  If we don't care about the *remainder*, we write:

$$\frac{6x^3 + 13x^2 + 13x + 7}{2x+1} = +3x^{+2} + (\ldots)$$

  where we write $+\ (\ldots)$ as a reminder that $\frac{6x^3 + 13x^2 + 13x + 7}{2x+1}$ is not exactly equal to $+3x^{+2}$ since there was a *remainder*.

– If we decide to *continue* the division, we begin a new cycle

CYCLE 2. Step I. We find the *second monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & \quad 3x^2 \;\; +5x \\ \cline{2-2} 2x+1 & 6x^3 + 13x^2 + 13x + 7 \\ & \underline{-6x^3 \;\; -3x^2} \\ & \qquad 10x^2 + 13x \end{array}$$

Step II. We get the *second opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & \quad 3x^2 \;\; +5x \\ \cline{2-2} 2x+1 & 6x^3 + 13x^2 + 13x + 7 \\ & \underline{-6x^3 \;\; -3x^2} \\ & \qquad 10x^2 + 13x \\ & \qquad \underline{-10x^2 \;\; -5x} \end{array}$$

Step III. We get the *second remainder* by oplussing the fir second st opposite product

$$\begin{array}{r|l} & \quad 3x^2 \;\; +5x \\ \cline{2-2} 2x+1 & 6x^3 + 13x^2 + 13x + 7 \\ & \underline{-6x^3 \;\; -3x^2} \\ & \qquad 10x^2 + 13x \\ & \qquad \underline{-10x^2 \;\; -5x} \\ & \qquad\qquad\quad 8x + 7 \end{array}$$

Step IV. We decide if we want to stop or continue the division

– If we decide to *stop* the division,

  * the *quotient* of the division is $+3x^{+2} + 5x$.

∗ the *remainder* of the division is $+8x+7$

If we don't care about the *remainder*, we write:

$$\frac{6x^3+13x^2+13x+7}{2x+1} = +3x^{+2}+5x+(\ldots)$$

where we write $+(\ldots)$ as a reminder that $\dfrac{6x^3+13x^2+13x+7}{2x+1}$ is not exactly equal to $+3x^{+2}+5x$ since there was a *remainder*.

– If we decide to *continue* the division, we begin a new cycle

CYCLE 3. Step I. We find the *third monomial* in the quotient by *short division*:

$$\begin{array}{r|l} & \quad 3x^2 \quad +5x+4 \\ \hline 2x+1) & \quad 6x^3+13x^2+13x+7 \\ & -6x^3 \quad -3x^2 \\ \hline & \qquad\quad 10x^2+13x \\ & \qquad -10x^2 \quad -5x \\ \hline & \qquad\qquad\qquad 8x+7 \end{array}$$

Step II. We get the *third opposite product* by writing the opposite of the result of the *full multiplication*

$$\begin{array}{r|l} & \quad 3x^2 \quad +5x+4 \\ \hline 2x+1) & \quad 6x^3+13x^2+13x+7 \\ & -6x^3 \quad -3x^2 \\ \hline & \qquad\quad 10x^2+13x \\ & \qquad -10x^2 \quad -5x \\ \hline & \qquad\qquad\qquad 8x+7 \\ & \qquad\qquad\quad -8x-4 \\ \hline \end{array}$$

Step III. We get the *third remainder* by oplussing the third opposite product

$$\begin{array}{r|l} & \quad 3x^2 \quad +5x+4 \\ \hline 2x+1) & \quad 6x^3+13x^2+13x+7 \\ & -6x^3 \quad -3x^2 \\ \hline & \qquad\quad 10x^2+13x \\ & \qquad -10x^2 \quad -5x \\ \hline & \qquad\qquad\qquad 8x+7 \\ & \qquad\qquad\quad -8x-4 \\ \hline & \qquad\qquad\qquad\qquad 3 \end{array}$$

Step IV. We decide if we want to stop or continue the division

– If we decide to *stop* the division,

∗ the *quotient* of the division is $+3x^{+2}+5x+4$.

∗ the *remainder* of the division is $+3$

If we don't care about the *remainder*, we write:

$$\frac{6x^3+13x^2+13x+7}{2x+1} = +3x^{+2}+5x+4+(\ldots)$$

where we write $+\ (\ldots)$ as a reminder that $\dfrac{6x^3+13x^2+13x+7}{2x+1}$ is not exactly equal to $+3x^{+2}+5x+4$ since there was a *remainder*.

– If we decide to *continue* the division, we begin a new cycle

- When writing the partial remainders, we do not write the monomials beyond those that result from subtracting the *partial product*.

EXAMPLE 12.

	$+4x$	$+3$		
$-3x^2\ +5x\ -2$)	$-12x^3$	$+11x^2$	$-16x$	$+1$
First *opposite* partial product:	$+12x^3$	$-20x^2$	$+8x$	
First remainder:		$-9x^2$	$-8x$	
Second *opposite* partial product:		$+9x^2$	$-15x$	$+6$
Second remainder:			$-23x$	$+7$

The danger here is that, when we do the next subtraction, we may subtract from 0 rather than from the monomial that was left unwritten in the partial remainder.

15.5 Comparison With Arithmetic Division

The procedure to divide polynomials is in fact a lot simpler than the procedure for dividing in ARITHMETIC:

- There is never any "carryover"
- The first term of each partial remainder always has coefficient 0
- There are no Trials in **Step I** because, when we divide the first monomial in the divisor into the first monomial of a partial remainder, we always get a coefficient for the corresponding monomial in the quotient and the worst that can happen is that this coefficient is a fraction.

15.6 Division in Ascending Order Of Exponents

Fortunately, the procedure is exactly the same as in the case of division in descending order of exponents and so we will just look at an example.

EXAMPLE 13. In order to compute $\dfrac{-12+23h-h^2-2h^3}{-3+2h}$, we divide $-3+2h$ into $-12+23h-h^2-2h^3$:

	$+4$	$-5h$	$-3h^2$	
$-3\ +2h$)	-12	$+23h$	$-h^2$	$-2h^3$
First *opposite* partial product:	$+12$	$-8h$		
First remainder:		$+15h$	$-h^2$	$-2h^3$
Second *opposite* partial product:		$-15h$	$+10h^2$	
Second remainder:			$+9h^2$	$-2h^3$
Third *opposite* partial product:			$-9h^2$	$+6h^3$
Third remainder:				$+4h^3$

- If we decide to *stop* the division,
  - the *quotient* of the division is $+4 - 5h - 3h^2$.
  - the *remainder* of the division is $+4h^3$. Observe that if we replace the unspecified numerator h by, say, 0.2, then the remainder is equal to $4 \bullet 0.2^3 = 4 \bullet 0.008 = 0.032$ which is indeed small.

  If we don't care about the *remainder*, we write:

$$\frac{-12 + 23h - h^2 - 2h^3}{-3 + 2h} = +4 - 5h - 3h^2 + [...]$$

  where we write $+$ [...] as a reminder that $\dfrac{-12 + 23h - h^2 - 2h^3}{-3 + 2h}$ is not exactly equal to $+4 - 5h - 3h^2$ since there was a *remainder*.
- If we were to decide to *continue* the division, we would begin a new cycle

EXAMPLE 14. In order to divide $2x^3 + 5x^2 - 6$ by $3x - 1$ we write (in the *anglo-saxon* tradition):

$$\begin{array}{r|rrrr}
 & & \frac{2}{3}x^2 & +\frac{17}{9}x & +\frac{17}{27} \\
\hline
3x-1 & 2x^3 & +5x^2 & & -6 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\
\cline{2-3}
 & & \frac{17}{3}x^2 & & \\
 & & -\frac{17}{3}x^2 & +\frac{17}{9}x & \\
\cline{3-4}
 & & & \frac{17}{9}x & -6 \\
 & & & -\frac{17}{9}x & +\frac{17}{27} \\
\cline{4-5}
 & & & & -\frac{145}{27}
\end{array}$$

The *quotient* is

$$+\tfrac{2}{3}x^2 + \tfrac{17}{9}x + \tfrac{17}{27}$$

The *remainder* is

$$-\tfrac{145}{27}$$

GRAPHIC ADJUSTING

15.7 Graphic Difficulties

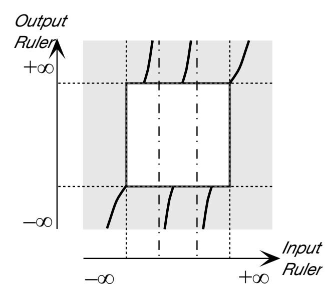
Finally, when there is one or more ∞-height bounded input(s), beginners often encounter difficulties when trying to interpolate smoothly the outlying graph of a rational function.

The difficulties are caused by the fact that, when we draw the local graph near ∞ and the local graphs near the ∞-height inputs from the local input-output rules, we are only concerned with drawing the local graphs themselves from the local input-output rules. In particular, when we draw the local graph near ∞ and the local graphs near the ∞-height inputs, we want to bend them enough to show the concavity but we often end up bending them *too much* to interpolate them.

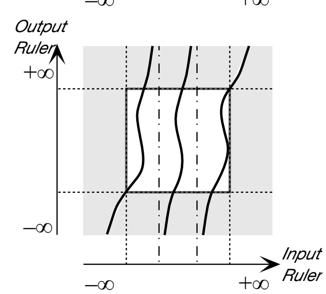
But then, what often happens as a result is that, when we want to interpolate, the local graphs may not line up well enough for us to interpolate them (smoothly).

EXAMPLE 15.

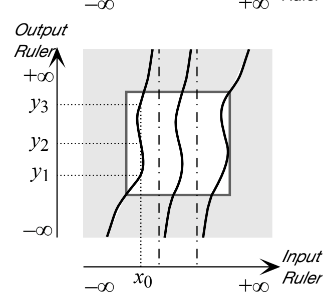
Given the rational function whose off-screen graph was drawn so as to show the concavity.



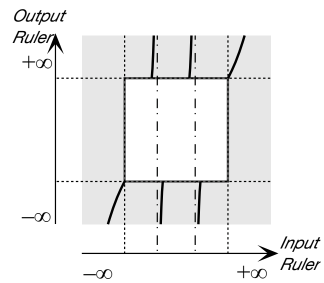
Here is what can happens when we attempt to interpolate



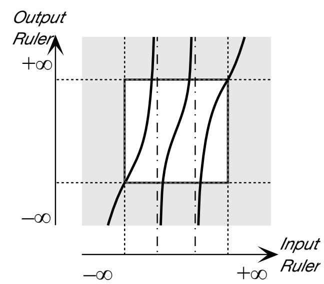
Of course, this is absolutely impossible since, according to this global graph, there would be inputs, such as x_0, with more than one output, y_1, y_2, . . . :



But if we unbend the local graphs just a bit as in



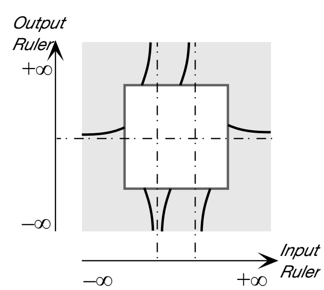
we have no trouble interpolating:



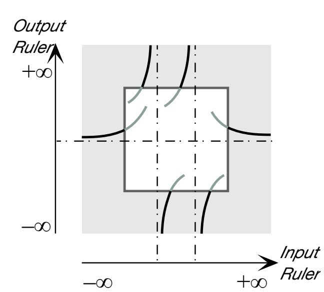
The way to avoid this difficulty is not to wait until we have to interpolate but to catch any problem as we draw the local graphs by mentally extending the local graphs slightly into the transitions.

EXAMPLE 16.

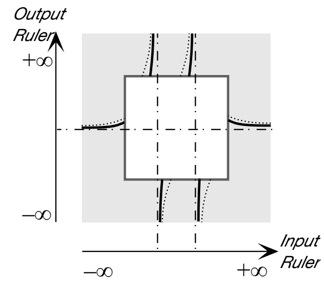
Given the rational function whose off-screen graph was drawn do as to show the concavity



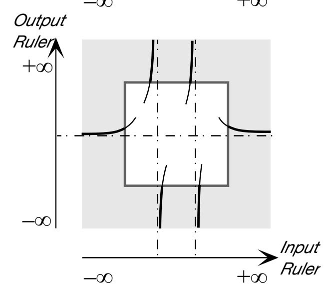
we can already see by extending the local graphs just a little bit into the transitions that this will cause a lot of trouble when we try to interpolate the local graph:



So, here, we bend the local graph near ∞ a little bit more and we unbend the local graphs near the ∞-height inputs a little bit:



We check again by extending the local graphs just a little bit into the transitions:



and indeed now we have no trouble interpolating: